

# Multiparameter Operator Systems with Three Parameters

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**Abstract:** For the multiparameter system of operators in three parameters the conditions of the existence of multiple basis of eigen and associated vectors in finite dimensional space is proved. The proof of this fact uses essentially the notion of the Resultant of two operator pencils, acting in, generally speaking, in different Hilbert spaces and the criterion of existence of common eigenvalues of several operator pencils, acting in Hilbert spaces.

**Keywords:** Eigen and Associated Vectors, Finite Dimensional Space, Multiparameter System of Operators, Nonlinear Algebraic System of Equations, Resultant-Operator of Two Pencils

## 1. Introduction

The founder of researches of spectral problems of the multiparameter selfadjoint systems was F.V. Atkinson [1]. Studied the outcomes which are available for multiparameter symmetrical differential systems, Atkinson has constructed the spectral theory of multiparameter systems in finite dimensional spaces. Further, by means of passage to the limit Atkinson has generalized the received outcomes on a case of multiparameter systems with the selfadjoint compact operators in infinite-dimensional Hilbert spaces. In the further, a design introduced by Atkinson for study of multiparameter systems in finite-dimensional spaces, it has appeared possible to build and in infinite-dimensional spaces that has allowed to construct the spectral theory of multiparameter systems in Hilbert spaces [2],[3], etc.

But, unfortunately, the technique of research in these works demands all operators in the system to be selfadjoint.

For non- selfadjoint multiparameter systems the investigated technique does not allow to solve the simplest problems of the spectral theory.

The paper is devoted to the study of nonlinear multiparameter system of operators in finite dimensional Hilbert spaces. Previously, for nonlinear algebraic of system of equations was built analog of determinant of Cramer and it is given a necessary and sufficient condition for the existence of solutions of nonlinear algebraic systems with a complex dependence on the variables [4]. For two-parameter systems in the abstract case, it is obtained the results to determine the conditions for finding the number of solutions [6].

In this paper describes the method of determining the conditions on the coefficients of the multiparameter system to establish the existence of its eigenvalues.

Consider a multiparameter system of operators in three parameter of the form.

$$\begin{aligned} A(\lambda, \mu, \xi)x &= (A_0 + \lambda A_1 + \dots + \lambda^{m_1} A_{m_1} + \mu A_{m_1+1} + \dots + \mu^{n_1} A_{m_1+n_1} + \xi A_{m_1+n_1+1})x = 0 \\ B(\lambda, \mu, \xi)y &= (B_0 + \lambda B_1 + \dots + \lambda^{m_2} B_{m_2} + \mu B_{m_2+1} + \dots + \mu^{n_2} B_{m_2+n_2} + \xi B_{m_2+n_2+1} + \dots + \xi^{r_2} B_{m_2+n_2+r_2})y = 0 \\ C(\lambda, \mu, \xi)z &= (C_0 + \lambda C_1 + \dots + \lambda^{m_3} C_{m_3} + \mu C_{m_3+1} + \dots + \mu^{n_3} C_{m_3+n_3} + \xi C_{m_3+n_3+1} + \dots + \xi^{r_3} C_{m_3+n_3+r_3})z = 0 \end{aligned} \quad (1)$$

when operators  $A_i, i=1, 2, \dots, m_1+n_1+1$  (correspondingly,  $B_i, i=1, 2, \dots, m_2+n_2+r_2$ , and  $C_i, i=1, 2, \dots, m_3+n_3+r_3$ ) act in Hilbert space  $H_i$  (correspondingly,  $H_2$  and  $H_3$ ).

## 2. Preliminary definitions and Remarks.

Let's reduce a series of known positions from the spectral

theory of multiparameter systems

*Definition 1.* [1,2,3]. An operator  $A_k^+ = A_k \otimes E_2 \otimes E_3$  (accordingly,  $B_k^+ = E_1 \otimes B_k \otimes E_3, C_k^+ = E_1 \otimes E_2 \otimes C_k$ ), where  $E_1$  (accordingly,  $E_2$  and  $E_3$ ) is identical operators in  $H_1$  (accordingly,  $H_2$  and  $H_3$ ), is called the operator, induced in  $H = H_1 \otimes H_2 \otimes H_3$  by the operators  $A_k$  (accordingly,  $B_k$  and  $C_k$ ).

