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Tychonoff's theorem as a direct application of Zorn's lemma

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Abstract: A simple proof of Tychonoff's theorem (the compactness of the product of compact spaces) as a direct application of Zorn's lemma is given. In contrast to the classical Cartan-Bourbaki proof which uses Zorn's lemma twice, our proof uses it only once.

Keywords: Tychonoff's Theorem, Compact Space, Product Space, Zorn's Lemma, Axiom of Choice, Ultrafilter Theorem

1. Introduction

Tychonoff's theorem [5] states that the product of an arbitrary family of compact topological spaces, with the product topology, is compact. In a standard Cartan-Bourbaki proof (cf. [1],[3],[6]) of this theorem, the ultrafilter theorem is used to obtain an ultrafilter (i.e. a maximal filter) G including a filter F on the product space, and the axiom of choice is used in selecting an element of the product space which G converges to. If we start from the axiom of choice, the flow of the argument is illustrated as follows.

$$\begin{array}{c}\text{Axiom of choice}\\ \Downarrow\\ \text{Zorn's lemma + Axiom of choice}\\ \Downarrow\\ \text{Ultrafilter theorem + Axiom of choice}\\ \Downarrow\\ \text{Tychonoff's theorem}\end{array}$$

This proof uses the axiom of choice twice. Alternatively, if we start from Zorn's lemma, the flow of the argument is slightly simpler.

$$\begin{array}{c}\text{Zorn's lemma}\\ \Downarrow\\ \text{Ultrafilter theorem + Axiom of choice}\\ \Downarrow\\ \text{Tychonoff's theorem}\end{array}$$

This proof also seems to be a roundabout, since Zorn's lemma is applied twice. Therefore, it is desirable to find a direct proof which uses Zorn's lemma only once.

$$\begin{array}{c}\text{Zorn's lemma}\\ \Downarrow\\ \text{Tychonoff's theorem}\end{array}$$

In this paper we provide a shortcut proof of Tychonoff's theorem as a direct application of Zorn's lemma without appealing to the axiom of choice or the ultrafilter theorem.

2. Preliminaries

In this section, we state basic definitions and lemmas about filters on sets and compact topological spaces required in our proof.

Zorn's Lemma. *Let P be a non-empty partially ordered set. Assume that for any non-empty chain (i.e. totally ordered subset) $C \subset P$, there exists an upper bound of C in P. Then P has a maximal element.*

Definition 2.1. *A set F consisting of subsets of a set X is called a filter on X, if the following conditions are satisfied:*

(1) $\phi \notin F$, $X \in F$.

(2) *For any* $U, V \subset X$, *if* $U \in F$ *and* $U \subset V$, *then* $V \in F$.

(3) *For any* $U, V \in F$, *we have* $U \cap V \in F$.

A filter F on a set X is called an ultrafilter or a maximal filter, if F is maximal with respect to the inclusion, that is, if any filter including F must be equal to F.

Note that with our definition, the power set $\mathrm{P}(X)$ is not a filter on X. Some authors call a filter with our definition a proper filter, and $\mathrm{P}(X)$ a nonproper filter.

Definition 2.2. *Let X, Y be two sets, and let $f: X \longrightarrow Y$ be a map.*

(1) *For a filter F on X, we define a filter $f(F)$ on Y by*

$$f(F) = \{V \subset Y | f^{-1}(V) \in F\}$$

This is the smallest filter on Y including the set of the images $\{f(U) | U \in F\}$. Namely, we have

$$f(F) = \{V \subset Y | \exists U \in F\ (f(U) \subset V)\}.$$

(2) *For a filter G on Y such that $f^{-1}(V) \neq \phi$ for all $V \in G$, we define a filter $f^{-1}(G)$ on X by*

$$f^{-1}(G) = \{U \subset X | \exists V \in G\ (f^{-1}(V) \subset U)\}.$$

This is the smallest filter on X including the set of the inverse images $\{f^{-1}(V) | V \in G\}$.

