



# Azimuthal Averaging for Rotational Electromagnetic Waves

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## To cite this article:

Jinsik Mok, Hyoung-In Lee. Azimuthal Averaging for Rotational Electromagnetic Waves. *World Journal of Applied Physics*.

Vol. 1, No. 2, 2016, pp. 30-36. doi: 10.11648/j.wjap.20160102.11

**Received:** October 9, 2016; **Accepted:** October 21, 2016; **Published:** November 14, 2016

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**Abstract:** We make use of the well-known integral representation of Bessel function in order to derive higher-order rotational electromagnetic waves. For this purpose, we employ the simplest weighting function in carrying out an azimuthal averaging of an E-parallel-H wave. To our surprise, the resulting wave turns out to describe interactions between two co-rotational waves with a half-cycle phase difference. In addition, we will provide both implications of the resulting waves concerning optical vortices and relevant technical applications.

**Keywords:** Integral Representation, Bessel Function, Electromagnetic Wave, Azimuthal Mode Index, Interference, Optical Vortex

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## 1. Introduction

The angular momentum of electromagnetic waves is not only a scientific curiosity but also a crucial ingredient in nanotechnology such as optical manipulation [1, 2]. In this respect, there is a surge of interests by optics community in the cylindrical electromagnetic (EM) waves with higher-order rotations.

For instance, rotational EM waves carrying a variety of angular momentums are employed for multiplexing EM waves for wireless communication with higher information density [3, 4]. As regards exact solutions to Maxwell's equations, three-dimensional Bessel beams are quite useful because of their non-diffractive axial propagations [4-8]. In comparison, two-dimensional (2D) Bessel waves in the absence of axial propagations are describable in terms of only two polar coordinates. In this sense, 2D Bessel waves are more akin to whispering-gallery waves [9, 10].

In the meantime, EM waves with their electric field and magnetic field parallel to each other have been analytically derived as a special solution to Maxwell's equations [5]. The authors of [5] derived the zeroth-order Bessel wave without rotations by taking an azimuthal averaging with a uniform weighting function. Furthermore, they suggested that higher-order Bessel waves could be derived by making a suitable choice of weighting function.

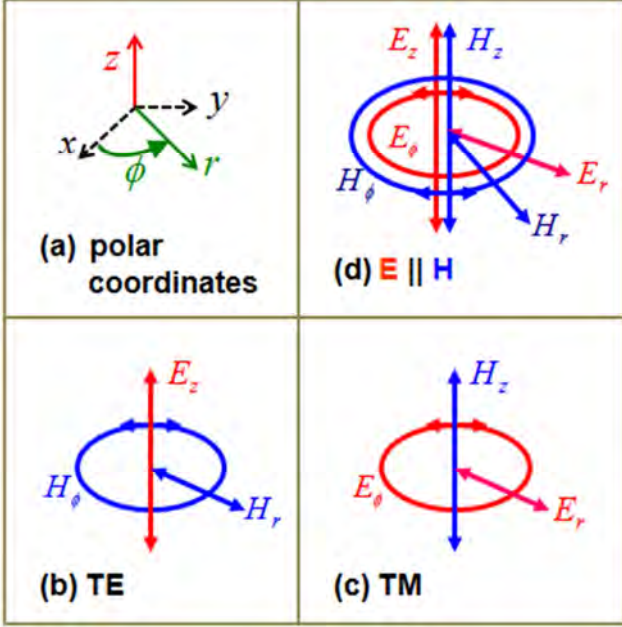
We are here to actually perform such an azimuthal averaging for higher-order rotational EM waves. To this goal, we have employed the simplest conceivable weighting function. As expected, we found higher-order rotational 2D Bessel waves. However, a rotational EM wave thus found turns out to be non-trivial because it disappears for odd-numbered azimuthal mode indices (AMIs). We found out that this even-numbered wave arises from two co-rotational EM waves with a half-cycle phase difference. As regards this parity issue, we will indicate further works to be done.

