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The Higher Derivation of the Hurwitz Zeta-function

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Abstract: In this paper, the Euler-Maclaurin Summation formula was researched, the purpose of research is to promote the application of the Hurwitz Zeta-function; Combination method of number theory special a function and Euler-Maclaurin Summation Formula was been used; By three derivatives of the the Euler-Maclaurin Summation formula, three formulas of Hurwitz Zeta-function were been given.

Keywords: Hurwitz Zeta-function, Euler Maclauring Summation, Logarithmic Derivative

1. Introduction and Main Results

In paper [1], many authors give many integral formulas of some Series by Zeta function, In paper [2], prefessor S. Kanemitsu exhibit the importance and usefulness of the Euler Maclaurn Summation formula by applying it to the sum.

$$L_u(x,a) = \sum_{0\le u\le x} (n+a)^u$$

In a Similar setting which appeared in the pursuit of the divisor problem [2] . In paper [3], Author give the formula $\frac{d}{du}L_u(x,a)$ and many beautiful formulas. In paper [5] we gave the formula $\frac{d^2}{du^2}L_u(x,a)$. In this paper, we can give $\frac{\mathrm{d}^3}{\mathrm{d}u^3}L_u(x,a)$ by an integral formula, then we give $\zeta'(-m,a)$, $\zeta''(-m,a)$ and $\zeta'''(-m,a)$. The papers [8-10] also give some results about Zeta function.

We use the following notation.

Notation $s=\sigma+it$ -- the complex variable $\Gamma(s)=\int_0^{+\infty}\mathrm{e}^{-t}t^{s-1}\mathrm{d}t$ -- the gamma function ($\sigma>0$). $\psi(s)=\frac{\Gamma'}{\Gamma}(s)=(\log\Gamma(s))'$ -- the digamma funtion.

Both of which are meromorphically continued to the whole complex plane with simple poles at non-positive integers;

$\zeta(s,a)=\sum_{n=0}^{+\infty}\frac{1}{(n+a)^s}$ -- Hurwitz Zeta function, $s>1$, $a>0$, the power taking the princial value $\zeta(s)=\zeta(s,1)$ -- the Riemann Zeta function, both of which are continued meromorphically over the complex plane with a simple pole at $s=1$,

$Br^{(\alpha)}(x)$ the generalized Bernoulli polynomial of degree r in x, defined through the generating function $\left(\frac{z}{\mathrm{e}^z-1}\right)^\alpha \mathrm{e}^{zx}=\sum_{r=0}^{+\infty}\frac{1}{r!}Br^{(\alpha)}(x)z^n \quad (|z|<2\pi)$ satisfying the addition formula

$$B_r^{(\alpha+\beta)}(x+y)=\sum_{r=0}^{\infty}\binom{r}{k}B_r^{(\alpha)}(x)B_{r-k}^{(\beta)}(y) \tag{1}$$

([1],Formula(24),p.61)with the properties $B_r^{(\alpha)}=B_r^{(\alpha)}(0)$, $B_r^{(1)}(x)=B_r(x)$, $B_r^{(1)}=B_r$, where $B_r(x)$ and $B_r=B_r(0)$ are the $r-$th Bernoulli polynomial and the $r-$th Bernoulli number defined by (1) with $\alpha=1$.

$\overline{B_r}(x)=B_r(\{x\})$ the $r-$th periodic Bernoulli polynomial.

Theorem1. for any $l\in\mathbb{N}$ with $l<\mathrm{Re}\,u+1$, $a>0$. we have

$$
\begin{aligned}
&\frac{\mathrm{d}^3}{\mathrm{d}u^3}L_u(x,a)=\sum_{0\le n\le x}(n+a)^u\log^3(x+a)\\
&=\sum_{r=1}^{l}\frac{(-1)^r}{r!}\overline{B_r}(x)(x+a)^{u-r+1}\frac{\Gamma(u+1)}{\Gamma(u+2-r)}\cdot\left\{\left[f(u,r)+\log(x+a)\right]^3+2f'(u,r)\log a+3f(u,r)f'(u,r)+f''(u,r)\right\}\\
&+\frac{(-1)^l}{l!}\frac{\Gamma(u+1)}{\Gamma(u+1-r)}\int_x^{+\infty}\overline{B}_r(t)(x+a)^{u-1}\cdot\left\{\left[f(u,l)+\log(x+a)\right]^3+3f(u,l)f'(u,l)+f'(u,l)\log(x+a)+f''(u,l)\right\}\mathrm{dt}\\
&+\begin{cases}
\frac{(x+a)^{u+1}}{u+1}\log^3(x+a)\\
\quad-\frac{3(x+a)^{u+1}}{(u+1)^2}\log^2(x+a)\\
\quad+\frac{6(x+a)^{u+1}}{(u+1)^3}\log(x+a)\\
\quad-\frac{6(x+a)^{u+1}}{(u+1)^4}-\zeta'''(-u,a) & u\ne 1\\
\frac{1}{4}\log^4(x+a)+\gamma_3(a) & u=1
\end{cases}
\end{aligned}
\tag{2}
$$