Lemma 2.3. *Let X, Y be two sets, and let $f: X \longrightarrow Y$ be a map. For a filter F on X and a filter G on Y such that $f^{-1}(V) \neq \phi$ for all $V \in G$, we have a following equivalence:*

$$f^{-1}(G) \subset F \Leftrightarrow G \subset f(F).$$

Proof: The equivalence follows since we have:

$$f^{-1}(G) \subset F$$

$$\Leftrightarrow \forall U \subset X (\exists V \in G (f^{-1}(V) \subset U) \Rightarrow U \in F)$$

$$\Leftrightarrow \forall V \in G (f^{-1}(V) \in F)$$

$$\Leftrightarrow G \subset f(F)$$

This is essentially the Galois connection between the power sets $\mathrm{P}(\mathrm{P}(X))$ and $\mathrm{P}(\mathrm{P}(Y))$ induced by the inverse image map $f^{-1}: \mathrm{P}(Y) \longrightarrow \mathrm{P}(X)$.

Lemma 2.4. *Let X, Y be two sets, and let $f: X \longrightarrow Y$ be a map. Let F and G be filters on X and Y, respectively. If $f(F) \subset G$, then there exists a filter H on X such that $F \subset H$ and $G = f(H)$.*

Proof: Note that for any $U \in F$ we have $f(U) \in f(F) \subset G$, and thus for any $V \in G$, we have $f(U) \cap V \neq \phi$. This implies that $U \cap f^{-1}(V)$ is not empty. Let H be the smallest filter on X including $F \cup f^{-1}(G)$, namely, let

$$H = \{W \subset X | \exists U \in F\ \exists V \in G\ (U \cap f^{-1}(V) \subset W)\}.$$

Then we have $F \subset H$ and $f^{-1}(G) \subset H$, which implies $G \subset f(H)$ by Lemma 2.3.

Note that we have

$$f(H) = \{Z \subset Y | \exists U \in F\ \exists V \in G\ (U \cap f^{-1}(V) \subset f^{-1}(Z))\}.$$

For any $U \in F, V \in G$, and $Z \subset Y$,

$$U \cap f^{-1}(V) \subset f^{-1}(Z) \Longrightarrow f(U) \cap V \subset Z \Longrightarrow Z \in G,$$

and thus we get $f(H) \subset G$.

Corollary. *Let X, Y be two sets, and let $f: X \longrightarrow Y$ be a map. For any ultrafilter (i.e. a maximal filter) F on X, $f(F)$ is also an ultrafilter on Y.*

Definition 2.5. *A topological space X is a set equipped with a map $N: X \longrightarrow \mathrm{P}(\mathrm{P}(X))$ which satisfies the following conditions:*

(1) *$N(x)$ is a filter on X for all $x \in X$.*

(2)*For all $x \in X$, x belongs to the intersection $\bigcap N(x)$. That is, $x \in U$ for all $U \in N(x)$.*

(3) *For all $x \in X$ and for all $U \in N(x)$, there exists an element $V \in N(x)$ such that $V \subset U$ and $U \in N(y)$ for each $y \in V$.*

An element of $N(x)$ is called a neighborhood of x.

Note that the empty set is a topological space since the empty map $\phi: \phi \longrightarrow \mathrm{P}(\mathrm{P}(\phi))$ satisfies the above definition.

Definition 2.6. *Let X be a topological space with the neighborhood filter $N(x)$ for $x \in X$. A filter F on X is said to converge to an element x of X, and we write $F \to x$, if $N(x) \subset F$.*

Lemma 2.7. *Let X, Y be two topological spaces, ane let $f: X \longrightarrow Y$ be a map. For an element $x \in X$, the following conditions are equivalent:*

(1) *f is continuous at x, that is, for any filter F on X, if $F \to x$ then $f(F) \to f(x)$.*

(2) *$f(N(x))$ converges to $f(x)$, that is, we have an inclusion $N(f(x)) \subset f(N(x))$.*

(3) *$f^{-1}(N(f(x))) \subset N(x)$.*

Definition 2.8. *A topological space X is called compact if for any filter F on X, there exists a filter G on X and an element x of X such that $F \subset G$ and $G \to x$.*

The above definition of compactness is equivalent to the standard definition using open covers. We adopted it to make this paper self-contained.