*Definition 2.* [1], [2], [3]  $(\lambda, \mu, \xi) \in C^3$  is an eigen value of the system(1) if there are such nonzero elements  $x \in H_1, y \in H_2, z \in H_3$  that equations (1) and also the equations

$$A(\lambda, \mu, \xi)^+ x \otimes y \otimes z = 0$$

$$B(\lambda, \mu, \xi)^+ x \otimes y \otimes z = 0$$

$$C(\lambda, \mu, \xi)^+ x \otimes y \otimes z = 0$$

are satisfied, then the decomposable tensor  $x \otimes y \otimes z$  is called the eigenvector of the multiparameter system (1).

*Definition 3.* [9],[10] A tensor  $z_{m_1, m_2, m_3}$  is called

the  $(m_1, m_2, m_3)$ -associated vector to an eigenvector  $z_{0,0,0} = x \otimes y \otimes z$  if there is the set of elements  $(z_{i,j,k})$  such that the following conditions (2) are satisfied

$$\text{Res}(A(\lambda), B(\lambda)) = \begin{pmatrix} A_0 \otimes E_2 & A_1 \otimes E_2 & \dots & A_n \otimes E_2 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot & \dots & \cdot \\ 0 & 0 & \dots A_0 \otimes E_2 & A_1 \otimes E_2 & \dots & A_n \otimes E_2 \\ E_1 \otimes B_0 & E_1 \otimes B_1 & \dots & E_1 \otimes B_m & \dots & 0 \\ \cdot & \cdot & \dots & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots E_1 \otimes B_0 & E_1 \otimes B_1 & \dots & E_1 \otimes B_m \end{pmatrix} \quad (4)$$

In a matrix  $\text{Res}(A(\lambda), B(\lambda))$  the number of rows with operators  $A_i$  is equal to leading degree of parameter  $\lambda$  in the operator pencil  $B(\lambda)$ , that is  $m$ ; the number of rows in matrix  $\text{Res}(A(\lambda), B(\lambda))$  with operators  $B_i$  coincides with the leading degree of parameter  $\lambda$  in the operator pencil  $A(\lambda)$ , that is  $n$ . One major application of the matrix theory is calculation of determinants. It turns out that a mapping is invertible if and only if the determinant of this matrix is not zero.

This definition is the generalization of the notion of the definition of resultant for two polynomials

$$f(x), g(x); f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0, \quad a_n \neq 0;$$

$$g(x) = b_m x^m + b_{m-1} x^{m-1} + \dots + b_0, \quad b_m \neq 0;$$

Resultant of these polynomials is the operator acting in the space  $R^{n+m}$  or  $C^{n+m}$  (probably, in some expansion of a field).

$$\begin{aligned} \sum_{0 \leq r_1 \leq k_1} \frac{1}{r_1! r_2! r_3!} \frac{\partial^{r_1+r_2+r_3}}{\partial \lambda^{r_1} \partial \mu^{r_2} \partial \xi^{r_3}} A^+(\lambda, \mu, \xi) z_{k_1-r_1, k_2-r_2, k_3-r_3} &= 0 \\ \sum_{0 \leq r_1 \leq k_1} \frac{1}{r_1! r_2! r_3!} \frac{\partial^{r_1+r_2+r_3}}{\partial \lambda^{r_1} \partial \mu^{r_2} \partial \xi^{r_3}} B^+(\lambda, \mu, \xi) z_{k_1-r_1, k_2-r_2, k_3-r_3} &= 0 \\ \sum_{0 \leq r_1 \leq k_1} \frac{1}{r_1! r_2! r_3!} \frac{\partial^{r_1+r_2+r_3}}{\partial \lambda^{r_1} \partial \mu^{r_2} \partial \xi^{r_3}} C^+(\lambda, \mu, \xi) z_{k_1-r_1, k_2-r_2, k_3-r_3} &= 0 \end{aligned} \quad (2)$$

$k_s \leq m_s; i = 1, 2, 3; s = 1, 2, 3.$

$(k_1, k_2, k_3)$  is arrangement from set of the whole nonnegative numbers on 3 with possible recurring and zero.

