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Criterion of Existence of Eigen Values of Linear Multiparameter Systems

Rakhshanda Dzhabarzadeh [1], Elnara Sultanova [2]

[1]Department of functional analysis of Institute of Mathematics and Mechanics of NAS of Azerbaijan, Baku
[2]Department of Applied Mathematics of Baku State Universiteties, Azerbaijan, Baku

Email address:
rakhshanda.dzhabarzade@rambler.ru (R. M. Dzhabarzadeh), elnare-sultanova@mail.ru (E. B. Sultanova)

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Abstract: It is considered the linear multiparameter system of operators when the number of equations may be more than the number of parameters. For such multiparameter systems the authors have proved the criterion of existence of eigen values. Under certain conditions, the authors a have proved that all components of the eigen values of the considered multiparameter systems are real numbers.

Keywords: Operator, Parameter, Eigenvalue, System, Multiparameter

1. Introduction

Spectral theory of operators is one of the important directions of functional analysis. The method of separation of variables in many cases turned out to be the only acceptable, since it reduces finding a solution of a complex equation with many variables to finding a solution of a system of ordinary differential equations, which are much easier to study ..F.V. Atkinson [1] studied the results available for multiparameter symmetric differential systems, built multiparameter spectral theory of selfadjoint systems in finite-dimensional Euclidean spaces. Further, by taking the limit Atkinson generalized the results obtained for the multiparameter systems with self-adjoint operators in finite dimensional space on the case of the multiparameter system with self-adjoint compact operators in infinite-dimensional Hilbert spaces. These multiparameter systems were studied of many mathematicians in the case when the number of parameters is the system is equal to the number of equations. Until recently time in [1],[2],[3] the main requisition to the operators, forming the multiparameter system were to be self-adjoint and bounded. Browne received the spectral expansion of Parseval- Atkinson when operators $B_{0,k}$ $(k=1,2,...,n)$ in (1) may be symmetric unbounded operator.

2. Preliminary Definitions and Remarks

Let be the linear multiparameter system in the form:

$$B_k(\lambda)x_k = (B_{0,k} + \sum_{i=1}^{n} \lambda_i B_{i,k})x_k = 0, \quad k = 1,2,...,n \qquad (1)$$

when operators $B_{k,i}$ act in the Hilbert space H_i

Definition 1. [1,2,3] $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \in C^n$ is an eigenvalue of the system (1) if there are such non-zero elements $x_i \in H_i$, $i = 1,2,...,n$ s that (1) is satisfied, and decomposable tensor $x = x_1 \otimes x_2 \otimes ... \otimes x_n$ is called the eigenvector corresponding to eigenvalue $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \in C^n$.

Definition 2.[1,2,3]

The operator $B_{s,i}^+$ is induced into the space $H_1 \otimes \cdots \otimes H_n$ by an operator $B_{s,i}$, acting in the space H_i ,, if on each decomposable tensor $x = x_1 \otimes ... \otimes x_n$ of tensor product space $H = H_1 \otimes ... \otimes H_n$ we have $B_{s,i}^+ x = x_1 \otimes ... \otimes x_{i-1} \otimes B_{s,i} x_i \otimes x_{i+1} \otimes ... \otimes x_n$, on all the other elements of $H = H_1 \otimes ... \otimes H_n$ the operator $B_{s,i}^+$ is defined on linearity and continuity.

Definition 3. For the system (1) in [1,2,3] analogue of the Cramer's determinants, when the number of equations is equal to the number of variables, is defined as follows: on decomposable tensor $x = x_1 \otimes ... \otimes x_n$ operators Δ_i are defined with help of the matrices.

$$\sum_{i=0}^{n}\alpha_i\Delta_i x=\otimes\begin{pmatrix}\alpha_0 & \alpha_1 & \alpha_2 & .... & \alpha_n\\ B_{0,1}x_1 & B_{1,1}x_1 & B_{2,1}x_1 & ... & B_{n,1}x_1\\ B_{0,2}x_2 & B_{1,2}x_2 & B_{2,2}x_2 & ... & B_{n,2}x_2\\ B_{0,3}x_3 & B_{1,3}x_3 & B_{2,3}x_3 & ... & B_{n,3}x_3\\ ... & ... & ... & ... & ...\\ B_{0,n}x_n & B_{1,n}x_n & B_{2,n} & ... & B_{n,n}\end{pmatrix} \quad (2)$$

