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On The Parameter Function Involving Gamma Function and Its Applications

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Abstract: In this paper, the complete monotonic parameter function involving the Gamma function is considered. The necessary and sufficient condition of the parameter *f* is presented. As an application, two meaningful inequalities of Gamma function are obtained.

Keywords: Gamma function, Completely Monotonicity, Inequalities

1. Introduction

The classical Gamma function in [1,3] is defined by

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt,(x>0) \tag{1}$$

The logarithmic derivative of the Gamma function in [1,3] is defined as follows

$$\Psi(x)=\frac{\Gamma'(x)}{\Gamma(x)},(x>0) \tag{2}$$

There are lots of literatures and applications about the completely monotonic functions and logarithmic completely monotonic functions, for example, [2,4,5,6-10] and the references therein.

The function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and

$$(-1)^n f^{(n)}(x)\geq 0 \tag{3}$$

If the inequality is strict for all $x\in I$ and $n\geq 0$ is said to be strictly completely monotonic function.

A positive function f is said to be logarithmic completely monotonic on an interval I if its logarithm $\ln f$ satisfies

$$(-1)^n\left[\ln f(x)\right]^{(n)}\geq 0 \tag{4}$$

If the inequality is strict for all $x\in I$ and $n>0$ is said to be strictly logarithmically completely monotonic function.

In this paper, we are about to consider the completely monotonic property of a parameter functions involving the Gamma function as follows.

2. Main Results

Theorem 2.1.

The function $f_\alpha(x)=1-\ln x+\frac{1}{x}\ln\Gamma(x+\alpha)$ is strictly completely monotonic on $(0,\infty)$ if and only if $1\leq\alpha\leq\beta$, where it exists $0<t_0<1$ such that $\beta=\frac{1}{t_0}-\frac{1}{e^{t_0}-1}$.

Proof. By using the Leibniz'rule,

$$\left[\mu(x)\nu(x)\right]^{(n)}=\sum_{k=0}^{n}\binom{n}{k}\mu^{(k)}(x)\nu^{(n-k)}(x). \tag{5}$$

We have

$$f_\alpha^{(n)}(x)=-(\ln x)^{(n)}+\left(\frac{1}{x}\ln\Gamma(x+\alpha)\right)^{(n)}. \tag{6}$$

Then we can obtain

$$f_{\alpha}^{(n)}(x) = -\frac{(-1)^{n-1}(n-1)!}{x^n} + \sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{x}\right)^{(n-k)}\left(\ln\Gamma(x+\alpha)\right)^{(k)}$$

$$= \frac{(-1)^n(n-1)!}{x^n} + \left(\frac{1}{x}\right)^{(n)}\ln\Gamma(x+\alpha) + \sum_{k=1}^{n}\binom{n}{k}\left(\frac{1}{x}\right)^{(n-k)}\Psi^{(k-1)}(x+\alpha)$$

$$= \frac{(-1)^n(n-1)!}{x^n} + \frac{(-1)^n n!}{x^{n+1}}\ln\Gamma(x+\alpha) + \sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{(n-k)}(n-k)!}{x^{n-k+1}}\Psi^{(k-1)}(x+\alpha) = \frac{(-1)^n n!}{x^{n+1}}h(x). \tag{7}$$

where we denotes

$$h(x) = \frac{x}{n} + \ln\Gamma(x+\alpha) + \sum_{k=1}^{n}\frac{(-1)^{(-k)}}{k!x^{-k}}\Psi^{(k-1)}(x+\alpha). \tag{8}$$

We can count the derivative of $h(x)$ as follows

$$h'(x) = \frac{1}{n} + \Psi(x+\alpha) - \Psi(x+\alpha) - x\Psi'(x+\alpha) + x\Psi'(x+\alpha) + \frac{1}{2}x^2\Psi''(x+\alpha)$$

$$+\cdots+\frac{(-1)^n}{(n-1)!}x^{(n-1)}\Psi^{(n-1)}(x+\alpha) + \frac{(-1)^n}{n!}x^n\Psi^{(n)}(x+\alpha) = \frac{(-1)^n}{n!}x^n\Psi^{(n)}(x+\alpha) + \frac{1}{n}. \tag{9}$$

Base on above, we have

$$\frac{1}{x^n}h'(x) = \frac{(-1)^n}{n!}\Psi^{(n)}(x+\alpha) + \frac{1}{nx^n}. \tag{10}$$

