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On the Riesz Sums in Number Theory

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Abstract: The Riesz means, or sometimes typical means, were introduced by M. Riesz and have been studied in connection with summability of Fourier series and of Dirichlet series [8] and [11]. In number-theoretic context, it is the Riesz sum rather than the Riesz mean that has been extensively studied. The Riesz sums appear as long as there appears the G-function. Cf. Remark 1 and [14]. As is shown below, the Riesz sum corresponds to integration while Landau's differencing is an analogue of differentiation. This integration-differentiation aspect has been the driving force of many researches on number-theoretic asymptotic formulas. Ingham's decent treatment [13] of the prime number theorem is one typical example. We state some efficient theorems that give asymptotic formulas for the sums of coefficients of the generating Dirichlet series not necessarily satisfying the functional equation.

Keywords: Riesz Sum, Riesz Mean, Dirichlet Series, Asymptotic Formula

1. Introduction

The Riesz means, or sometimes typical means, were introduced by M. Riesz and have been studied in connection with sum ability of Fourier series and of Dirichlet series [8] and [11]. Given an increasing sequence $\{\lambda_k\}$ of real numbers and a sequence $\{\alpha_k\}$ of complex numbers, the Riesz sum of order μ is defined as in [8, p.2] and [11, p.21] by

$$\begin{aligned} A^{\mu}(x) = A_{\lambda}^{\mu}(x) &= \sum_{\lambda_k \le x}{}' (x-\lambda_k)^{\mu} \alpha_k \\ &= \mu\int_0^x (x-t)^{\mu-1} A_{\lambda}(t)\,\mathrm{d}t \qquad (1.1) \\ &= \mu\int_0^x (x-t)^{\mu-1}\,\mathrm{d}A_{\lambda}(t) \end{aligned}$$

where

$$A_{\lambda}(x) = A_{\lambda}^{0}(x) = \sum_{\lambda_k \le x}{}' \alpha_k \qquad (1.2)$$

where the prime on the summation sign means that when $\lambda_k = x$, the corresponding term is to be halved.

(1.1) or rather normalized $\frac{1}{\Gamma(\mu+1)}A^{\mu}(x)$ which appears in (G-8-2) is called the Riesz sum of order μ If $\frac{1}{\Gamma(\mu+1)}A^{\mu}(x)$ approaches a limit A as $x\to\infty$, the sequence $\{\alpha_k\}$ is called Riesz summable or (R,μ,λ) summble to A, which is called the Riesz mean of the sequence. Sometimes the negative order Riesz sum is considered, in which case the sum is taken over all n which are not equal to x.

In number theory it is often the case that the main study of research is the behavior of an arithmetic function whose generating function is given explicitly, say in the form of a Dirichlet series or an Euler product. Then the problem amounts to extracting the essential main term from the data on generating functions.

In this number-theoretic context, it is the Riesz sum rather than the Riesz mean that has been extensively studied. The Riesz sums appear as long as there appears the $G_{1,1}^{1,0}$ Cf. Remark 1. There is some mention on the divisor problem in [7] in the light of the Riesz sum and there are enormous amount of literature on the Riesz sums and we shall not dwell on well-known cases very in detail. We are concerned with the case where the generating function does not necessarily satisfy the functional equation and concentrate on asymptotic formulas rather than exact identities.

An example is given.

Recall the definition of the periodic Bernoulli polynomial etc. ([14, p.170]). Then

$$B_1(x)-\psi(x)=\sum_{n\le x}{}' 1 := A(x)$$

say, or

$$B_1(x)-\overline{B_1}(x)=\begin{cases} A(x), & x\notin\mathbb{Z} \\ A(x)+\frac{1}{2}, & x\in\mathbb{Z}\end{cases}$$

Integration of both sides amounts to (1.1):

$$\frac{1}{2}B_2(t)-\frac{1}{2}\overline{B_2}(t)=\int_0^x\left(B_1(t)-\overline{B_1}(t)\right)\mathrm{d}t$$
$$=\int_0^x A(t)\,\mathrm{d}t=\sum_{n\le x}(x-n)$$

where we used $B_2(0)=\overline{B_2}(0)\left(=B_2=\frac{1}{6}\right)$.

The application of the Riesz sum comes into play through Perron's formula (1.6) below, sometimes in truncated form. The application of the truncated first order Riesz sum appears on [10, p.105] and a truncated general order Riesz sum is treated in [13] in both of which the functional equation is not assumed. Riesz sums with the functional equation can be found e.g. in [9], where by differencing, the asymptotic formula for the original sum is deduced. The principle goes back to Landau [15] in which one can find the integral order Riesz sum and its reduction to the original partial sum by differencing.