$$f(u,r)=\psi(u+1)-\psi(u+2-r),\ f(u,l)=\psi(u+1)-\psi(u+1-l),$$

$$\gamma_3(a)=\frac{1}{2a}\log^3 a-\frac{1}{4}\log^4 a-\int_0^{+\infty}\frac{\overline{B_1}(t)}{(t+a)^2}\left(\log^3(t+a)-3\log^4(t+a)\right)\mathrm{d}t$$

Corollary 1. For $m\in\mathbb{N}\cup\{0\}$, $\operatorname{Re}a>0$ and $m+2\le l\in\mathbb{N}$. We have

$$
\begin{aligned}
\zeta'(-m,a)&=\frac{1}{m+1}a^{m+1}\log a-\frac{1}{(m+1)^2}a^{m+1}-\frac{1}{2}a^m\log a+\frac{1}{12}a^{m-1}\log a+\sum_{r=4}^{m+1}\frac{B_r}{r}\left(\sum_{j=0}^{r-2}(-1)^j\binom{m}{j}\frac{1}{r-1-j}+\binom{m}{r-1}\log a\right)a^{m-r+1}\\
&+\frac{1}{m+1}\sum_{r=m+2}^{l}B_r\left(\sum_{j=0}^{r-1}(-1)^j\binom{r-m-2}{j}\frac{1}{r-j}\right)a^{m-r+1}+(-1)^{l+1}\int_0^{+\infty}\left(\sum_{j=0}^{l-1}(-1)^j\binom{l-m-1}{j}\frac{1}{l-j}B_l(t)(t+a)^{m-l}\right)\mathrm{d}t
\end{aligned}
\tag{3}
$$

Corollary 2. For $m\in\mathbb{N}\cup\{0\}$, $\operatorname{Re}a>0$ and $m+2\le l\in\mathbb{N}$. We have

$$
\begin{aligned}
\zeta''(-m,a)&=a^m\log a-\frac{1}{m+1}a^{m+1}\log^2 a+\frac{2a^{m+1}}{(m+1)^2}\log a-\frac{2a^{m+1}}{(m+1)^3}\\
&-\sum_{r=1}^{l}\frac{(-1)^r}{r!}B_ra^{m-r+1}\frac{m!}{(m+1-r)!}\cdot\left(\left(\sum_{k=0}^{r-1}\frac{1}{m+k-l+1}\log a\right)^2-\sum_{k=0}^{r-1}\frac{1}{(m+k-r)^2}\right)\\
&-\frac{(-1)^l}{l!}\frac{m!}{(m-l)!}\int_0^{+\infty}\overline{B_l}(t)(t+a)^{m-l}\cdot\left(\left(\sum_{k=1}^{r-1}\frac{1}{m+k-l}\log a\right)^2+\log^2(t+a)-\sum_{k=1}^{l}\frac{1}{(m+k-r)^2}\right)\mathrm{d}t
\end{aligned}
\tag{4}
$$

Corollary 3. For $m\in\mathbb{N}\cup\{0\}$, $\operatorname{Re}a>0$ and $m+2\le l\in\mathbb{N}$. We have

$$\zeta'''(-m,a)=-a^3\log^3 a+\frac{a^{m+1}}{m+1}\log^3 a-\frac{3a^{m+1}}{(m+1)^2}\log^2 a-\frac{6a^{m+1}}{(m+1)^3}\log a-\frac{6a^{m+1}}{(m+1)^4}+\sum_{r=1}^{l}\frac{(-1)^r}{r!}B_r\frac{m!}{(m+1-r)!}a^{m-r+1}$$
$$\cdot\left(\left(\sum_{k=1}^{r-1}\frac{1}{m+k-l+1}+\log a\right)^3-2\sum_{k=1}^{r-1}\frac{\log a}{(m+k-r+1)^2}-3\sum_{k_1=1}^{r-1}\sum_{k_2=1}^{r-1}\frac{1}{(m+k_2-r+1)(m+k_1-r+1)^2}+\sum_{k=1}^{r-1}\frac{1}{(m+k-r+1)^3}\right)$$
$$+\frac{(-1)^l}{l!}\frac{m!}{(m-l)!}a^{m-1}\int_0^{+\infty}\overline{B}_l(t)\left(\left(\sum_{k=1}^{l}\frac{1}{m+k-l}+\log a\right)^3\right.$$
$$\left.-3\sum_{k_1=1}^{l}\sum_{k_2=1}^{l}\frac{1}{(m+k_2-r+1)(m+k_1-r+1)^2}-\sum_{k=1}^{l}\frac{1}{(m+k-l)^2}\log a+\sum_{k=1}^{l}\frac{1}{(m+k-l)^3}\right)\mathrm{d}t \tag{5}$$