Our presentation goes as follows. Section 2 deals with zeroth-order 2D Bessel waves. Section 3 derives higher-order 2D Bessel waves. Section 4 provides discussions, followed by Section 5 on Conclusion.

## 2. Azimuthal Averaging for Zeroth-Order Bessel Waves

By definition, the usual zeroth-order 3D Bessel beams display no azimuthal rotations. But, they exhibit axial propagation [4]. Moreover, the non-zero field components in this case are either of the two types: [i] transverse-electric (TE) wave: non-zero axial electric field, non-zero radial

electric field, and non-zero azimuthal magnetic field, [ii] transverse-magnetic (TM) wave: non-zero axial magnetic field, non-zero radial magnetic field, and non-zero azimuthal electric field [4, 8]. See Fig. 1 (a) for the coordinate systems employed. Figures 1 (b) and (c) illustrate the TE and TM waves with their respective non-zero field components.



**Fig. 1.** A schematic figure. (a) Cartesian coordinates with the  $z$ -axis referring to the transverse direction. The two-dimensional polar coordinates  $(r, \phi)$  are defined on the cross-sectional  $XY$ -plane.

In contrast, the zeroth-order Bessel wave under this study exhibits neither axial propagations nor azimuthal rotations. In this regard, the TE and TM waves shown in Fig. 1 may be interpreted to be either three- or two-dimensional.

Maxwell's equations can be reformulated for free-space wave propagations in terms of the usual dimensionless variables and parameters [10]. Besides, we assume our wave to be time-harmonic with the common phase factor  $e^{-i\omega t}$ , where  $i \equiv \sqrt{-1}$  and  $\omega$  is a constant frequency.

As a particular solution to Maxwell's equations, we obtain the electric-field vector  $\vec{E}$  parallel to the magnetic-field vector  $\vec{H}$  at all the positions. We call this wave a "E-parallel-H (or  $\vec{E} \parallel \vec{H}$  in a format of equation)" wave [5]. Notice however that the two field vectors are out of phase by a quarter cycle or  $\pi/2$ . Hence, a treatment solely of  $\vec{E}$  is sufficient. Figure 1 (d) schematically displays an  $\vec{E} \parallel \vec{H}$  wave, where all of the six field components are non-zero in general.

We further consider two-dimensional cylindrical EM waves described by polar coordinates  $(r, \phi)$ , where  $r$  and  $\phi$  are respectively the radial and azimuthal coordinates as depicted in Fig. 1 (a). As with most of two-dimensional EM waves, the axial component  $E_z$  is governed by Helmholtz

equation, while both radial component  $E_r$  and azimuthal component  $E_\phi$  are related to  $E_z$  [8, 10, 11].

The fundamental solution of the  $\vec{E} \parallel \vec{H}$  wave has been derived by [5], where  $E_z$  is proportional to  $\cos[r \cos(\tau - \phi)]$ . Here,  $\tau$  is a real parameter modifying the independent variable  $\phi$ . Consider then the following azimuthal averaging.

$$B_0(r, \phi) \equiv \frac{1}{2\pi} \int_0^{2\pi} w_0(\tau) \cos[r \cos(\tau - \phi)] d\tau. \quad (1)$$

Here,  $w_0(\tau)$  is a weighting function of zeroth order. By assuming the uniform weighting function  $w_0(\tau) = 1$ , we obtain the following.

$$B_0(r, \phi) = \frac{1}{2\pi} \int_0^{2\pi} \cos[r \cos(\tau - \phi)] d\tau. \quad (2)$$

By changing the integration variable such that  $(\tau - \phi) \rightarrow s$  and setting temporarily  $F(r, s) = \cos[r \cos(s)]$ ,

$$\begin{aligned} 2\pi B_0(r, \phi) &= \int_{-\phi}^{2\pi-\phi} \cos[r \cos(\tau - \phi)] d(\tau - \phi) \\ &= \int_{-\phi}^{2\pi-\phi} F(r, s) ds \\ &= \int_0^{2\pi} F(r, s) ds + \int_{2\pi}^{2\pi-\phi} F(r, s) ds + \int_{-\phi}^0 F(r, s) ds \end{aligned} \quad (3)$$