The general formula for the difference operator of order $\alpha\in\mathbb{N}$ with difference $y\ge 0$ is given by

$$\Delta_y^\alpha f(x)=\sum_{\nu=0}^{\alpha}(-1)^{\alpha-\nu}\binom{\alpha}{\nu}f(x+\nu y). \qquad (1.3)$$

If f has the $\alpha-\mathrm{th}$ derivative $f^{(\alpha)}$, then

$$\Delta_y^\alpha f(x)=\int_x^{x+y}\mathrm{d}t_1\int_x^{x+y}\mathrm{d}t_2\cdots\cdots\int_{t_{\alpha-1}}^{t_{\alpha-1}+y}f^{(\alpha)}(t_\alpha)\,\mathrm{d}t_\alpha. \qquad (1.4)$$

The Riesz kernel which produces the Riesz sum is defined by

$$G_{1,1}^{1,0}\left(z\,\middle|\,\begin{matrix}a\\b\end{matrix}\right)=\begin{cases}\frac{1}{\Gamma(a-b)}z^b(1-z)^{a-b-1}, & |z|<1\\ \frac{1}{2} & z=1, a=b+1\\ 0, & |z|>1\end{cases} \qquad (1.5)$$

Remark 1. Notes on (1.5). Let $\mu\ge 0$ denote the order of the Riesz mean and set $b=0, a=\mu+1$. Then (1.5) reads ($c>0$)

$$\frac{1}{2\pi i}\int_c\frac{\Gamma(s)}{\Gamma(s+\mu+1)}z^{-s}\mathrm{d}s=\begin{cases}\frac{1}{\Gamma(\mu+1)}(1-z)^\mu, |z|<1\\ \frac{1}{2}, & \mu=0, z=1\\ 0, & |z|>1\end{cases} \qquad (1.6)$$

This can be found in Hardy-Riesz [11] and Chandrasekharan and Minakshisundaram [8] and used in the context of Perron's formula

$$\frac{1}{\Gamma(\mu+1)}\sum_{\lambda_k\le x}{}'(x-\lambda_n)^\mu=\frac{1}{2\pi i}\int_c\frac{\Gamma(s)F(s)x^{s+\mu}}{\Gamma(s+\mu+1)}z^{-s}\mathrm{d}s, \quad (1.7)$$

where the left-hand side sum is called the Riesz sum of order μ and denoted $A_\lambda^\mu(x)$ as mentioned above and

$$F(s)=\sum_{k=1}^{\infty}\frac{\alpha_k}{k^s}.$$

The special case of (1.6) with $\mu=0$ is known as the discontinuous integral whose truncated form can be found e.g. in Davenport [10, pp.109-110]. This and the general case (1.6) can be proved by the method of residues, distinguishing the cases $|z|<1$ and $|z|>1$.

Here as above, the prime on the summation sign means that when $\lambda_k=x$, the corresponding term is to be halved, and this halving comes from the peculiarity of the discontinuous integral.

If the order $\mu\in\mathbb{N}$, then the right-hand side member of (1.6) is

$$\frac{1}{2\pi i}\int_c\frac{1}{s(s+1)\cdots\cdots(s+\mu)}z^{-s}\mathrm{d}s,$$

and the Riesz sum amounts to the μ times integration of the original sum $A_\lambda(x)$. Thus Landau's differencing is an analogue of the integration and differentiation.

In view of this integration-differentiation aspect there are a number of cases in which the Riesz sum appears in disguised form. Especially, when there is a gamma factor $\frac{\Gamma(s)}{\Gamma(s+\mu+1)}$ or $\binom{\mu+1}{0}$ involved.

The very special case $z=1, a+b=1$ of (1.5) (and of the corresponding logarithmic case ($z=1, m=1$) presents excessive complexities in notation, so that we follow Hardy and Riesz [11] to use (1.7) by suppressing the prime on the summation. We are to bear this special case in mind although not explicitly stated.

Remark 2. The Riesz summability is useful in summability of (Fourier) series. We recall e.g. the well-known result that the Riesz summability implies Abel summability [11, Theorem 24]. There is an explicit formula known for the transition.

Lemma 1.1. The sum of the Abel mean $\sum_{k=1}^{\infty} a_n e^{-\lambda_k s}$ at all points of the sector $|\arg s| \le u \le \frac{\pi}{2}$ other than the origin is

$$\frac{1}{\Gamma(\mu+1)}\int_0^{\infty} s^{\mu+1} e^{-s\tau} A^{\mu}(\tau)\,\mathrm{d}\tau, \tag{1.8}$$

Where $A^{\mu}(x)$ is the $\mu-\mathrm{th}$ Riesz sum defined in (1.1).