where $\alpha_0,\alpha_1,...,\alpha_n$ are arbitrary complex numbers, under the expansion of the determinant means its formal expansion, when the element $x=x_1\otimes x_2\otimes...\otimes x_n$ is the tensor products of elements $x_1,x_2,...,x_n$ If $\alpha_k=1,\alpha_i=0,\ i\neq k$, then right side of (2) is equal to $\Delta_k x$, where $x=x_1\otimes x_2\otimes...\otimes x_n$ On all the other elements of the space H operators Δ_i are defined by linearity and continuity. $E_s(s=1,2,...,n)$ is the identity operator of space H_s . Suppose that for all $x\in H,\ x\neq 0,\ (\Delta_0 x,x)\geq\delta(x,x)>0,\ \delta>0$, and all $B_{i,j}$ are selfadjoint bounded operators in the space H_j. Inner product [.,.] is defined as follows; if $x=x_1\otimes x_2\otimes...\otimes x_n$ and $y=y_1\otimes y_2\otimes...\otimes y_n$ are decomposable tensors, then $[x,y]=(\Delta_0 x,y)$ where (x,y) is the inner product in the space H_i. On all the other elements of the space H the inner product is defined on linearity and continuity. In space H with such a metric all operators $\Gamma_i=\Delta_0^{-1}\Delta_i$ are self adjoint.

Spectrum of the system(1)[see[1],[2]) coincides with the joint spectrum of the operators $\Gamma_i=\Delta_0^{-1}\Delta_i$ so in this metric operators $\Gamma_i=\Delta_0^{-1}\Delta_i$ are self adjoint operators. Moreover, the joint spectrum of operators $\Gamma_i=\Delta_0^{-1}\Delta_i$ contains only real numbers. In all these works number of parameters is equal to the number of equations.

Consider the multiparameter system of operators when the number of operators in it is more than the number of parameters.

Let be

$$B_k(\lambda)x_k=(B_{0,k}+\sum_{i=1}^{n}\lambda_i B_{i,k})x_k=0,\quad k=1,2,...,l;\ l>n \quad (3)$$

And $B_{s,i}$ are the self-adjoint bounded operators acting in the Hilbert space H_j, $H_1\otimes\cdots\otimes H_l,\ l>n$

System (4) differs from system (1) so the number of equations is more than the number of parameters

Let $B^{+}_{j,k}$ be the operators, induced into the space $H=H_1\otimes...\otimes H_l$ by the operators $B_{j,k}$, correspondingly. So we have the system

$$B_k^{+}(\lambda)x_1\otimes...\otimes x_l=(B^{+}_{0,k}+\sum_{i=1}^{n}\lambda_i B^{+}_{i,k})x_1\otimes...\otimes x_l=0,\quad k=1,2,...,l;\ l>n \quad (4)$$

We form of the equations of the system (1) various multiparameter systems consisting of n equations and n parameters. If several equations have been left out of groups, supplement them with any of the equations of the system (1). Each group represents a multiparameter system previously studied, since the number of parameters is identical to the number of variables

Let be the groups N. Denote the operators $B_{j,k}$ entering in s -th multiparameter system though $B^{s}_{j,,k}$ .We have the systems

$$B_s^{r}(\lambda)x_{(r-1)n+s}=(B^{r}_{0,s}+\sum_{i=1}^{n}\lambda_i B^{r}_{i,s})x_{(r-1)n+s}=0,\quad r=1,2,...,N;\ l>n$$
$$s=1,2,...,N; \quad (5)$$

We will consider each group separately. For each group the following results hold. From]1],[2] follows that if the operators $x\in H,\ x\neq 0,\ (\Delta_0^{r}x,x)\geq\delta(x,x)>0,\ \delta>0,\ r=1,...,N$ all operators $B_{j,k}$ are self adjoint operator in corresponding spaces then the parameters of each multiparameter systems are separated and we have the following equalities

$\Gamma_i^{r}x_r=\lambda_i^{r}x_r\ \ i=1,...,n;\ r=1,...,N,x\in H_1\otimes...\otimes H_l.$ In fact, that all components of eigenvalues in each multiparameter system are real numbers, so in this metric of the space $H=H_1\otimes...\otimes H_l$ operators Γ_i^{r} for all $i=1,...,n;\ r=1,...,N$ are the selfadjoint operators. Consequently, for all meanings of I we have $\Delta_i^{r}x_r=\lambda_i^{r}\Delta_0^{r}x_r\quad r=1,...,N,x\in H_1\otimes...\otimes H_l.$