Consider the two off-period terms (the second and third terms).

$$\begin{aligned} &\int_{2\pi}^{2\pi-\phi} \cos[r \cos(s)] ds \\ &= \int_{2\pi}^{2\pi-\phi} \cos[r \cos(2\pi + s)] d(2\pi + s) \\ &= \int_0^{-\phi} \cos[r \cos(p)] dp = - \int_{-\phi}^0 \cos[r \cos(s)] ds \end{aligned} \quad (4)$$

Therefore, the two off-period terms (namely, the second and third terms) in Eq. (3) cancel each other, thereby leading to the following.

$$2\pi B_0(r) = \int_0^{2\pi} \cos[r \cos(s)] ds. \quad (5)$$

Hence, the dependence on  $\phi$  in Eq. (2) is eliminated in Eq. (5).

By changing the integration variables similarly and setting temporarily  $G(r, s) = \cos[r \sin(s)]$ ,

$$\begin{aligned}
2\pi B_0(r) &= \int_{-\pi/2}^{2\pi-\pi/2} \cos\left[r \cos\left(p + \frac{1}{2}\pi\right)\right] d\left(p + \frac{1}{2}\pi\right) \\
&= \int_{-\pi/2}^{2\pi-\pi/2} G(r, p) dp \\
&= \int_0^{2\pi} G(r, p) dp + \int_{-\pi/2}^0 G(r, p) dp + \int_{2\pi}^{2\pi-\pi/2} G(r, p) dp
\end{aligned} \quad (6)$$

Consider once more the two off-period terms (namely, the second and third terms).

$$\begin{aligned}
&\int_{2\pi}^{2\pi-\pi/2} \cos[r \sin(p)] dp \\
&= \int_{2\pi}^{2\pi-\pi/2} \cos[r \sin(2\pi + q)] d(2\pi + q) \\
&= \int_0^{-\pi/2} \cos[r \sin(q)] dq = - \int_{-\pi/2}^0 \cos[r \sin(p)] dp
\end{aligned} \quad (7)$$

Therefore, the two off-period terms in Eq. (6) cancel each other, thereby leading finally to the following.

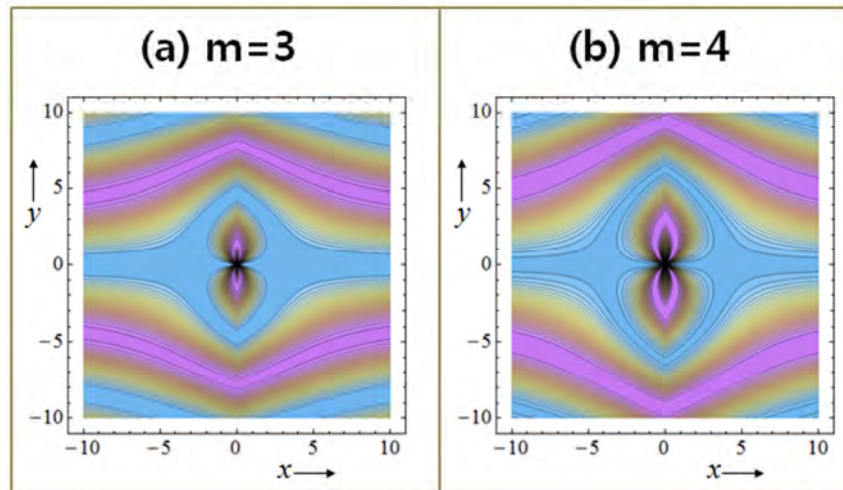
$$2\pi B_0(r) = \int_0^{2\pi} \cos[r \cos(s)] ds = \int_0^{2\pi} \cos[r \sin(s)] ds. \quad (8)$$

We now consult the following integral representation of Bessel function [12-15].

$$J_m(r) = \frac{1}{2\pi} \int_0^{2\pi} \cos[m\tau - r \sin(\tau)] d\tau. \quad (9)$$

Here,  $m \in \mathbb{Z}$  denoting integers is called the "azimuthal mode index (AMI)". As a consequence, Eq. (8) leads to  $B_0(r) = J_0(r)$  for  $w_0(\tau) = 1$ , indicating a zeroth-order Bessel wave, as has already been obtained in [5].