2. Lemmas and Proof

Lemma 1[6]. Suppose that f is of the Class C' in the closed interval $[a,b](a<b)$.then

$$\sum_{0\le n\le x}f(n)=\int_0^x f(t)\,\mathrm{d}t+\sum_{r=1}^{l}\frac{(-1)^r}{r!}\left(\overline{B}_r(x)f^{r-1}(x)-\overline{B}_r(x)f^{r-1}(0)\right)+\frac{(-1)^{l+1}}{l!}\int_0^x\overline{B}_l(t)f^{(l)}(t)\,\mathrm{d}t \tag{6}$$

Lemma 2. Let $L_u(x,a)=\sum_{0\le u\le x}(n+a)^u$. Then, for any $l\in\mathbb{N}$ with $l>\operatorname{Re}u+1$, $\operatorname{Re}a>0$ we have

$$L_u(x,a)=\sum_{r=1}^{l}\frac{\Gamma(u+1)}{\Gamma(u+2-r)}\frac{(-1)^r}{r!}\overline{B}_r(x)(x+a)^{u-r+1}+\frac{(-1)^r}{r!}\frac{\Gamma(u+1)}{\Gamma(u+1-r)}\int_x^{+\infty}\overline{B}_l(t)(t+a)^{u-l}\,\mathrm{d}t$$
$$+\begin{cases}\dfrac{1}{u+1}(x+a)^{u+1}+\zeta(-u,a), & u\ne-1\\ \log(x+a)-\psi(a), & u=-1\end{cases} \tag{7}$$

Also the asymptotic formula

$$L_u(x,a)=\sum_{r=1}^{l}\frac{(-1)^r}{r!}\binom{u}{r-1}\overline{B}_r(x)(x+a)^{u-r+1}+O\left(x^{\operatorname{Re}u-l}\right)+\begin{cases}\dfrac{1}{u+1}(x+a)^{u+1}+\zeta(-u,a), & u\ne-1\\ \log(x+a)-\psi(a), & u=-1\end{cases} \tag{8}$$

hods true as $x\to+\infty$. On the other hand, formula (7) with $x=0$ yields the integral representation

$$\zeta(-u,a)=a^u-\frac{1}{u+1}a^{u+1}-\sum_{r=1}^{l}\frac{(-1)^2}{r}\binom{u}{r-1}B_r a^{u-r+1}+(-1)^{l+1}\binom{u}{l}\int_0^{+\infty}\overline{B}_l(t)(t+a)^{u-1}\,\mathrm{d}t \tag{9}$$

Which is true for all $u\ne1$,and l can be any natural number satisfying $l>\operatorname{Re}u+1$; the integral being absolutely convergent in the region $\operatorname{Re}u<l-1$, where it is analytic except at $u=-1$.

Proof of Lemma 2.

Since the $r-$th derivative of $f(t)=(t+a)^u$ is

$$f^r(t)=\binom{u}{r}(r-1)!(t+a)^{u-r}=\frac{\Gamma(u+1)}{\Gamma(u+1-r)}(t+a)^{u-r} \tag{10}$$

We derive from (6) that

$$L_u(x,a)=(-1)^{l+1}\binom{u}{l}\int_x^{+\infty}\overline{B}_l(t)(t+a)^{u-l}\,\mathrm{d}t-\sum_{r=1}^{l}\frac{\Gamma(u+1)}{\Gamma(u+2-r)}\frac{(-1)^r}{r!}B_r(x)a^{u-r+1}+\sum_{r=1}^{l}\frac{\Gamma(u+1)}{\Gamma(u+2-r)}\frac{(-1)^r}{r!}\overline{B}_l(x)(x+a)^{u-r+1}$$

$$+\frac{(-1)^l}{l!}\frac{\Gamma(u+1)}{\Gamma(u+1-l)}\int_x^{+\infty}\overline{B_l}(t)(t+a)^{u-l}\,\mathrm{d}t+a^u+\begin{cases}\frac{1}{u+1}(x+a)^{u+1}-\frac{1}{u+1}a^{u+1}, & u\neq -1\\ \log(x+a)-\log a, & u=-1\end{cases} \tag{11}$$

For any natural number $l>\Re(u)+1$.