Combining (6) with the following formula,

$$\frac{(n-1)!}{x^n} = \int_0^\infty t^{n-1}e^{-xt}dt. \tag{11}$$

we have

$$\frac{1}{x^n}h'(x) = \frac{(-1)^{2n+1}}{n!}\int_0^\infty \frac{t^n}{1-e^{-t}}e^{-(x+\alpha)t}dt + \frac{1}{n!}\int_0^\infty t^{n-1}e^{-xt}dt = \frac{1}{n!}\int_0^\infty\left[t^{n-1}e^{-xt} - \frac{t^n}{1-e^{-t}}e^{-(x+\alpha)t}\right]dt = \frac{1}{n!}\int_0^\infty\left[1-\frac{te^{-\alpha t}}{1-e^{-t}}\right]t^{n-1}e^{-xt}dt \tag{12}$$

Here we let

$$g(t) = 1 - \frac{te^{t-\alpha t}}{e^t - 1}. \tag{13}$$

Since $0 < t/e^t - 1 < 1$ for $t > 0$, then

$$0 < \frac{te^{t-\alpha t}}{e^t-1} < e^{t-\alpha t}, \tag{14}$$

especially, we have

$$1 - e^{t-\alpha t} < g(t) < 1 - \frac{te^{t-\alpha t}}{e^t-1} < 1. \tag{15}$$

If we need $h(x) \geq 0$, then we should have

$$1\text{-}e^{t-\alpha t} \geq 0, \tag{16}$$

while $t > 0$. In this case, we can obtain

$$t(1-\alpha) \leq 0, \tag{17}$$

then α should satisfies the following necessary and sufficient condition

$$\alpha \geq 1, \tag{18}$$

and the function $h(x)$ is strictly increasing, then for any $x > 0$, we have

$$h(x) > h(0) = \ln\Gamma(\alpha) \geq 0, \tag{19}$$

on $x > 0$, and $\alpha \geq 1$.

On the other hand, we find

$$\lim_{t\to 0} g(t) = 1-\lim_{t\to 0}\frac{te^{t-\alpha t}}{e^t-1} = 1-\lim_{t\to 0}\frac{e^{t-\alpha t}+t(1-\alpha)e^{t-\alpha t}}{e^t} = 0 \tag{20}$$

and

$$\lim_{t\to\infty} g(t) = 1-\lim_{t\to\infty}\frac{te^{t-\alpha t}}{e^t-1} = 1-\lim_{t\to\infty}\frac{e^{t-\alpha t}+t(1-\alpha)e^{t-\alpha t}}{e^t} = 1-\lim_{t\to\infty}\frac{1+t(1-\alpha)}{e^{\alpha t}} = 1-\lim_{t\to\infty}\frac{1-\alpha}{\alpha e^{\alpha t}} = 1, \tag{21}$$

for $\alpha > 0$.

We can also count $g'(t)$ as follows

$$g'(t) = \frac{e^{2t-t\alpha}(1-t\alpha)-e^{t-\alpha t}(1+t-t\alpha)}{(e^t-1)^2} \tag{22}$$

then it means one must have $g'(t)\ge 0$, while in this second condition, we get

$$e^t(1-t\alpha)\ge 1+t-t\alpha. \tag{23}$$

In this case, we can find that α should satisfies the necessary and sufficient condition as following

$$\alpha \le \frac{1}{t}-\frac{1}{e^t-1}, t>0. \tag{24}$$

Consider first condition $\alpha \ge 1$ with the second one, we have

$$1\le\alpha\le\frac{1}{t}-\frac{1}{e^t-1}, t>0. \tag{25}$$

But we should have

$$\frac{1}{t}-\frac{1}{e^t-1}\ge 1, t>0. \tag{26}$$

We can find that t should satisfies

$$-\ln(1-t)\le t<1. \tag{27}$$

In fact, we can describe the condition (26) into the following form

$$e\ge\left(1+\frac{1}{\frac{1}{t}-1}\right)^{\frac{1}{t}}. \tag{28}$$

We have the following fact that

$$\lim_{t\to 0}\left(1+\frac{1}{\frac{1}{t}-1}\right)^{\frac{1}{t}} = \lim_{\frac{1}{t}\to\infty}\left(1+\frac{1}{\frac{1}{t}-1}\right)^{\frac{1}{t}} = e\cdot \tag{29}$$

Base on the fact above, we can get a result that there exist a $0<t_0<1$ such that the (28) holdsfor $\alpha\ge 1$, we denote β as follows

$$1\le\alpha\le\beta=\frac{1}{t_0}-\frac{1}{e^{t_0}-1}, 0<t_0<1. \tag{30}$$

Then the theorem follows directly. We can easily get the following corollary.