*Remark 1.* We gave here the definition of the associated vector for the case when the number of parameters in the system is equal to 3 (in [10] the definition is given for the common case)

*Definition 4.* [11],[13]

Let

$$\begin{aligned} A(\lambda) &= A_0 + \lambda A_1 + \lambda^2 A_2 + \dots + \lambda^n A_n, \\ B(\lambda) &= B_0 + \lambda B_1 + \lambda^2 B_2 + \dots + \lambda^m B_m \end{aligned} \quad (3)$$

be two operator pencils depending on the same parameter  $\lambda$  and acting, generally speaking, in different Hilbert spaces  $H_1, H_2$ , correspondingly. Resultant of two operator pencils  $A(\lambda)$  and  $B(\lambda)$  is the operator, presented by the determinant (4) and acting in the space  $(H_1 \otimes H_2)^{n+m}$  - direct sum of  $n + m$  copies of tensor product  $H_1 \otimes H_2$  of spaces

$$\text{Res}(f, g) = \begin{pmatrix} a_n & a_{n-1} & \dots & a_1 & a_0 & 0 & \dots & \dots & 0 & 0 \\ 0 & a_n & \dots & a_2 & a_1 & a_0 & \dots & \dots & 0 & 0 \\ \cdot & \cdot & \dots & \cdot & \cdot & \cdot & \dots & \dots & \cdot & \cdot \\ \cdot & \cdot & \dots & \cdot & \cdot & \cdot & \dots & \dots & \cdot & \cdot \\ 0 & 0 & \dots & 0 & 0 & a_n & \dots & \dots & a_1 & a_0 \\ b_m & b_{m-1} & \dots & b_3 & b_2 & b_1 & b_0 & \dots & 0 & 0 \\ 0 & b_m & \dots & b_4 & b_3 & b_2 & b_1 & b_0 & \dots & 0 & 0 \\ \cdot & \cdot & \dots & \cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ \cdot & \cdot & \dots & \cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & b_m & b_{m-1} & \dots & b_1 & b_0 \end{pmatrix}$$

In a matrix  $\text{Res}(f, g)$  the number of rows with coefficients  $a_i$  to equally leading degree of unknown  $x$  of a polynomial  $g(x)$ , that is  $m$ , the number of rows of a matrices with numbers  $b_i$  coincides with the leading degree

of unknown  $x$  of a polynomial  $f(x)$ , that is  $n$ . Continuant of this Resultant is a polynomial from coefficients and equal to zero then and only then when polynomials also have a common roots (probably, in some expansion of a field).

The study of multiparameter system (1) of equations is spent with help of following result from [5]:

### 3. On common Eigenvalues of Several Operator Bundles

Let the  $n$  bundles depending on the same parameter  $\lambda$

$$\{B_i(\lambda) = B_{0,i} + \lambda B_{1,i} + \dots + \lambda^{k_i} B_{k_i,i}, \quad i = 1, 2, \dots, n\}$$

$B_i(\lambda)$  - operator bundles acting in Hilbert space  $H_i$  correspondingly. Suppose that  $k_1 \geq k_2 \geq \dots \geq k_n$ . In the space  $H^{k_1+k_2}$  (the direct sum of  $k_1+k_2$  tensor product  $H = H_1 \otimes \dots \otimes H_n$  of spaces  $H_1, H_2, \dots, H_n$ ) are introduced the operators  $R_i$  ( $i = 1, \dots, n-1$ ) with the help of operational matrices (4).

Let  $B_i(\lambda)$  be the operational bundles acting in a finite dimensional Hilbert space  $H_i$ , correspondingly.

$$\begin{aligned} A(\lambda, \mu)x &= (A_0 + \lambda A_1 + \dots + \lambda^{m_1} A_{m_1} + \mu A_{m_1+1} + \dots + \mu^{n_1} A_{m_1+n_1})x = 0 \\ B(\lambda, \mu)y &= (B_0 + \lambda B_1 + \dots + \lambda^{m_2} B_{m_2} + \mu B_{m_2+1} + \dots + \mu^{n_2} B_{m_2+n_2})y = 0 \end{aligned} \quad (5)$$

in tensor product  $H_1 \otimes H_2$  of finite-dimensional spaces  $H_1$  and  $H_2$  is studied.

Dimension of space  $H$  is the product of dimensions of spaces  $H_1$  and  $H_2$ . In (5) linear operators  $A_i$  ( $i = 0, 1, \dots, m_1 + n_1$ ) act in finite-dimensional space  $H_1$ ; and linear operators  $B_i$  ( $i = 0, 1, \dots, m_2 + n_2$ ) act in finite-dimensional space  $H_2$ .