Figure 2 shows the integrand  $\cos[m\tau - r \sin(\tau)]$  of Eq. (9) on the Cartesian coordinates with the corresponding polar coordinates  $(r, \tau)$ . For simplicity, it is fixed that  $m = 3$  and  $m = 4$  respectively on panels (a) and (b) as examples of odd and even AMIs. The blue and red colors denote respectively positive and negative values, whereas the color intensity implies how large absolute values are. As a result, we find here that  $\cos[m\tau - r \sin(\tau)]$  does not exhibit any rotational feature for both odd and even AMIs.



**Fig. 2.** The integrand  $\cos[m\tau - r \sin(\tau)]$  evaluated on the Cartesian coordinates with the corresponding polar coordinates  $(r, \tau)$  for (a)  $m = 3$  (odd) and (b)  $m = 4$  (even).

### 3. Azimuthal Averaging for Higher-Order Bessel Waves

For the first time to the best of our knowledge, we are to perform the azimuthal averaging by taking non-constant weighting function  $w_m(\tau) = \exp(im\tau)$ . Therefore, the AMI  $m$  refers truly to angular periodicity of all the field variables. We stress that the choice  $w_m(\tau) = \exp(im\tau)$  is the simplest conceivable one in expectation of higher-order rotational EM waves for  $m \neq 0$ . Other weighting functions such as piecewise constant or nonlinear ones can be tried as long as they satisfy the  $2\pi$ -periodicity.

Hence, Eq. (1) takes the following form.

$$B_m(r, \phi) = \frac{1}{2\pi} \int_0^{2\pi} \exp(im\tau) \cos[r \cos(\tau - \phi)] d\tau. \quad (10)$$

We carry out this integration in the following several steps. Firstly, we introduce the complementary angle  $\phi$  satisfying  $\phi + \phi = \frac{1}{2}\pi$  so that  $\cos(\tau - \phi) = \sin(\tau + \phi)$ .

$$B_m(r, \phi) = \frac{1}{2\pi} \int_0^{2\pi} \exp(im\tau) \cos[r \sin(\tau + \phi)] d\tau. \quad (11)$$

The reason for constructing the last form is that the

integrand  $\cos[m\tau - r \sin(\tau)]$  is employed for Eq. (9) in defining  $J_m(r)$  instead of  $\cos[m\tau - r \cos(\tau)]$ .

Secondly, we substitute  $\exp(im\tau) = \cos(m\tau) + i\sin(m\tau)$  into Eq. (11) to get two integrals as follows.

$$B_m(r, \varphi) = \frac{1}{2\pi} \int_0^{2\pi} \cos(m\tau) \cos[r \sin(\tau + \varphi)] d\tau + i \frac{1}{2\pi} \int_0^{2\pi} \sin(m\tau) \cos[r \sin(\tau + \varphi)] d\tau. \quad (12)$$

Let us define a pair of auxiliary functions.

$$\begin{cases} U_m^\pm(r, \varphi; \tau) \equiv \cos[ms \pm r \sin(\tau + \varphi)] \\ V_m^\pm(r, \varphi; \tau) \equiv \sin[ms \pm r \sin(\tau + \varphi)] \end{cases}. \quad (13)$$

Applying the summation rules of trigonometric functions, Eq. (12) becomes the following.

$$\begin{aligned} B_z(r, \varphi) &= \frac{1}{2\pi} \frac{1}{2} \int_0^{2\pi} [U_m^+(r, \varphi; \tau + \varphi) + U_m^-(r, \varphi; \tau + \varphi)] d\tau \\ &+ i \frac{1}{2\pi} \frac{1}{2} \int_0^{2\pi} [V_m^+(r, \varphi; \tau + \varphi) + V_m^-(r, \varphi; \tau + \varphi)] d\tau \end{aligned} \quad (14)$$