2. Riesz Sums

Definition 1. Let μ be a real number (mostly we assume that it is nonnegative) and let $\{\lambda_n\} = \{\lambda_n\}_{n=1}^{\infty}$, $\{\ell_n\} = \{\ell_n\}_{n=1}^{\infty}$ be arbitrary sequences of real numbers strictly increasing to infinity such that $\lambda_1 \ge 1, \ell_1 \ge 0$. Let $\{a_n\}_{n=1}^{\infty}$ ne any sequence of complex numbers. Then we write

$$\begin{cases} A_{\lambda}^{\mu}(x) = \dfrac{1}{\Gamma(\mu+1)} \displaystyle\sum_{\lambda_n \le x} a_n (x-\lambda_n)^{\mu}, \\ A_{1}^{\mu}(x) = \dfrac{1}{\Gamma(\mu+1)} \displaystyle\sum_{\ell_n \le x} a_n (x-\ell_n)^{\mu} \end{cases} \tag{2.1}$$

and refer to $A_{\lambda}^{\mu}(x)$ *(resp.* $A_{1}^{\mu}(x)$*) as the Riesz sum of order* μ *of the second (resp. first) kind associated to the series* $\sum_{n=1}^{\infty} a_n \lambda_n^{-s}$ *(resp.* $\sum_{n=1}^{\infty} a_n \ell_n^{-s}$ *), where absolute convergence of the series is assumed in some half-plane* $\mathrm{Re}\, s = \sigma > \sigma_a$. *For the special choice of* λ *(resp.* ℓ*), i.e.* $\lambda_n = n$ *or* $= N\mathfrak{a}$, $N\mathfrak{a}$ *denoting the norm of the integral ideal* $\mathfrak{a}$ *(resp.* $\ell_n = \log n$ *or* $= \log N\mathfrak{a}$*), we dennote the corresponding Riesz sum* $A_{\lambda}^{\mu}(x)$ *(resp.* $A_{1}^{\mu}(x)$*) by* $A_{a}^{\mu}(x)$ *(resp.* $A_{1}^{\mu}(x)$ *) and refer to it as the arithmetic (resp/ logarithmic) Riesz sum of order* μ *associated to the series* $\sum_{n=1}^{\infty} a_n n^{-s}$ *or* $\sum_{\mathfrak{a}} a_{\mathfrak{a}} \mathfrak{a}^{-s}$.

Theorem 2.1. *(Kanemitsu) Let* σ_a *denote the abscissa of absolute convergence of the Dirichlet series*

$$F(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s} \tag{2.2}$$

Which we may assume $\sigma_a > 0$ *without loss of generality, for* $\sigma > \sigma_a$ *and for* $b > \sigma_a$ *let* $B(b) = \sum_{n=1}^{\infty} |a_n| \lambda_n^{-b}$ *denote a Majorant of* $F(s)$. *Suppose that* $F(s)$ *can be continued analytically to a meromorphic function in some region R0 extending vertically from top to bottom fo the complex plane and bounded on the left by a piecewise smooth Jordan curve*

$$\Gamma : \sigma = f(t), \quad 0 < f(t) < b, \tag{2.3}$$

and that all the poles of $F(s)$ lying in R' are contained in a finite part of R' and are not on Γ. Take a subregion R whose boundary consists of the line segments $\overline{AB}, \overline{CD}$, overline DA and that part of BC of Γ with $|t| \le T$ with T large enough for all the poles of $F(s)$ are contained in R and is to be taken as (2.7). I.e. $\overline{AB} : -K \le \sigma \le b, t = -T$, $\overline{CD} : -K \le \sigma \le b, t = T$, *and* $\overline{DA} : \sigma = -K, -T \le t \le T$, *where* $K > 0$ *is a constant large enough. Suppose that* $F(s)$ *satisfies the following growth conditions: there exists a constant* $\mu < \mu + 1$ *such that*

$$F(s) = O\left(T^{v+\varepsilon}\right) \quad on \quad \overline{AB} \text{ and } \overline{CA}; \tag{2.4}$$

$$F(s) = O\left(|t|\,\mathrm{V}(t)\right) \quad on \quad \Gamma \text{ if } |t| \ge t_0; \tag{2.5}$$

$$F(s) = O\left(\mathrm{W}\left(f(t), t_0\right)\right) \quad on \quad \Gamma \text{ if } |t| \le t_0; \tag{2.6}$$