Now we note that the last integral is in (11) clearly $O\left(x^{\Re(u)-l}\right)$ by the mean value theorem for partial integration. Thus, taking the limit as $x\to+\infty$ the case when $\Re(u)<-l$, we conclude that the constant term on the right-hand side of (11) must coincide with the left-hand side.

$$\lim_{x\to\infty}L_u(x,a)=\zeta(-u,a)$$

That is, (9) follows.

Then, by analytic continuation; (9) can be show n to hold true for all $u\in C\setminus\{-1\}$.

At the same time, this gives a generic definition of $\zeta(-u,a)$;

$$\zeta(-u,a)=\lim_{N\to\infty}\left(\sum_{a=0}^{N}(n+a)^u-\frac{1}{u+1}(N+a)^{u+1}-\sum_{r=1}^{l}\frac{\Gamma(u+1)}{\Gamma(u+1-r)}\frac{(-1)^r}{r!}(N+a)^{u-r+1}\right. \tag{12}$$

Where $l\in\mathbb{N}$ satisfies $l>\Re(u)+1$.

On the other hand, for $u=-1$. Formula (11) implies that the constant term is

$$\log a+\frac{1}{2a}-\sum_{r=2}^{l}\frac{B_r}{r!}a^{-r}+\int_0^\infty\overline{B_l}(t)(t+a)^{-l-i}dt\ ,$$

Which must be equal to

$$\lim_{N\to\infty}\left(\sum_{n=0}^{N}(n+a)^{-1}-\log(N+a)\right),$$

Which we denote by $\gamma(a)=\gamma_0(a)$.

Upon replacing the constant term by $\zeta(-u,a)$ in (11) gives (7), which then entails (8) on replacing the integral by the above estimate $O\left(x^{\Re(u)-l}\right)$. this completes the proof of Lemma 2.

Lemma 3.[3] For any complex u and $a>0$

$$\frac{\mathrm{d}}{\mathrm{d}u}L_u(x,a)=M_u(x,a):=\sum_{0\le n\le x}(n+a)^u\log(n+a)=\sum_{r=1}^{l}\frac{\Gamma(u+1)}{\Gamma(u+2-r)}\frac{(-1)^r}{r!}\overline{B_r}(x)a^{u-r+1}\cdot\{\psi(u+1)-\psi(u+2-l)+\log(x+a)\}$$

$$+\frac{(-1)^r}{r!}\frac{\Gamma(u+1)}{\Gamma(u+1-r)}\int_x^{+\infty}\overline{B_l}(t)(t+a)^{u-l}\,\mathrm{d}t\cdot\{\psi(u+1)-\psi(u+1-l)+\log(x+a)\}+\begin{cases}\frac{1}{u+1}(x+a)^{u+1}\log(x+a)\\ \quad-\frac{1}{(u+1)^2}(x+a)^{u+1}+\zeta'(-u,a), & u\neq-1\\ \frac{1}{2}\{\log(x+a)\}^2+\gamma_1(a), & u=-1\end{cases} \tag{13}$$

Lemma 4.[5] For any complex u and $a>0$

$$\frac{\mathrm{d}^2}{\mathrm{d}u^2}L_u(x,a)=\sum_{0\le n\le x}(n+a)^u\log^2(n+a)=\sum_{r=1}^{l}\frac{\Gamma(u+1)}{\Gamma(u+2-r)}\frac{(-1)^r}{r!}\overline{B_r}(x)(x+a)^{u-r+1}\cdot\left\{\left[\psi(u+1)-\psi(u+2-l)+\log(x+a)\right]^2\right.$$
$$\left.+\psi'(u+1)-\psi'(u+2-l)\right\}+\frac{(-1)^l}{l!}\frac{\Gamma(u+1)}{\Gamma(u+1-r)}\int_x^{+\infty}\overline{B_l}(t)(t+a)^{u-l}\cdot\left\{\left[\psi(u+1)-\psi(u+1-l)+\log(x+a)\right]^2\right.$$
$$\left.+\psi'(u+1)-\psi'(u+1-l)\right\}\mathrm{d}t+\begin{cases}\dfrac{(x+a)^{u+1}}{u+1}\{\log(x+a)\}^2-\dfrac{2(x+a)^{u+1}}{(u+1)^2}\{\log(x+a)\}^2 \\ \quad+\dfrac{2(x+a)^{u+1}}{(u+1)^3}+\zeta''(-u,a), & u\ne-1 \\ \dfrac{1}{3}\{\log(x+a)\}^3+\gamma_2(a), & u=-1\end{cases}\tag{14}$$

Lemma 5.[7] For any $u\in\mathbb{C}$ and $n\in\mathbb{N}$, we have

$$\sum_{k=0}^{n-1}(-1)^k\binom{u}{k}\frac{1}{n-k}=(-1)^{n-1}\binom{u}{n}\left(\psi(u+1)-\psi(u+1-n)\right)\tag{15}$$

Which is the most convenient for most cases. but if $0\ge u\in\mathbb{Z}$, the right-hand side is table interpreted as $(-1)^n\binom{u}{n}\left(\psi(n-u)-\psi(-u)\right)$.