If  $f_1 \otimes f_2 \in H_1 \otimes H_2$  and  $g_1 \otimes g_2 \in H_1 \otimes H_2$  then inner product of these elements in space  $H_1 \otimes H_2$  is defined by means of the formulae

$$[f_1 \otimes f_2, g_1 \otimes g_2]_{H_1 \otimes H_2} = (f_1, g_1)_{H_1} \cdot (f_2, g_2)_{H_2} \quad (6)$$

This definition of inner product is spread to other elements of tensor product space  $H_1 \otimes H_2$  on linearity.

If  $H_1 \otimes H_2$  is the Hilbert space then the inner product (6) is spread to other elements of  $H_1 \otimes H_2$  on linearity and continuity.

By means of the approach stated in [5,6], we can establish completeness, multiple completeness of system of eigen and associated vectors, a possibility of multiple decompositions on system of eigen and associated vectors of multiparameter system (5).

**Theorem 2.** [6] Let operators  $A_i$  ( $i = 0, 1, \dots, m_1 + n_1$ ) also  $B_i$  ( $i = 0, 1, \dots, m_2 + n_2$ ) act in finite-dimensional spaces  $H_1$

$$R_{i-1} = \begin{pmatrix} B_{0,1}^+ & B_{1,1}^+ & \dots & B_{k_1,1}^+ & \dots & 0 \\ 0 & B_{0,1}^+ & B_{1,1}^+ \dots & B_{k_1-1,1}^+ & B_{k_1,1}^+ \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots B_{0,1}^+ & B_{1,1}^+ & \dots & B_{k_1,1}^+ \\ B_{0,i}^+ & B_{1,i}^+ & \dots & B_{k_i,i}^+ & 0 \dots & 0 \\ 0 & B_{0,i}^+ & B_{1,i}^+ \dots & \cdot & B_{k_i,i}^+ \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots B_{0,i}^+ & B_{1,i}^+ & \dots & B_{k_i,i}^+ \end{pmatrix} \quad (4)$$

$i = 2, 3, \dots, n$

The number of rows with operators  $B_{s,1}$ ,  $s = 0, 1, \dots, k_1$  in the matrix  $R_{i-1}$  is equal to  $k_2$  and the number of rows with operators  $B_{s,i}$ ,  $s = 0, 1, \dots, k_i$  is equal to  $k_1$ . We designate  $\sigma_p(B_i(\lambda))$  the set of eigenvalues of an operator  $B_i(\lambda)$ . From [5] we have the result:

**Theorem 1.**  $\bigcap_{i=1}^n \sigma_p(B_i(\lambda)) \neq \{\theta\}$  if and only if  $\bigcap_{i=1}^{n-1} \text{Ker} R_i \neq \{\theta\}$ , ( $\text{Ker} B_{k_1} = \{\theta\}$ ).

In particular, in [6] the two-parameter system operators

and  $H_2$ , accordingly, and one of three following conditions is fulfilled:

- $\max(m_1 n_2, m_2 n_1) = m_1 n_2$ ,  $\text{Ker} A_{m_1} = \{\theta\}$ ,  $\text{Ker} B_{m_2+n_2} = \{\theta\}$ ;  $A_{m_1}, B_{m_2+n_2}$  are self-adjoint operators everyone in their space
- $\max(m_1 n_2, m_2 n_1) = m_2 n_1$ ,  $\text{Ker} B_{m_2} = \{\theta\}$ ,  $\text{Ker} A_{m_1+n_1} = \{\theta\}$   $A_{m_1+n_1}, B_{m_1}$  - self-adjoint operators, acting in the corresponding spaces
- $m_1 n_2 = m_2 n_1$ ,  $\text{Ker}(A_{m_1}^{n_2} \otimes B_{m_1+n_1}^{n_1} + (-1)^{n_1 n_2} A_{m_1+n_1}^{n_2} \otimes B_{m_2}^{n_1})$ ,  $A_{m_1}, B_{m_2}, A_{m_1+n_1}, B_{m_2+n_2}$  are the self-adjoint operators, acting everyone in finite-dimensional space, correspondingly.

Then the  $\max(m_1 n_2, m_2 n_1)$ -fold basis of system of eigen and associated vectors of (5) takes place.