For all fixed r we have n equations on one parameter. Further we use the results of the work [11]. We give some notions, necessary for understanding of

3. The Notion of Resultant's Analog of Two Operator Pencils

Let

$$A(\lambda)=A_0+\lambda A_1+\lambda^2 A_2+...+\lambda^n A_n,$$
$$B(\lambda)=B_0+\lambda B_1+\lambda^2 B_2+...+\lambda^m B_m$$

be two operator pencils depending on the same parameters λ and acting, generally speaking, in various Hilbert spaces H_1,H_2, correspondingly.[4,12]

Resultant of two operator pencils $A(\lambda)$ and $B(\lambda)$ is the operator, presented by the determinant (2) and acting in the space $(H_1 \otimes H_2)^{n+m}$ - direct sum of $n+m$ copies of tensor product $H_1 \otimes H_2$ of spaces H_1 and H_2.

$$\operatorname{Re}s(A(\lambda),B(\lambda)) =$$
$$\begin{pmatrix} A_0\otimes E_2 & A_1\otimes E_2 & \dots & A_n\otimes E_2 & \dots & 0 \\ . & . & \dots & . & \dots & . \\ 0 & 0 & \dots A_0\otimes E_2 & A_1\otimes E_2 & \dots & A_n\otimes E_2 \\ E_1\otimes B_0 & E_1\otimes B_1 & \dots & E_1\otimes B_m & .. & 0 \\ . & . & \dots & . & \dots & . \\ . & . & \dots E_1\otimes B_0 & E_1\otimes B_1 & \dots. & E_1\otimes B_m \end{pmatrix} \quad (6)$$

In a matrix $\operatorname{Re}s(A_1(\lambda),A_2(\lambda))$ the number of rows with operators A_i is equal to leading degree of parameter λ in the operator pencil $B(\lambda)$, that is m, the number of rows in matrices with operators B_i coincides with the leading degree of parameter λ of the pencil $A(\lambda)$, that is n. One major application of the matrix theory is calculation of determinants, a central concept in linear algebra. It turns out that a mapping is invertible if and only if the determinant is non-zero.

The concept of abstract analog of resultant for two operator pencils when they have the identical degrees concerning parameter, has been given in work of Khayniq [9], for operator pencils, generally speaking, with the different degrees of parameter the abstract analog of a Resultant is studied by Balinskii [4].

Let all operators A_i $(i=0,1,...,n)$ (correspondingly, B_i $(i=0,1,...,m)$ are bounded in the Hilbert space (correspondingly, H_2) and operator A_n or B_m is invertible.

By [2] follows that the existence non-zero kernel of the operator $\operatorname{Re}s(A(\lambda),B(\lambda))$ is the necessary and sufficient conditions for the existing the common point of spectra of operators $A(\lambda)$ and $B(\lambda)$, if the spectrum of each operator $A(\lambda)$ and $B(\lambda)$ contains only eigen values. In this case a common point of spectra of these operators $A(\lambda)$ and $B(\lambda)$ is their eigenvalue.

4. Necessary and Sufficient Conditions of Existence of Eigen Values of Several Operator Polynomials

Consider [7,11]

$$\begin{cases} B_i(\lambda) = B_{0,i} + \lambda B_{1,i} + \dots + \lambda^{k_i} B_{k_i,i} \\ i=1,2,...,n \end{cases} \quad (7)$$

when $B_i(\lambda)$ is an operator bundle with a discrete spectrum, acting in Hilbert space H_i ($i=1,2,...,n$). Without loss of generality, we assume that. $k_1 \ge k_2 \ge \dots \ge k_n$. $H^{k_1+k_2}$ is the direct sum of k_1+k_2 copies of tensor product space $H = H_1 \otimes \dots \otimes H_n$.