Let us define another pair of auxiliary functions.

$$\begin{cases} C_m^\pm(r, \varphi; s) \equiv \cos[ms \pm r \sin(s) - m\varphi] \\ S_m^\pm(r, \varphi; s) \equiv \sin[ms \pm r \sin(s) - m\varphi] \end{cases}. \quad (15)$$

Making the substitution  $(\tau + \varphi) \rightarrow s$ , Eq. (14) is transformed into the following.

$$\begin{aligned} B_m(r, \varphi) &= \frac{1}{2\pi} \frac{1}{2} \int_\varphi^{2\pi+\varphi} [C_m^+(r, \varphi; s) + C_m^-(r, \varphi; s)] ds \\ &+ i \frac{1}{2\pi} \frac{1}{2} \int_\varphi^{2\pi+\varphi} [S_m^+(r, \varphi; s) + S_m^-(r, \varphi; s)] ds \end{aligned} \quad (16)$$

Hence, we obtained four terms for  $B_m(r, \varphi)$ . In addition, notice that the integration bounds are off-period.

Another trivial-looking yet important step is to utilize the following shift relations for use in azimuthal averaging by shifting the integration interval from  $\varphi \leq s \leq \varphi + 2\pi$  to  $0 \leq s \leq 2\pi$ , whereby all the participating integrals are not altered.

$$\begin{aligned} &\int_\varphi^{2\pi+\varphi} \left\{ \begin{aligned} &\cos[ms \pm r \sin(s)] \\ &\sin[ms \pm r \sin(s)] \end{aligned} \right\} ds \\ &= \int_0^{2\pi} \left\{ \begin{aligned} &\cos[ms \pm r \sin(s)] \\ &\sin[ms \pm r \sin(s)] \end{aligned} \right\} ds \end{aligned} \quad (17)$$

In fact, we have made use of such techniques already in coping with both Eqs. (3) and (6). As a result of Eq. (17), Eq. (16) becomes the following.

$$\begin{aligned} B_m(r, \varphi) &= \frac{1}{2\pi} \frac{1}{2} \int_0^{2\pi} [C_m^+(r, \varphi; s) + C_m^-(r, \varphi; s)] ds \\ &+ i \frac{1}{2\pi} \frac{1}{2} \int_0^{2\pi} [S_m^+(r, \varphi; s) + S_m^-(r, \varphi; s)] ds \end{aligned} \quad (18)$$

Consider the following integral.

$$\begin{aligned} &\int_0^{2\pi} \sin[ms \pm kr \sin(s)] ds \\ &= \int_0^\pi \sin[ms \pm kr \sin(s)] ds \\ &+ \int_\pi^{2\pi} \sin[ms \pm kr \sin(s)] ds \end{aligned} \quad (19)$$

We evaluate the second integral as follows.

$$\begin{aligned} &\int_\pi^{2\pi} \sin[ms \pm kr \sin(s)] ds \\ &= \int_{-\pi}^0 \sin[m(p + 2\pi) \pm kr \sin(p + 2\pi)] d(p + 2\pi) \\ &= \int_{-\pi}^0 \sin[mp \pm kr \sin(p)] dp \\ &= \int_\pi^0 \sin[-mq \pm kr \sin(-q)] d(-q) \\ &= \int_\pi^0 \sin[mq \pm kr \sin(q)] dq \\ &= - \int_0^\pi \sin[mq \pm kr \sin(q)] dq \end{aligned} \quad (20)$$

This integral in Eq. (20) cancels out the first integral in Eq. (19), thus leading Eq. (19) to the following null-integral relation,

$$\int_0^{2\pi} \sin[ms \pm kr \sin(s)] ds = 0. \quad (21)$$

The validity of this relation can be alternatively seen from the periodic properties of both  $\sin(s)$  and

$\sin[ms \pm kr \sin(s)]$  over  $0 \leq s \leq 2\pi$  as is the case with  $\int_0^{2\pi} \cos(s) ds = 0$  and  $\int_0^{2\pi} \sin(s) ds = 0$ .