At the study of the system (1) we will be used essentially the results of [5,6]:

### 4. Three Parameter System of Operators in Finite Dimensional Hilbert Spaces

In (1) we fix parameters  $\lambda, \mu$ . Let be  $\lambda = \lambda_0, \mu = \mu_0$ . Then the system (1) contains three operator bundles depending on one parameter  $\xi$ . Using the result of the theorem 2, we build operators  $R_1$  и  $R_2$ . They are the Resultants of operator bundles  $A(\lambda_0, \mu_0, \xi)$  and  $B(\lambda_0, \mu_0, \xi)$ ,

and also operator bundles  $A(\lambda_0, \mu_0, \xi)$  and  $C(\lambda_0, \mu_0, \xi)$ , Introduce the notations:  
correspondingly.

$$\begin{aligned}\tilde{A}(\lambda_0, \mu_0) &= A_0 + \lambda_0 A_1 + \dots + \lambda_0^{m_1} A_{m_1} + \mu_0 A_{m_1+1} + \dots + \mu_0^{n_1} A_{m_1+n_1} \\ \tilde{B}(\lambda_0, \mu_0) &= B_0 + \lambda_0 B_1 + \dots + \lambda_0^{m_2} B_{m_2} + \mu_0 B_{m_2+1} + \dots + \mu_0^{n_2} B_{m_2+n_2} \\ C(\lambda_0, \mu_0) &= C_0 + \lambda_0 C_1 + \dots + \lambda_0^{m_3} C_{m_3} + \mu_0 C_{m_3+1} + \dots + \mu_0^{n_3} C_{m_3+n_3}\end{aligned}\quad (6)$$

Let  $r_2 \geq r_3$ . For the polynomials  $A(\lambda_0, \mu_0, \xi)$  and  $B(\lambda_0, \mu_0, \xi)$  Resultant  $R_1$  has a form

$$R_1 = R\{A(\lambda_0, \mu_0, \xi), C(\lambda_0, \mu_0, \xi)\} = \begin{pmatrix} \tilde{A}^+(\lambda_0, \mu_0) & A_{m_1+n_1+1}^+ & \cdot & \dots & \cdot \\ 0 & \tilde{A}^+(\lambda_0, \mu_0) & A_{m_1+n_1+1}^+ & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ B^+(\lambda_0, \mu_0) & B_{m_3+n_3+1}^+ & B_{m_3+n_3+2}^+ & \dots & B_{m_3+n_3+r_3}^+ \end{pmatrix} \quad (7)$$

In the matrices of the operator  $R_1$  the number of rows with  $\tilde{A}^+(\lambda_0, \mu_0), A_{m_1+n_1+1}^+$  equal to  $r_2$  and the number of rows with  $B^+(\lambda_0, \mu_0), B_{m_3+n_3+1}^+, \dots, B_{m_3+n_3+r_3}^+$  is equal to 1.

By analogy operator  $R_2$  in the space  $(H_1 \otimes H_2 \otimes H_3)^{r_2+1}$  is determined with the help of matrices

$$R_1 = R\{A(\lambda_0, \mu_0, \xi), C(\lambda_0, \mu_0, \xi)\} = \begin{pmatrix} \tilde{A}^+(\lambda_0, \mu_0) & A_{m_1+n_1+1}^+ & \cdot & \dots & \cdot \\ 0 & \tilde{A}^+(\lambda_0, \mu_0) & A_{m_1+n_1+1}^+ & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ C^+(\lambda_0, \mu_0) & C_{m_3+n_3+1}^+ & C_{m_3+n_3+2}^+ & \dots & C_{m_3+n_3+r_3}^+ \end{pmatrix} \quad (8)$$

In the matrix of the operator  $R_2$  the number of rows with  $\tilde{A}^+(\lambda_0, \mu_0), A_{m_1+n_1+1}^+$  is equal to  $r_2$  and the number of rows with  $C^+(\lambda_0, \mu_0), C_{m_3+n_3+1}^+, \dots, C_{m_3+n_3+r_3}^+$  is equal to 1

Operators  $R_1$  and  $R_2$  act in the same space  $(H_1 \otimes H_2 \otimes H_3)^{r_2+1}$  - direct sum of the  $r_2+1$  copies of tensor product space  $H_1 \otimes H_2 \otimes H_3$ .

Continuants of Resultants  $R_1$  and  $R_2$  are equal to zero if and only if matrices of operators  $R_1$  and  $R_2$  have nonzero

kernels.