where V, W *are positive, integrable and* $V(y) = O\left(y^{\varepsilon}\right)$ *as* $y \to \infty$, *and* t_0 *is some constant. Then with*

$$T = x^{\alpha} \tag{2.7}$$

with a constant $\alpha > 0$ *and if* $f(t)$ *is given by*

$$(\ \mathcal{L} = \log_{\vartriangle}(|t|+2)\ )$$

$$f(t) = \beta - \psi(t) \ge \eta > 0 \tag{2.8}$$

$$\psi(t) = A\mathcal{L}^{-a'}(\log \mathcal{L})^{-b'}, \quad |t| \ge t_0 \tag{2.9}$$

with constants $a', b', A, \vartriangle, \beta$ *such that* $a' \ge 0, b' \ge 0$, $A > 0, \vartriangle \ge 1, \beta$. *Then for any* $\mu > \tau$ *and with (2.7) we have the asymptotic formula*

$$A_{\lambda}^{\mu}(x) = Q_{\mu}(x) + R_{\lambda,I}^{\mu}(x), \tag{2.10}$$

provided that

$$\log_{\vartriangle} \ll (\log x)^{1/(1+a')-\eta} \tag{2.11}$$

for some $\eta > 0$, *where* $Q_{\mu}(x)$ *is the sum of the residues of* $\frac{\Gamma(s)}{\Gamma(s+\mu+1)} F(s) x^{s+\mu}$ *in* R,

$$R_{\lambda,I}^{\mu}(x) = O\left(x^{\mu+b}\left(x^{\alpha(v+\varepsilon-\mu-1)} + x^{-\alpha\mu} B(b)\right)\right) \tag{2.12}$$

$$+O\left(x^{\mu+u} W(\beta, \vartriangle)\right) + O\left(x^{\mu+\beta} \delta_A(x)\right) \tag{2.13}$$

where

$$\begin{aligned}\delta(x) &= \delta_{A,a',b'}(x) \\ &= \exp\left(-A(\log x)^{1/(a'+1)}(\log\log x)^{-b'/(a'+1)}\right)\end{aligned} \tag{2.14}$$

and $u = \max_{|t|\le 0} f(t)$.

Theorem 2.2. *Under the same condition as in Theorem 2.1, if* $f(t)$ *is given by*

$$f(t) = \beta = \cos n \tan t \tag{2.15}$$

Then with a constant $\alpha > 0$ *and we have the asymptotic formula*

$$A_\lambda^\mu(x) = Q_\mu(x) + R_{\lambda,II}^\mu(x), \tag{2.16}$$

where $Q_\mu(x)$ *is the sum of the residues of*

$\dfrac{\Gamma(s)}{\Gamma(s+\mu+1)}F(s)x^{s+\mu}$ *in* R,

$$\begin{aligned}R_{\lambda,II}^\mu(x) &= O\left(x^{\mu+b}\left(x^{\alpha(\nu+\varepsilon-\mu-1)} + x^{-\alpha\mu}B(b)\right)\right) \\ &+ O\left(x^{\mu+u}W(\beta,\vartriangle)\right).\end{aligned} \tag{2.17}$$

Corollary 2.1. *Suppose that the conditions of Theorem 2.1 are satisfied and let* q *be the maximum of the real parts of poles of* $F(s)$ *in* R, *and* r *be the maximum order of poles with real parts* q, *and define* θ *to be 1or 0 according as* $F(s)$ *has a pole in* $R^{/}$ *or not* θ' *to be 1 or 0 according as* $a_n \ge 0$ *or not . then*

$$\begin{aligned}A_\lambda^0(x) &= \theta Q_0(x) + \theta' O\left(\sum_{x\le\lambda\le x+\mu y}|a_n|\right) \\ &+ \theta O\left(yx^{q-1}\log^{r-1}x\right)\end{aligned} \tag{2.18}$$

$$+O\left(R_{\lambda,i}^\mu(x)\right), \tag{2.19}$$

where $\mu \in \mathbb{N}$ *and* $i = I$ *or* $i = II$ *according to the choice of* $f(t)$.

Similar results hold for the logarithmic Riesz sums with the following replacement to be made: Instead of $Q_\mu(x)$ we have $P_\mu(x)$ the sum of the residues of $\dfrac{1}{s^\mu}F(s)e^{xs}$ in R, instead of $B(b)$ we have $B^*(b) = \sum_{n=1}^{\infty}|a_n|\exp{-b\ell_n}$, $T = e^{\alpha x}$. We state a very convenient corollary.