Now, both  $C_m^\pm(r, \varphi; s)$  and  $S_m^\pm(r, \varphi; s)$  in Eq. (15) are expanded out respectively into two terms according to the trigonometric sum rules such that

$$\begin{aligned} \cos[ms \pm r \sin(s) - m\varphi] \\ = \cos(m\varphi) \cos[ms \pm r \sin(s)] \\ + \sin(m\varphi) \sin[ms \pm r \sin(s)] \end{aligned} \quad (22a)$$

$$\begin{aligned} \sin[ms \pm r \sin(s) - m\varphi] \\ = \cos(m\varphi) \sin[ms \pm r \sin(s)] \\ - \sin(m\varphi) \cos[ms \pm r \sin(s)] \end{aligned} \quad (22b)$$

Consequently, the null-integral relation in Eq. (15) helps to simplify Eq. (18) into the following.

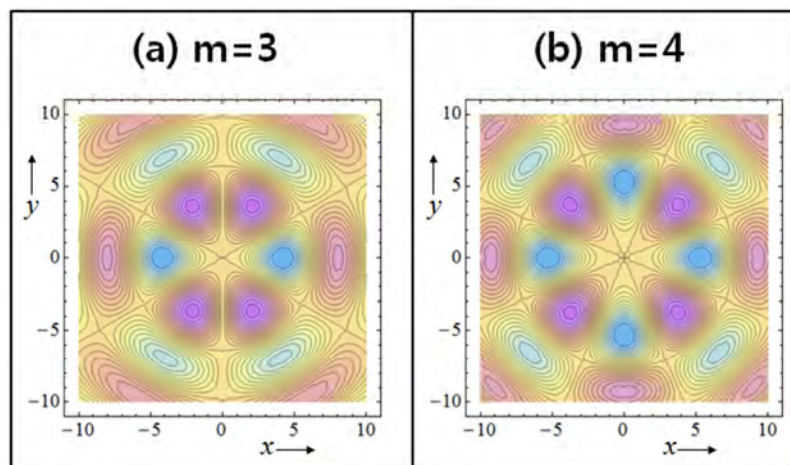
$$B_m(r, \varphi) = \frac{1}{2\pi} \frac{1}{2} [B_m^1(r, \varphi) - iB_m^2(r, \varphi)]. \quad (23)$$

Here,

$$B_m^1(r, \varphi) = \cos(m\varphi) \left[ \int_0^{2\pi} \cos[ms + r \sin(s)] ds + \int_0^{2\pi} \cos[ms - r \sin(s)] ds \right]. \quad (24a)$$

$$B_m^2(r, \varphi) = \sin(m\varphi) \left[ \int_0^{2\pi} \cos[ms + r \sin(s)] ds + \int_0^{2\pi} \cos[ms - r \sin(s)] ds \right]. \quad (24b)$$

Therefore,



**Fig. 3.** The azimuthally integrated periodic function  $\cos(m\varphi)J_m(r)$  evaluated on the Cartesian coordinates with the corresponding polar coordinates  $(r, \phi)$  for (a)  $m=3$  (odd) and (b)  $m=4$  (even).

$$B_m(r, \varphi) = \frac{1}{2} e^{-im\varphi} \frac{1}{2\pi} \left\{ \int_0^{2\pi} \cos[ms + r \sin(s)] ds + \int_0^{2\pi} \cos[ms - r \sin(s)] ds \right\}. \quad (25)$$

Through the integral representation of Bessel functions in Eq. (9) along with  $J_{-m}(r) = (-1)^m J_m(r)$ , the above Eq. (25) ends up with the following.

$$\begin{aligned} B_m(r, \varphi) &= \frac{1}{2} e^{-im\varphi} [J_{-m}(r) + J_m(r)] \\ &= \frac{1}{2} [(-1)^m + 1] e^{-im\varphi} J_m(r) \end{aligned} \quad (26)$$

Reverting back to the azimuthal angle via  $\varphi + \phi = \frac{1}{2}\pi$ , the above is recast as follows.