So at the decompositions of continuants of Resultants  $R_1$  and  $R_2$  we obtain very bulky forms, in obtained equations we will operate with the leading degrees of parameters in  $R_1$  and  $R_2$ . Further we will work only with the members having the leading degrees. We can write the decompositions of continuants of Resultants  $R_1$  and  $R_2$  in the forms, correspondingly:

$$\dots + \lambda_0^{m_1 r_2} (A_{m_1}^{r_2})^+ B_{m_2+n_2+r_2}^+ + \dots + \mu_0^{n_1 r_2} (A_{m_1+n_1}^{r_2})^+ B_{m_2+n_2+r_2}^+ - \dots - \lambda_0^{m_2} (A_{m_1+n_1+1}^{r_2})^+ B_{m_2}^+ - \dots - \mu_0^{r_2} (A_{m_1+n_1+1}^{r_2})^+ B_{m_2+n_2}^+ \quad (9)$$

$$\dots + \lambda_0^{m_1 r_3} (A_{m_1}^{r_2})^+ C_{m_3+n_3+r_3}^+ + \dots + \mu_0^{n_1 r_2} (A_{m_1+n_1}^{r_2})^+ C_{m_3+n_3+r_3}^+ - \dots - \lambda_0^{m_2} (A_{m_1+n_1+1}^{r_2})^+ C_{m_3}^+ - \dots - \mu_0^{r_2} (A_{m_1+n_1+1}^{r_2})^+ C_{m_3+n_3}^+ \quad (10)$$

Expression (9) is the decomposition of the continuant of the Resultant of the operator bundles  $A(\lambda_0, \mu_0, \xi)$  and  $B(\lambda_0, \mu_0, \xi)$  from (1), when  $\lambda = \lambda_0$  and  $\mu = \mu_0$ . Expression (10) is the decomposition of the continuant of the Resultant of operator bundles  $A(\lambda_0, \mu_0, \xi)$  and  $C(\lambda_0, \mu_0, \xi)$ . Let the couple  $(\lambda_1, \mu_1)$  is the eigen value of the (9), then from the definition of Resultant follows that at these meanings of

parameters  $\lambda = \lambda_1$  and  $\mu = \mu_1$  we have that operator pencils  $A(\lambda_1, \mu_1, \xi)$  and  $B(\lambda_1, \mu_1, \xi)$  have a common point of spectra  $\xi_1(\lambda_1, \mu_1)$ . By analogy if the couple  $(\lambda_2, \mu_2)$  is the eigen value of the (10), then from the definition of Resultant follows that at these meanings of parameters  $\lambda = \lambda_2$  and  $\mu = \mu_2$  we have that operator pencils  $A(\lambda_2, \mu_2, \xi)$  and

$C(\lambda_2, \mu_2, \xi)$  have a common point  $\xi_2(\lambda_2, \mu_2)$  of spectra of these operator bundles.

Now we consider the system of equations

$$\begin{aligned} & (\dots + \lambda_0^{m_1 r_2} (A_{m_1}^{r_2})^+ B_{m_2+n_2+r_2}^+ + \dots + \mu_0^{n_1 r_2} (A_{m_1+n_1}^{r_2})^+ B_{m_2+n_2+r_2}^+ - \dots - \lambda_0^{m_2} (A_{m_1+n_1+1}^{r_2})^+ B_{m_2}^+ - \dots - \mu_0^{r_2} (A_{m_1+n_1+1}^{r_2})^+ B_{m_2+n_2}^+) \tilde{u} = 0 \\ & (\dots + \lambda_0^{m_1 r_3} (A_{m_1}^{r_3})^+ C_{m_3+n_3+r_3}^+ + \dots + \mu_0^{n_1 r_3} (A_{m_1+n_1}^{r_3})^+ C_{m_3+n_3+r_3}^+ - \dots - \lambda_0^{m_3} (A_{m_1+n_1+1}^{r_3})^+ C_{m_3}^+ - \dots - \mu_0^{r_3} (A_{m_1+n_1+1}^{r_3})^+ C_{m_3+n_3}^+) \tilde{u} = 0 \end{aligned} \quad (11)$$

If the  $(\lambda, \mu)$  is the eigenvalue of the system (11) and  $\tilde{u}$  is the corresponding eigen vector of (11) then  $\text{Ker} R_1 \cap R_2 \neq \{\emptyset\}$ . So the [8], [9] are decompositions of Resultants  $R_1$  and  $R_2$ , correspondingly, and (11) is satisfied then the element  $\tilde{u}$  is the first component of the element of  $\text{Ker} R_1 \cap R_2$  and also the eigen vector of the system (1) or its associated vector in definition of which no the differential on  $\lambda$  and  $\mu$ . All the associated vectors of the system (11) are also the associated vectors of the system (1).