Introduce the operators R_i $(i=1,2,...,n-1)$ with help of the operator matrices (5)

$$R_{i-1} = \begin{pmatrix} B^+_{0,1} & B^+_{1,1} & \cdots & B^+_{k_1,1} & 0 & 0 & \cdots & 0 \\ 0 & B^+_{0,1} & \cdots & B^+_{k_1-1,1}, & B^+_{k_1,1} & 0 & \cdots & 0 \\ . & . & \cdots & . & . & . & \cdots & . \\ . & . & \cdots & B^+_{0,1} & B^+_{1,1} & . & \cdots & B^+_{k_1,1} \\ B^+_{0,i} & B^+_{1,i} & \cdots & B^+_{k_i-1,i} & B^+_{k_2-1,i} & 0 & \cdots & 0 \\ 0 & B^+_{0,i} & \cdots & B^+_{k_2-2,i} & B^+_{k_2-1,i} & B^+_{k_2,i} & \cdots & 0 \\ . & . & \cdots & . & . & . & \cdots & . \\ 0 & 0 & 0 & 0 & B^+_{0,i} & B^+_{1,i} & \cdots & B^+_{k_2,i} \end{pmatrix} \quad (8)$$

Rows with the operators $B^+_{i,1}$ $(i=0,1,...,k_1)$ are repeated k_2 time and rows with the operators $B^+_{s,i}(s=0,1,...,k_2)$, $B_{k_i+1,i} = \dots = B_{k_2,i} = 0$ are repeated $\max(m_1n_2, m_2n_1) = m_2n_1$ time. The operator $B^+_{i,s}(i=1,2,...,k_s; s=2,...,k_i)$ is induced by an operator $B_{i,s}$ acting in the space H_k into the space $H_1 \otimes \cdots \otimes H_n$ by the formulae

$$B^+_{i,s} = E_1 \otimes \cdots \otimes E_{s-1} \otimes B_{i,s} \otimes E_{s+1} \otimes \cdots \otimes E_n$$

Denote $\sigma_p(B_i(\lambda))$ the set of eigen values of operator $B_i(\lambda)$.

Theorem 2. [7],[11]. Let all operators $B_{j,k}$ are bounded in the corresponding spaces H_k, the operator $B_{k_1,1}$ has an inverse. Spectrum of each operator pencil $B_i(\lambda)$ contains only eigen values.

Then $\bigcap_{i=1}^{n} \sigma_p(B_i(\lambda)) \ne \{\vartheta\}$ if and only if

$$\bigcap_{i=2}^{n} KerR_i \ne \{\theta\},\ (KerB_{k_1} = \{\theta\}). \quad (9)$$

This result is obtained in [7], [11].

Each group we study separately. Results for any group will be same. Without loss of generality of proof we consider the first group. Further, we assume that all operators $B_{i,k}$ are induced in space $H = H_1 \otimes \dots \otimes H_l$.

In order not to complicate the entry of a further induced in the space $H = H_1 \otimes \dots \otimes H_l$ of operators $B_{i,k}$ leave designation $B_{i,k}$ .Denote though Δ^r_i operators, constructing for $r-$ th multiparameter system on the rules (3). All operators Δ^r_i act in the space $H = H_1 \otimes \dots \otimes H_l$.

Theorem3. Let all operators $B_{i,k}$ in the system are self adjoint and $x \in H$, $x \ne 0$, $(\Delta^r_0 x, x) \ge \delta_r(x,x) > 0$, $\delta_r > 0$, $r = 1,...,N$

Then for all multiparameter systems with n parameters the following equalities

$$\Gamma_i^r x_r = \lambda_i^r x_r, \; i = 1,2,...,n; \;\; r = 1,2,...,N \quad (10)$$

Or

$$\Delta_i^r x_r = \lambda_i^r \Delta_0^r x_r, \; i = 1,2,...,n; \;\; r = 1,2,...,N \quad (11)$$

hold.

Proof of the Theorem3

It is clear that the conditions of the theorem3 mean the fulfilling of the conditions of the theorem from the work[1],[2]. In the adopting metric of the tensor space $H = H_1 \otimes ... \otimes H_l$ operators Γ_i^r are selfadjoint operators in the space $H = H_1 \otimes ... \otimes H_l$ and, moreover, the separation of the parameters in the form(10) takes place.

Let the inner product [.,.] is defined as follows; if $x = x_1 \otimes x_2 \otimes ... \otimes x_n$ and $y = y_1 \otimes y_2 \otimes ... \otimes y_n$ are decomposable tensors, then $[x, y] = \sum_{i=1}^{l} (x_i, y_i)$ where (x_i, y_i) is the inner product in the space H_i On all the other elements of the space H the inner product is defined on linearity and continuity In this metric of the space H operators Γ_i^r are not self adjoint, but all operators Δ_i^r $(i = 0,1,...,n; r = 1,2,...,N)$ are self-adjoint and the formulas (11) take place.