Corollary. *Any ultrafilter on a compact space converges.*

Definition 2.9. *For an arbitrary family of sets $\{X_i\}_{i \in I}$, the product set is defined as*

$$\prod_{i \in I} X_i = \left\{ x: I \to \bigcup_{i \in I} X_i \,\middle|\, \forall i \in I\ (x(i) \in X_i) \right\}.$$

An element of the product set is called a choice map of $\{X_i\}_{i \in I}$.

For each index $i \in I$, the projection map pr_i from the product $X = \prod_{i \in I} X_i$ onto X_i is defined by $\mathrm{pr}_i(x) = x(i)$ for $x \in X$.

For example, we have a bijection between $\prod_{i \in \{1,2\}} X_i$ and $X_1 \times X_2$ by assigning a pair $(x(1), x(2))$ to a choice map $x: \{1,2\} \to X_1 \cup X_2$.

Definition 2.10. *For an arbitrary family of topological spaces $\{X_i\}_{i \in I}$, the product topology on the product set $X = \prod_{i \in I} X_i$ is the weakest topology making the projections $\{\mathrm{pr}_i: X \to X_i\}_{i \in I}$ continuous. Namely, the neighborhood filter of $x \in X$ is the smallest filter including the union:*

$$\bigcup_{i \in I} \{ \mathrm{pr}_i{}^{-1}(U) | U \in N(\mathrm{pr}_i(x)) \}$$

or $\bigcup_{i\in I}\mathrm{pr}_i{}^{-1}\left(N\left(\mathrm{pr}_i(x)\right)\right)$.

Lemma 2.11. *For any filter* F *on the product space* $X=\prod_{i\in I}X_i$, *and an element* x *of* X, *the following two conditions are equivalent.*

(1) $F\to x$

(2) $\mathrm{pr}_i(F)\to\mathrm{pr}_i(x)$ *for all* $i\in I$

Proof: The lemma follows from Definition 2.10 and Lemma 2.3, as we have:

$$N(x)\subset F$$

$$\Leftrightarrow \mathrm{pr}_i{}^{-1}\left(N\left(\mathrm{pr}_i(x)\right)\right)\subset F \text{ for all } i\in I$$

$$\Leftrightarrow N\left(\mathrm{pr}_i(x)\right)\subset \mathrm{pr}_i(F) \text{ for all } i\in I.$$

3. The Product of Two Compact Spaces

In this section, we present a proof of the compactness of the product of two compact spaces. Note that Zorn's lemma is not required in the proof.

Proposition 3.1. *For any two compact topological spaces* X_1 *and* X_2, *the product space* $X_1\times X_2\cong\prod_{i\in\{1,2\}}X_i$ *is compact.*

Proof: Let F be a filter on the product $X_1\times X_2$. Since X_1 is a compact space, there exists a filter G_1 on X_1 and an element x_1 of X_1 such that $\mathrm{pr}_1(F)\subset G_1$ and $G_1\to x_1$, by Definition2.8. It follows from Lemma 2.4 there exists a filter H_1 on $X_1\times X_2$ such that $F\subset H_1$ and $G_1=\mathrm{pr}_1(H_1)$. By Definition 2.8 again, as X_2 is a compact space, there exists a filter G_2 on X_2 and an element x_2 of X_2 such that $\mathrm{pr}_2(H_1)\subset G_2$ and $G_2\to x_2$. It follows from Lemma 2.4 there exists a filter H_2 on $X_1\times X_2$ such that $H_1\subset H_2$ and $G_2=\mathrm{pr}_2(H_2)$. As the filter F is included in H_2 and $\mathrm{pr}_1(H_2)\to x_1, \mathrm{pr}_2(H_2)\to x_2$ implies the convergence of H_2 to an element (x_1,x_2), the product space is compact, by Definition 2.8.