Operator pencils (9) and (10) form the nonlinear two parameter system of the kind of the system (5).

We apply the results of the theorem 2 to the system (9), (10).

**Theorem 3.** Let operators  $A_i (i = 0, 1, \dots, m_1 + n_1 + 1)$ ,  $B_i (i = 0, 1, \dots, m_2 + n_2 + r_2)$  and also  $C_i (i = 0, 1, \dots, m_3 + n_3 + r_3)$  act in finite-dimensional spaces  $H_1$ ,  $H_2$  and  $H_3$ , accordingly, and one of three following conditions is fulfilled:

a)  $\max(m_1 r_2 r_3, m_2 r_3, m_3 n_1 r_2) = m_1 r_2 r_3$ ,  
 $\text{Ker} A_{m_1} = \{\emptyset\}$ ,  $\text{Ker} A_{m_1+n} = \{\emptyset\}$ ,  $\text{Ker} B_{m_2+n_2+r_2} = \{\emptyset\}$ ,  $\text{Ker} C_{m_3+n_3} = \{\emptyset\}$  ;  
 $A_{m_1}^{r_2} A_{m_1+n_1+1}^{r_2} \otimes B_{m_2+n_2} \otimes C_{m_3+n_3}$  are self-adjoint operator in the space  $H_1 \otimes H_2 \otimes H_3$

b)  $\max(m_1 r_2 r_3, m_2 r_3, m_3 n_1 r_2) = m_2 r_3$ ,  
 $\text{Ker} B_{m_2} = \{\emptyset\}$ ,  $\text{Ker} A_{m_1+n_1+1} = \{\emptyset\}$ ,  $\text{Ker} C_{m_3+n_3} = \{\emptyset\}$

$A_{m_1+n_1+1}^{2r_2} \otimes B_{m_2} \otimes C_{m_3+n_3}$ , are the self-adjoint operator, acting in finite-dimensional space  $H_1 \otimes H_2 \otimes H_3$ , correspondingly.

c)  $\max(m_1 r_2 r_3, m_2 r_3, m_3 n_1 r_2) = m_3 n_1 r_2$   
 $\text{Ker} B_{m_2+n_2+r_2} = \{\emptyset\}$ ,  $\text{Ker} A_{m_1+n_1+1} = \{\emptyset\}$ ,  $\text{Ker} C_{m_3} = \{\emptyset\}$ ,  $\text{Ker} A_{m_1+n_1} = \{\emptyset\}$   
 $A_{m_1+n_1}^{r_2} A_{m_1+n_1-1}^{r_2} \otimes B_{m_2+n_2+r_2} \otimes C_{m_3}$  is selfadjoint operator in the space  $H_1 \otimes H_2 \otimes H_3$ , Then the eigen and associated vectors of the three parameter system of operators in finite dimensional space form the multiple basis in  $H_1 \otimes H_2 \otimes H_3$ .

Proof of the theorem3 uses the approach applying of the proof of the theorem2. We fix one of the parameters of the system (9),(10). For the definite let it is the parameter  $\lambda$ . Then we have two operator bundles depending on the parameter  $\mu$ . Construct the resultant of (9) and (10), when parameter  $\lambda$  is fixed. It is clear that the decomposition of this resultant is the operator bundle. In fact, the leading degree of the parameter  $\lambda$  depends on values of numbers  $m_1, m_2, m_3, n_1, n_2, n_3, r_1, r_2, r_3$ . Conditions a), b) and c) are only some variants of possibility. If in each of the cases a), b) and

c) operator coefficient at the parameter  $\lambda$  with the greatest degree  $\lambda$  is selfadjoint operator with the zero kernel then the system of eigen and associated vectors of this bundle forms the multiple basis in the tensor product space. The multiple of the system of eigen and associated vectors coincides with the leading degree of the parameter  $\lambda$ . In works [8] and [9] it is proved that the system of eigen and associated vectors of decomposition of resultant coincides with the system of eigen and associated vectors of two parameter system (9),(10).

So the (9) and (10) are also decompositions of corresponding of resultants, we obtain that the system of eigen and associated vectors of (9) and (10) coincides with the system of eigen and associated vectors of the three parameter system (1).

## 5. Conclusion

In this paper the conditions of existence of multiple basis on eigen and associated vectors of three parameter operator system in finite dimensional spaces are proved.

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