Remark All λ_i^r; $i = 1,2,...,n$; $r = 1,2,...,N$ are real numbers.

Theorem4. Let the conditions of the theorem3 is fulfilled.. Operators Δ_0^s

$s = 1,2,...,N)$ have inverse, spectrum of the bundle (6) contains only eigen values, then the multiparameter system (4) have a common real eigenvalue if and only if

$$\bigcap_{i=1}^{n}\bigcap_{r=1}^{N} Ker\{\Delta_i^1\Delta_0^r - \Delta_0^1\Delta_i^r\} \neq \{\vartheta\}$$

Proof of the Theorem4.

We have n systems of operator equations

$$\begin{matrix} \Gamma_1^1 x_1 = \lambda x_1 & & \Delta_1^1 x_1 = \lambda\Delta_0^1 x_1 \\ \Gamma_1^2 x_2 = \lambda x_2 & , \text{ or} & \Delta_1^2 x_2 = \lambda\Delta_0^2 x_2 \\ \dots\dots\dots & & \dots\dots\dots \\ \Gamma_1^N x_n = \lambda x_N & & \Delta_1^N x_N = \lambda\Delta_0^N x_N \end{matrix} \quad (12)$$

$$\begin{matrix} \Gamma_n^1 x_1 = \lambda x_1 & & \Delta_n^1 x_1 = \lambda\Delta_0^1 x_1 \\ \Gamma_n^2 x_2 = \lambda x_2 & , \text{ or} & \Delta_n^2 x_2 = \lambda\Delta_0^2 x_2 \\ \dots\dots\dots & & \dots\dots\dots \\ \Gamma_1^N x_{nn} = \lambda x_N & & \Delta_n^N x_N = \lambda\Delta_0^N x_N \end{matrix}$$

For any multiparameter system from (5) the separation of parameters holds.

$$\begin{matrix} \Gamma_1^1 x_1 = \lambda x_1, \dots, \Gamma_n^1 x_1 = \lambda x_1 \\ \Gamma_1^2 x_2 = \lambda x_2, \dots, \Gamma_n^2 x_2 = \lambda x_2 \\ \Gamma_1^N x_N = \lambda x_N, \dots, \Gamma_n^N x_N = \lambda x_N \end{matrix} \quad (13)$$

Each system in (12) contains N operator equations in one parameter. All operator in (12) and (13) act in the space $H = H_1 \otimes ... \otimes H_l$.

The common eigenvalue of the system (12) when $r = 1$ is the first component of the eigen value $(\lambda_1, ..., \lambda_n)$ of the system (4).

The proof at the fact is analogous to the proof of the theorem 2. Construct the resultants of operator equations $\Delta_1^1 x_1 = \lambda\Delta_0^r x_1$ and $\Delta_1^s x_2 = \lambda\Delta_0^2 x_2$, $s = 2,...,N$. We have $N-1$ operator equations with one parameter. From the results of the theorem 2 we have that $\bigcap_{s=2}^{N} Ker(\Delta_1^1\Delta_0^s - \Delta_0^1\Delta_1^s) \neq \{\theta\}$, $(Ker\Delta_0^s = \{\theta\})$. By analogy for other systems in (12) at $i = 2,3,...,n$ we have the conditions $\bigcap_{s=2}^{N} Ker(\Delta_i^1\Delta_0^s - \Delta_0^1\Delta_i^s) \neq \{\theta\}$, $(Ker\Delta_0^s = \{\theta\}, s = 1,2,...,N)$.

Are the necessary and sufficient for the existence of the common eigenvalue of each operator groups.

Condition $\bigcap_{i=1}^{n}\bigcap_{r=1}^{N} Ker\{\Delta_i^1\Delta_0^r - \Delta_0^1\Delta_i^r\} \neq \{\vartheta\}$ allows to collect the separate common components of the eigen values of the systems and to obtain the common eigenvalue of the multiparameter system (4).

Remark. If $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ is an eigenvalue of the system (4), then all λ_i $i = 1,2,...,n$ are real numbers.

5. Conclusion

In this work the necessary and sufficient conditions of existence of eigenvalue of the multiparameter system, when the number of operator equations is more than the number of parameters in it.

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