3. The Proof of Theorem and Corollaries

First, we now go on to the proof Theorem.
From lemma 4 (14), when $u\ne-1$. Let

$$M_1(u)=\sum_{r=1}^{l}\frac{\Gamma(u+1)}{\Gamma(u+2-r)}\frac{(-1)^r}{r!}\overline{B_r}(x)(x+a)^{u-r+1}\cdot\left\{\left[\psi(u+1)-\psi(u+2-l)+\log(x+a)\right]^2+\psi'(u+1)-\psi'(u+2-l)\right\}$$

$$M_2(u)=\frac{(-1)^l}{l!}\frac{\Gamma(u+1)}{\Gamma(u+1-l)}\int_x^{+\infty}\overline{B_l}(t)(t+a)^{u-l}\cdot\left\{\left[\psi(u+1)-\psi(u+2-l)+\log(x+a)\right]^2+\psi'(u+1)-\psi'(u+1-l)\right\}\mathrm{d}t$$

$$M_3(u)=\frac{(x+a)^{u+1}}{u+1}\{\log(x+a)\}^2-\frac{2(x+a)^{u+1}}{(u+1)^2}\{\log(x+a)\}^2+\frac{2(x+a)^{u+1}}{(u+1)^3}+\zeta''(-u,a)$$

So $\frac{\mathrm{d}^2}{\mathrm{d}u^2}Lu(x,a)=M_1(u)+M_2(u)+M_3(u)$. We give derivation

$$\frac{\mathrm{d}^3}{\mathrm{d}u^3}L_u(x,a)=M_1'(u)+M_2'(u)+M_3'(u)\tag{16}$$

When $u\ne-1$.

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\Gamma(u+1)}{\Gamma(u+2-r)}\right)=\frac{\Gamma(u+1)}{\Gamma(u+2-r)}\{\psi(u+1)-\psi(u+2-r)\}$$

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\Gamma(u+1)}{\Gamma(u+1-l)}\right)=\frac{\Gamma(u+1)}{\Gamma(u+1-l)}\{\psi(u+1)-\psi(u+1-l)\}$$

Let $f(u,r)=\psi(u+1)-\psi(u+2-r)$,

$$f(u,l)=\psi(u+1)-\psi(u+1-l)$$

$$M_1'(u)=\sum_{r=1}^{l}\frac{(-1)^r}{r!}\overline{B_r}(x)(x+a)^{u-r+1}\frac{\Gamma(u+1)}{\Gamma(u+2-r)}\cdot\left\{\left[f(u,r)+\log(x+a)\right]^3+2f'(u,r)\log a+3f(u,r)f'(u,r)+f''(u,r)\right\} \quad (17)$$

$$M_2'(u)=\frac{(-1)^l}{l!}\frac{\Gamma(u+1)}{\Gamma(u+1-l)}\int_x^{+\infty}\overline{B_l}(t)(t+a)^{u-l}\cdot\left\{\left(f(u,l)+\log(x+a)\right)^3+3f(u,l)f'(u,l)+f'(u,l)\log(x+a)+f''(u,l)\right\}\mathrm{d}t \quad (18)$$

$$M_3'(u)=\frac{(x+a)^{u+1}}{u+1}\log^3(x+a)-\frac{3(x+a)^{u+1}}{(u+1)^2}\log^2(x+a)+\frac{6(x+a)^{u+1}}{(u+1)^3}\log(x+a)-\frac{6(x+a)^{u+1}}{(u+1)^4}-\zeta'''(-u,a) \quad (19)$$

Put (17), (18) and (19) in (16), we can get the poof of theorem with $u\neq -1$

When $u=-1$ let $f(t)=\frac{1}{t+a}\log^3(t+a)$

In Lemma 1 (6)

$$\begin{aligned}f^{(k)}(t)&=(-1)^k k!(t+a)^{-k-1}\left(\log^3(t+a)-3\sum_{l=1}^{k}\frac{1}{l}\log^2(t+a)+3\sum_{l=1}^{k-1}\frac{1}{l}\log(t+a)-\sum_{l=1}^{k-2}\frac{1}{l}\right.\\&=(-1)^k k!(t+a)^{-k-1}\left\{\log^3(t+a)-3\left(\Psi(k)-\Psi(1)\right)\log^2(t+a)\right)+3\left(\Psi(k-1)-\Psi(1)\right)\log(t+a)-\left(\Psi(k-2)-\Psi(1)\right)\right\}\end{aligned} \quad (20)$$