$$\begin{aligned} B_m(r, \phi) &= \frac{1}{2} [(-1)^m + 1] e^{-im(\pi/2)} e^{im\phi} J_m(r) \\ &= \frac{1}{2} [e^{-im(\pi/2)} e^{im\pi} + e^{-im(\pi/2)}] e^{im\phi} J_m(r) \\ &= \frac{1}{2} [e^{im(\pi/2)} + e^{-im(\pi/2)}] e^{im\phi} J_m(r) \end{aligned} \quad (27)$$

Here, we have utilized the identity  $(-1)^m = (e^{i\pi})^m = e^{im\pi}$ . As a consequence, Eq. (27) reads as follows.

$$B_m(r, \phi) = \cos\left(\frac{1}{2}m\pi\right) e^{im\phi} J_m(r). \quad (28)$$

Figure 3 shows the azimuthally integrated periodic function  $\text{Re}[e^{im\phi} J_m(r)] = \cos(m\phi) J_m(r)$  of Eq. (28) on the Cartesian coordinates with the corresponding polar coordinates  $(r, \phi)$  in correspondence to Fig. 2. The color scheme here is the same as for Fig. 2. Consequently, we find here that  $\cos(m\phi) J_m(r)$  does exhibit rotational features for both odd and even AMIs.

Combined with the temporal phase factor  $e^{-i\omega t}$ , the electric-field component  $E_z$  in the axial direction finally assumes the following form.

$$\begin{aligned} E_z(r, \phi, t) &= e^{-i\omega t} B_m(r, \phi) \\ &= \cos\left(\frac{1}{2}m\pi\right) e^{i(m\phi - \omega t)} J_m(r). \end{aligned} \quad (29)$$

We have expected the result to be  $E_z(r, \phi, t) \approx e^{i(m\phi - \omega t)} J_m(r)$ , since this form represents 2D Bessel wave with an azimuthally rotational wave speed represented by the exponent  $(m\phi - \omega t)$  in Eq. (29). In contradiction, we have Eq. (29), which includes an additional factor  $\cos(\frac{1}{2}m\pi)$ . Therefore, Eq. (29) can be alternatively put into the following.

$$E_z(r, \phi, t) = \begin{cases} e^{i(m\phi - \omega t)} J_m(r), & m = \text{even} \\ 0, & m = \text{odd} \end{cases}. \quad (30)$$

The non-existence of waves for  $m$  being odd can be understood from  $\frac{1}{2}e^{im(\pi/2)}e^{i(m\phi - \omega t)} + \frac{1}{2}e^{-im(\pi/2)}e^{i(m\phi - \omega t)}$  as seen from both Eqs. (27) and (29). In other words, Eq. (27) implies an interference between the two waves rotating in the same azimuthal direction. Hence, the net phase difference between the two co-rotational waves is  $m\pi$ , but the net phase difference per azimuthal mode index (AMI) is  $m\pi/m = \pi$ .

The other electric-field components are easily derived from Eq. (29) via Maxwell's equations adapted to the  $\vec{E} \parallel \vec{H}$  wave.

$$\begin{aligned} E_r(r, \phi, t) &= -\frac{1}{r} \frac{\partial E_z}{\partial \phi} = -i \frac{m}{r} \cos\left(\frac{1}{2}m\pi\right) e^{i(m\phi - \omega t)} J_m(r) \\ E_\phi(r, \phi, t) &= \frac{\partial E_z}{\partial r} = \cos\left(\frac{1}{2}m\pi\right) e^{i(m\phi - \omega t)} \frac{dJ_m(r)}{dr}. \end{aligned} \quad (31)$$

## 4. Discussions

For  $m = 0$ , the weighted integral in Eq. (10) reduces to the weighted integral Eq. (2). Correspondingly, the higher-order Bessel function  $B_m(r, \phi) = \cos(\frac{1}{2}m\pi) e^{im\phi} J_m(r)$  in Eq. (28) reduces for  $m = 0$  to the zeroth-order Bessel function  $B_0(r) = J_0(r)$ .