Corollary 2.2. *Suppose that the conditions of Theorem 2.1 are satisfied and suppose that we have the asymptotic formula*

$$\sum_{\ell_n\le x}|a_n| = x^{q'}\log^{r'-1}x\left(C+o(1)\right), \tag{2.20}$$

with $C > 0$ *a constant. Then*

$$\begin{aligned}A_\ell^0(x) &= \theta P_0(\log x) + \theta' O\left(\delta x^{q'}\log^{r'-1}x\right) \\ &+ \theta O\left(\delta^{1/\mu}x^q\log^{r-1}x\right) \\ &+ O\left(\delta - x^{-a\mu}(b-q')^{[r']+1} + x^{b+(\upsilon+\varepsilon-\mu-1)\alpha}\right) \\ &+ O\left(\delta^{-1}x^u W(\beta,\vartriangle)\right) + O\left(x^\beta\delta\right),\end{aligned} \tag{2.21}$$

where $\mu \in \mathbb{N}$ *and all* δ^c *'s amount to the reducing factor* δ *with possibly different constants* A, a' *etc. in (2.14).*

3. Examples

Example 3.1. Let K be an algebraic number field of degree n with discriminant d. Let $\mathfrak{D} = \mathfrak{D}_k$ be the ring of algebraic integers in K and let $\mathfrak{f}$ be an artitrary, fixed non-zero ideal of $\mathfrak{D}$. Let $A_\mathfrak{f}$ be the group of all fractional ideals with numerators and denominators relatively prime to $\mathfrak{f}$, and $H^*(\mathfrak{f})$ denote the ray class group of K, *i.e.* the quotient of $A_\mathfrak{f}$ modulo the group $S_\mathfrak{f}$ of principal ideals (α) with totally positive α such that $\alpha \equiv 1(\bmod \mathfrak{f})$.We define the Möbius function $\mu(\mathfrak{a})$ on ideals in the same manner as in the rational case and for $\mathfrak{C} \in H^*(\mathfrak{f})$ we put

$$M(x,\mathfrak{C}) = \sum_{\substack{N\mathfrak{a}\le\mathfrak{x} \\ \mathfrak{a}\in\mathfrak{C}}}\mu(\mathfrak{a}) \tag{3.1}$$

Then we have

Theorem 3.1. (A version of the Siegle-Walfisz prime ideal theorem) If

$$\vartriangle := N\mathfrak{f}|d| \ll \log^A x \tag{3.2}$$

with an arbitrary constant A however large it may be, we have

$$M(x,\mathfrak{C}) = O_{n,A}\left(x\exp\left(-a\sqrt{\log x}\right)\right) \tag{3.3}$$

with a constant $a = a(n,A) > 0$ depending at most on n and A and so is the O – constant.

With Theorem 3.1 at hand, we may obtain generalizations of asymptotic formulas in [5], [16] with sharp estimate on the error term.

Example 3.2. For $\mu \in \mathbb{N}$ set

$$M_\ell^\mu(x) = \frac{1}{\mu!}\sum_{N\mathfrak{a}\le x}\frac{\mu(\mathfrak{a})}{N\mathfrak{a}}\left(\log\frac{x}{N\mathfrak{a}}\right)^\mu \tag{3.4}$$

Since $F(s) = \zeta_K(s+1)^{-1}$ where $\zeta_K(s) = \sum_{\mathfrak{a}\ne 0}\dfrac{1}{N\mathfrak{a}}$ indicates the Dedekind zeta-function of K, we have

$$P_\mu(\log x) = \sum_{n=1}^{\mu}\frac{a_n}{(\mu-n)!}(\log x)^{(\mu-n)}, \tag{3.5}$$

Where a_n are the Laurent coefficients of the Dedekind zeta:

$$\zeta_K(s)^{-1} = \sum_{n=1}^{\infty} a_n (s-1)^n. \tag{3.6}$$

$$M_\ell^\mu(x) = \sum_{n=1}^{\mu} \frac{a_n}{(\mu-n)!} (\log x)^{(\mu-n)} + O(\delta_A(x)). \tag{3.7}$$

4. Quwllenangaben

The Riesz sums may be thought of as integration or Abelian process ([4]) while differencing the Riesz sum to deduce a formula for the Riesz sum of order 0 corresponds to differentiation or Tauberian process.

The logarithmic Riesz sums also appeared in various context and we refer to [1] and [2] for them for which the generating function satisfies the functional equation.

For general modular relations, we refer to [9], [3] and the most comprehensive [14]. In the last ref., the Riesz sums are treated in Chapter 6. Some extracts and generalization have been made in [17].

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