$$\int_0^x f(t)dt=\int_0^x\frac{1}{t+a}\log^3(t+a)dt=\frac{1}{4}\log^4(x+a)-\frac{1}{4}\log^4 a \quad (21)$$

$$\Psi(u+1)-\Psi(u+2-r)=-\sum_{k=1}^{r}\frac{1}{l}=\Psi(1)-\Psi(r+1),$$

$$\frac{\Gamma(u+1)}{\Gamma(u+2-r)}=\binom{-1}{r-1}(r-1)!,\ \frac{\Gamma(u+1)}{\Gamma(u+1-l)}=\binom{-1}{l}l!,$$

$$f(u,r)=\psi(u+1)-\psi(u+2-r)=\sum_{k=1}^{r-1}\frac{1}{u+k-r+1}=\sum_{k=1}^{r-1}\frac{1}{k-r}=-\sum_{k=1}^{r-1}\frac{1}{k}=\Psi(1)-\Psi(r)$$

$$f^{/}(u,r)=-\sum_{k=1}^{r-1}\frac{1}{u+k-r+1}=-\sum_{k=1}^{r-1}\frac{1}{(k-r)^2}=-\sum_{k=1}^{r-1}\frac{1}{k^2}$$

$$f''(u,r)=2\sum_{k=1}^{r-1}\frac{1}{(k-r)^3}=-2\sum_{k=1}^{r-1}\frac{1}{k^3}$$

$$\begin{aligned}&\sum_{r=1}^{l}\frac{(-1)^r}{r!}\overline{B_r}(x)f^{(r-1)}(x)=-\sum_{r=1}^{l}\frac{\overline{B_r}(x)}{r}(x+a)^{-r}\left\{\log^3(x+a)-3\left(\Psi(k-1)-\Psi(1)\right)\log^2(x+a)+3\left(\Psi(k-2)-\Psi(1)\right)\log(x+a)\right.\\&\left.-\left(\Psi(k-3)-\Psi(1)\right)\right\}\\&=\sum_{r=1}^{l}\frac{(-1)^r}{r!}\overline{B_r}(x)(x+a)^{u-r+1}\binom{-1}{r-1}(r-1)!\cdot\left[\left(\Psi(1)-\Psi(r)+\log(x+a)\right)^3-2\sum_{k=1}^{r-1}\frac{1}{k^2}\log a+3\left(\Psi(1)-\Psi(r)\right)\left(-\sum_{k=1}^{r-1}\frac{1}{k^2}\right)+f''(u,r)\right]\end{aligned}$$

$$=\sum_{r=1}^{l}\frac{(-1)^r}{r!}\overline{B_r}(x)(x+a)^{u-r+1}\binom{-1}{r-1}(r-1)!\cdot\left\{\left[\left(\Psi(1)-\Psi(r)\right)+\log(x+a)\right]^3-2\sum_{k=1}^{r-1}\frac{1}{k^2}\log a-3\sum_{k=1}^{r-1}\frac{1}{k^2}-2\sum_{k=1}^{r-1}\frac{1}{k^3}\right\} \quad (22)$$

$$\sum_{r=1}^{l}\frac{(-1)^r}{r!}\overline{B_r}(x)f^{(r-1)}(0)=\sum_{r=1}^{l}\frac{(-1)^r}{r!}\overline{B_r}(x)a^{-r}\binom{-1}{r-1}(r-1)!\left(\left[\left(\Psi(1)-\Psi(r)\right)+\log a\right]^3-2\sum_{k=1}^{r-1}\frac{1}{k^2}\log a-3\sum_{k=1}^{r-1}\frac{1}{k^2}-2\sum_{k=1}^{r-1}\frac{1}{k^3}\right) \quad (23)$$

We can us the same why to get

$$\begin{aligned}&\frac{(-1)^{l+1}}{l!}\int_0^x \overline{B_l}(t) f^{(l)}(t)\mathrm{d}t\\&=(-1)^l\binom{-1}{l}\int_x^{+\infty}\overline{B_l}(t)(t+a)^{-1-l}\cdot\left\{(\Psi(1)-\Psi(r)+\log(x+a))^3-3(\Psi(1)-\Psi(r))\sum_{k=1}^{r-1}\frac{1}{k^2}-\sum_{k=1}^{r-1}\frac{1}{k^2}\log(x+a)-2\sum_{k=1}^{r-1}\frac{1}{k^3}\right\}\mathrm{d}t\end{aligned} \tag{24}$$

Put (21),(22),(23) and (24) in (6), we get the proof with $u=-1$
We now complete the proof theorem.
Second, we go on to the Proof of Corollary 1.
Since those terms of the sum with $r\geq m+2$ for
$\frac{\mathrm{d}}{\mathrm{d}u}L_u(x,a)$ have singularities at non-positive integers, we have to take the limit as the limit as $u\to m$ of

$$\frac{\Gamma(u+1)}{\Gamma(u+2-r)}\left(-\psi(u+2-r)\right) \text{ or } \frac{\Gamma(u+1)}{\Gamma(u+1-l)}\left(-\psi(u+1-l)\right)$$

as the case may be, on noting that other terms vanish because of simple zeros of $\Gamma(z)^{-1}$ at non-positive integers.