The azimuthal averaging finally obtained in Eq. (29) has

already been suggested by [5], but it has not been carried out there. The reason for them not to carry out the actual azimuthal integrations for  $m \neq 0$  is supposed to be due to too many algebraic steps required.

From analytical viewpoints, the integral in Eq. (2) turns out not to depend on the particular azimuthal location, viz., independent of  $\phi$  as seen from Eq. (8). This azimuthal independence of the zeroth-order Bessel wave goes like an idling situation of a mechanical coupling device. On the contrary, the integral in Eq. (10) turns out to strongly depend on the particular azimuthal location  $\phi$ . Moreover, the double dependence on the cosine function in the original integrand  $\cos[r \cos(\tau - \phi)]$  in Eq. (10) halves the cyclic dependence on  $\phi$ , viz., with the net phase difference of a half-cycle or  $\pi$  as seen from Eq. (27). In other words, the weighting factor  $w_m(\tau) = \exp(im\tau)$  multiplies this net phase difference by the factor of  $m$ , i.e.,  $m\pi$ . This situation is again analogous to another kind of mechanical coupling device, where an array of  $m$  couplings work one after the next.

The mathematical steps leading to the azimuthal averaging in Eqs. (2) and (10) imply that a rectangular array of an infinite number of helical vortices are averaged in the azimuthal direction. As a result, a single cylindrical optical vortex emerges [5, 15].

The rare chance of a cylindrical optical vortex to be established can be understood when we closely examine all the underlying assumptions for Eq. (1) to Eq. (29). They are temporal coherence, azimuthal coherence, equal-magnitude constituent waves in interference with a net phase difference of a half-cycle or  $\pi$ , etc. By analogy, a meteorological tornado or a dust devil arises from gathering pertinent coherent energy from its surroundings. But, as we all know, tornadoes or dust devils occur in extremely rare ways although they are normally devastating once properly formed.

As a future work, we wish to expand out on the solution in Eq. (29) to investigate further detailed characteristics of our  $\vec{E} \parallel \vec{H}$  wave in the context of multiplexing waves of varied azimuthal mode indices. This issue has its applications in the field of wireless communications in free space [3]. As a numerical part of future work, we propose to work out a variety of azimuthal averaging  $w_m(\tau)$ . For instance, we may invent piecewise continuous functions for  $w_m(\tau)$  as long as the latter is periodic with a period of  $2\pi/m$ . These weighting functions are expected to lead to heavy computations though. Another issue will be the interactions among multiple-order 2D Bessel waves [16].

As stated earlier, our interests in rotational electromagnetic waves stemmed from our desire to better understand optical vortices. In this regard, the generation of optical vortices is our key concern, for which the two-dimensional TE and TM waves might be combined somehow to ultimately form either  $\vec{E} \parallel \vec{H}$  waves or propagating waves along the transversal or axial  $z$ -direction. Technically speaking, a successful

generation of axially propagating Bessel beams is necessary for efficient transports of both energy and information [4, 15, 17, 18].

Among numerous applications relevant to rotational electromagnetic waves, we like to introduce a work done on metamaterials [19]. It is well known that chiral or one-way transports of electromagnetic energy can be realized by fabricating periodic meta-atoms composed of split-ring unit cells. Here, diverse light scattering phenomena take part in propagations of electromagnetic waves through such metamaterials. It is shown in [19] that the multiple higher-order Bessel waves play a prime role in not only understanding how the chiral transports are at work but also leading to precise quantification for technical designs.

## 5. Conclusion

The azimuthal averaging is performed on the fundamental solution, which represents electromagnetic waves where the electric-field and magnetic-field vectors are parallel to each other. For this purpose, a simple weighting function is employed to add azimuthal rotations. The resulting wave exhibits two co-rotational electromagnetic waves combined in strong coherence but with a half-cycle phase difference. In addition, implications to optical vortices are provided along with a few areas of real applications.

## Acknowledgements

This study has been supported by the National Research Foundation (NRF) of Republic of Korea (Grant Number: NRF-2015R1D1A1A01056698).

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