Corollary 2.2. The function $g_\alpha(x)=\sqrt[x]{\Gamma(x+\alpha)}/x$ is strictly logarithmically completely monotonic on $(0,\infty)$ if and only if $1\le\alpha\le\beta$, , where it exists $0<t_0<1$ such that $\beta=\frac{1}{t_0}-\frac{1}{e^{t_0}-1}$.

3. Two Applications

Following the notation above, by applying the complete monotonicity of $g_\alpha(x)$, we can obtain the following inequalities.

Theorem 3.1. If $n>0$, $1\le\alpha\le\beta$, then

$$0<\left(\frac{\sqrt[n+1]{\Gamma(n+\alpha+1)}}{\sqrt[n]{\Gamma(n+\alpha)}}\right)^n<e, \tag{31}$$

and

$$e^{-1}<\left(\frac{\sqrt[n]{\Gamma(n+\alpha)}}{\sqrt[n+1]{\Gamma(n+\alpha+1)}}\right)^{n+1}, \tag{32}$$

where it exists $0<t_0<1$ such that $\beta=\frac{1}{t_0}-\frac{1}{e^{t_0}-1}$.

Proof. Consider the complete monotonicity of $g_\alpha(x)$, we have

$$\frac{\sqrt[a]{\Gamma(a+\alpha)}}{a}<\frac{\sqrt[b]{\Gamma(b+\alpha)}}{b}, \tag{33}$$

for $a>b>0$. It means

$$\frac{\sqrt[a]{\Gamma(a+\alpha)}}{\sqrt[b]{\Gamma(b+\alpha)}}<\frac{a}{b}. \tag{34}$$

Let $a=n+1, b=n>0$, we get

$$\frac{\sqrt[n+1]{\Gamma(n+\alpha+1)}}{\sqrt[n]{\Gamma(n+\alpha)}} < \frac{n+1}{n}, \tag{35}$$

then

$$0 < \lim_{n\to\infty}\left(\frac{\sqrt[n+1]{\Gamma(n+\alpha+1)}}{\sqrt[n]{\Gamma(n+\alpha)}}\right)^n < \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = e. \tag{36}$$

Using the following fact

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{n+1} < \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)^{n+1} = e^{-1}. \tag{37}$$

we get

$$e^{-1} < \lim_{n\to\infty}\left(\frac{\sqrt[n]{\Gamma(n+\alpha)}}{\sqrt[n+1]{\Gamma(n+\alpha+1)}}\right)^{n+1}. \tag{38}$$

Theorem 3.2. If $x > 0$, $1 \le \alpha \le \beta$, then we have

$$0 < \prod_{k=0}^{\infty}\frac{\sqrt[x+k+1]{\Gamma(x+\alpha+k+1)}}{\sqrt[x+k]{\Gamma(x+\alpha+k)}} < 1. \tag{39}$$

Especially, for a enough large positive integer n, we have

$$0 < \frac{\sqrt[n+2]{(n+[\alpha]+1)!}}{\sqrt[n+1]{(n+[\alpha])!}} < 1, \tag{40}$$

where it exists $0 < t_0 < 1$ such that $\beta = \frac{1}{t_0} - \frac{1}{e^{t_0}-1}$.

Proof. Using the completely monotonic property of $g_\alpha(x)$ again, we get

$$0 < \lim_{x\to\infty}\prod_{k=0}^{\infty}\frac{\sqrt[x+k+1]{\Gamma(x+\alpha+k+1)}}{\sqrt[x+k]{\Gamma(x+\alpha+k)}} < \lim_{x\to\infty}\prod_{k=0}^{\infty}\frac{x+k+1}{x+k} = 1, \tag{41}$$

for all $x > 0$. If $x = n$ is a positive integer, and $k = 0$, we have a classical useful inequality

$$0 < \frac{\sqrt[n+2]{\Gamma(n+[\alpha]+2)}}{\sqrt[n+1]{\Gamma(n+[\alpha]+1)}} = \frac{\sqrt[n+2]{(n+[\alpha]+1)!}}{\sqrt[n+1]{(n+[\alpha])!}} < \frac{n+2}{n+1}, \tag{42}$$

then

$$0 < \lim_{n\leftarrow\infty}\frac{\sqrt[n+2]{(n+[\alpha]+1)!}}{\sqrt[n+1]{(n+[\alpha])!}} < \lim_{n\to\infty}\frac{n+2}{n+1} = 1. \tag{43}$$

for a enough large n.

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