By the fundamental difference equation satisfied by the gamma function we deduce for $\nu\in\mathbb{N}\cup\{0\}$ that

$$\Gamma(z)=\frac{\Gamma(z+\nu+1)}{z\cdot\cdots\cdot(z+\nu)} \tag{25}$$

We contend that

$$\lim_{z\to-\nu}\frac{\psi(z)}{\Gamma(z)}=-(-1)^\nu\,\nu! \tag{26}$$

To prove this it suffices to note from (20) that in taking the limit

$$\lim_{z\to-\nu}\frac{\psi(z)}{\Gamma(z)}=\lim_{z\to-\nu}\frac{\Gamma'(z)}{\Gamma(z)^2}$$

Only one term counts, i.e., it is equal to

$$-\lim_{z\to-\nu}\frac{\Gamma(z+\nu+1)}{z\cdot\cdots\cdot(z+\nu-1)\left[(z+\nu)\Gamma(z)\right]^2}$$

Which is $-(-1)^\nu\,\nu!$ on recalling the fact
$\mathrm{Re}\,s_{z=-\nu}\Gamma(z)=\frac{(-1)^\nu}{\nu!}$, which in fact follows from (25). Hence

$$\lim_{u\to m}\frac{\Gamma(u+1)}{\Gamma(u+2-r)}\left(-\psi(u+2-r)\right)=(-1)^{r-m}\frac{r!}{(m+1)(m+2)\binom{r}{m+2}} \tag{27}$$

and

$$\lim_{u\to m}\frac{\Gamma(u+1)}{\Gamma(u+1-l)}\left(-\psi(u+1-l)\right)=\frac{(-1)^{l-m}}{(m+1)\binom{l}{m+1}}$$

Hence we conclude that

$$\zeta'(-m,a)=\frac{1}{m+1}a^{m+1}\log a-\frac{1}{(m+1)^2}a^{m+1}-\frac{1}{2}a^m\log a+\frac{1}{12}a^{m-1}(1+m\log a)$$

$$+\sum_{r=4}^{m+1}\frac{B_r}{r}\binom{m}{r-1}\left(\frac{1}{m}+\cdots+\frac{1}{m-(r-2)}+\log a\right)a^{m-r+1}+\frac{(-1)^m}{m+1}\sum_{r=m+2}^{l}B_r\frac{1}{(m+2)\binom{r}{m+2}}a^{m-r+1}$$

$$+\frac{(-1)^m}{m+1}\sum_{r=m+2}^{l}B_r\frac{1}{(m+2)\binom{r}{m+2}}a^{m-r+1}+\frac{(-1)^m}{(m+1)\binom{l}{m+1}}\int_0^{+\infty}B_l(t)(t+a)^{m-l}\,\mathrm{d}t \tag{28}$$

In order to deduce (3) from (28) we have recourse to Lemma 5 and its counterpart

$$\sum_{j=0}^{n-1}(-1)^j\binom{u}{j}\frac{1}{n-j}=(-1)^u\frac{1}{(n-u)\binom{n}{u}}$$

for $n>u\in\mathbb{N}$ ([7,Formula(2.6)]). Substituting the formulas

$$(-1)^{r-2}\binom{m}{r-1}\big(\psi(m+1)-\psi(m+2-r)\big)=\sum_{j=0}^{r-2}(-1)^j\binom{m}{j}\frac{1}{r-1-j}\quad(r\le m+1)$$

$$\frac{1}{(m+2)\binom{r}{m+2}}=(-1)^{r-m}\sum_{j=0}^{r-1}(-1)^j\binom{r-m-2}{j}\frac{1}{r-j}\quad(r\ge m+2)$$

and

$$\frac{1}{(m+1)\binom{l}{m+1}}=(-1)^{l-m-1}\sum_{j=0}^{l-1}(-1)^j\binom{l-m-1}{j}\frac{1}{l-j}$$

in (28), we now complete the proof of Corollary 1.
Because proof method of Corollary 2 and Corollary 3 is similar. So, in here, we only give proof of Corollary 3.
From [4]

$$\psi(z+1)-\psi(z)=\frac{1}{z}\quad(z\neq 0) \tag{29}$$

We any easily deduce that

$$\psi^{(2\nu)}(z+m)-\psi^{(\nu)}(z)=(-1)^{(\nu)}\nu!\sum_{k=1}^{m}\frac{1}{(z+k-1)^{\nu+1}} \tag{30}$$

So

$$f(u,r)=\psi(u+1)-\psi(u+2-r)=\sum_{k=1}^{r-1}\frac{1}{u+k-r+1},$$

$$f(u,l)=\psi(u+1)-\psi(u+1-l)=\sum_{k=1}^{r-1}\frac{1}{u+k-l},$$

$$f'(u,r)=-\sum_{k=1}^{r-1}\frac{1}{(u+k-r+1)^2},\ f'(u,l)=-\sum_{k=1}^{r-1}\frac{1}{(u+k-l)^2},$$

$$f''(u,r)=2\sum_{k=1}^{r-1}\frac{1}{(u+k-r+1)^2},\ f''(u,l)=2\sum_{k=1}^{r-1}\frac{1}{(u+k-l)^2}.$$

Let $x=0$, $\overline{B_r}(x)=B_r$, $u\neq -1$

$$\sum_{0\le n\le x}(n+a)^u\log(x+a)=a^u\log^3 a=\sum_{r=}^{l}\frac{(-1)^r}{r!}B_r\frac{\Gamma(u+1)}{\Gamma(u+2-r)}a^{u-r+1}\left\{\left(\sum_{k=1}^{r-1}\frac{1}{u+k-r+1}+\log a\right)^3\right.$$

$$-2\sum_{k=1}^{r-1}\frac{\log a}{(u+k-r+1)^2}-3\sum_{k_1=1}^{r-1}\sum_{k_2=1}^{r-1}\frac{1}{(u+k_2-r+1)(u+k_1-r+1)^2}+\left.\sum_{k=1}^{r-1}\frac{1}{(u+k-r+1)^3}\right\}+\frac{(-1)^l}{l!}\frac{\Gamma(u+1)}{\Gamma(u+1-l)}a^{u-1}\int_0^{+\infty}\overline{B_l}(t)$$

$$\left\{\left(\sum_{k=1}^{l}\frac{1}{u+k-l}+\log a\right)^3-3\sum_{k_1=1}^{l}\sum_{k_2=1}^{l}\frac{1}{(u+k_2-l)(u+k_1-l)^2}-\sum_{k=1}^{l}\frac{\log a}{(u+k-l)^2}+\sum_{k=1}^{l}\frac{1}{(u+k-l)^3}\right\}\mathrm{d}t$$

$$+\frac{a^{u+1}}{u+1}\log^3 a-\frac{3a^{u+1}}{(u+1)^2}\log^2 a+\frac{6a^{u+1}}{(u+1)^3}\log a-\frac{6a^{u+1}}{(u+1)^4}-\zeta'''(-u,a)$$

When $u\to m$

$$\frac{\Gamma(u+1)}{\Gamma(u+2-r)}=\frac{m!}{(m+1-r)!},\frac{\Gamma(u+1)}{\Gamma(u+1-l)}=\frac{l!}{(m-l)!}$$

Then

$$\zeta'''(-m,a)=-a^3\log^3 a+\frac{a^{m+1}}{m+1}\log^3 a-\frac{3a^{m+1}}{(m+1)^2}\log^2 a-\frac{6a^{m+1}}{(m+1)^3}\log a-\frac{6a^{m+1}}{(m+1)^4}$$

$$+\sum_{r=1}^{l}\frac{(-1)^r}{r!}B_r\frac{m!}{(m+1-r)!}a^{m-r+1}\cdot\left(\left(\sum_{k=1}^{r-1}\frac{1}{m+k-l+1}+\log a\right)^3-2\sum_{k=1}^{r-1}\frac{\log a}{(m+k-r+1)^2}\right.$$

$$\left.-3\sum_{k_1=1}^{r-1}\sum_{k_2=1}^{r-1}\frac{1}{(m+k_2-r+1)(m+k_1-r+1)^2}+\sum_{k=1}^{r-1}\frac{1}{(m+k-r+1)^3}\right)+\frac{(-1)^l}{l!}\frac{m!}{(m-l)!}a^{m-1}\int_0^{+\infty}\overline{B_l}(t)\left(\left(\sum_{k=1}^{l}\frac{1}{m+k-l}+\log a\right)^3\right.$$

$$\left.-3\sum_{k_1=1}^{l}\sum_{k_2=1}^{l}\frac{1}{(m+k_2-r+1)(m+k_1-r+1)^2}-\sum_{k=1}^{l}\frac{1}{(m+k-l)^2}\log a+\sum_{k=1}^{l}\frac{1}{(m+k-l)^3}\right)\mathrm{d}t$$

We now complete the proof of Corollary 3.

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