4. The Proof

In this section, we prove Tychonoff's theorem as a direct application of Zorn's lemma. The idea is to construct an ultrafilter and an element of the product space simultaneously by taking a suitable partially ordered set.

Theorem 4.1. *For an arbitrary family of compact topological spaces* $\{X_i\}_{i\in I}$, *any filter on the product space* $\prod_{i\in I}X_i$ *with the product topology is included in a convergent ultrafilter.*

Proof: Let F be a filter on the product set $\prod_{i\in I}X_i$. Let P be the set of pairs (G,x) where G is a filter on $\prod_{i\in I}X_i$ including F, and $x\colon J\to\bigcup_{i\in I}X_i$ is a map with $J\subset I$ satisfying $x(j)\in X_j$ and $\mathrm{pr}_j(G)\to x(j)$ for all $j\in J$. If we introduce a partial order on P by:

$$(G,x)\leqq(H,y)\Leftrightarrow G\subset H\wedge x\subset y,$$

then P satisfies the assumption of Zorn's lemma. Namely, $(F,\phi)\in P$, and for any non-empty chain $C\subset P$, its suprimum $\sup(C)=(H,y)$ with

$$H=\bigcup\{G|\exists x((G,x)\in C)\},$$

and

$$y=\bigcup\{x|\exists G((G,x)\in C)\}$$

is an element of P. Therefore a maximal element (G,x) of P exists. Note that if G is included in a filter H, then since $(G,x)\leqq(H,x)$, we have $(G,x)=(H,x)$, and G must be equal to H. Thus G is an ultrafilter. If $x\colon J\to\bigcup_{i\in I}X_i$, with $J\neq I$, then there is an element $i\in I$ with $i\notin J$. Since $\mathrm{pr}_i(G)$ is an ultrafilter, by the corollary of Lemma 2.4, and X_i is compact, $\mathrm{pr}_i(G)$ converges to an element p of X_i by the corollary of Definition 2.8. This implies that the pair $(G,x\cup\{(i,p)\})$ is an element of P and is strictly bigger than (G,x), contradicting its maximality. Thus $J=I$ and therefore, x is an element of the product space $\prod_{i\in I}X_i$. As $\mathrm{pr}_i(G)\to\mathrm{pr}_i(x)$ for all $i\in I$, the ultrafilter G converges to x, by Lemma 2.11.

As a corollary, now we have:

Tychonoff's Theorem. *For an arbitrary family of compact topological spaces* $\{X_i\}_{i\in I}$, *the product space* $\prod_{i\in I}X_i$ *with the product topology is compact.*

Proof: Since any filter on $\prod_{i\in I}X_i$ is included in a convergent filter, the product space is compact, by Definition 2.8.

Appendix

As an appendix, we present well-known proofs of the axiom of choice and the ultrafilter theorem as applications of Zorn's lemma. (Cf. [2],[4].)

Lemma 5.1. *For a family of sets* $\{X_i\}_{i\in I}$, *let* C *be any non-empty chain consisting of maps* $x\colon J\to\bigcup_{i\in I}X_i$ *with* $J\subset I$ *satisfying* $x(j)\in X_j$ *for all* $j\in J$ *(i.e. partial choice maps of* $\{X_i\}_{i\in I}$*). The union* $\bigcup C$ *is also a partial choice map.*

Proof: If (j,a) and (j,b) are elements of $\bigcup C$, then there exist two partial choice maps $x,y\in C$ such that $x(j)=a$, and $y(j)=b$. Since $x\subset y$ or $x\supset y$, we have $a=b$, and this implies that $\bigcup C$ is a partial map. Furthermore, if $\bigcup C\,(j)=a$, then there exists a partial choice map $x\in C$ such that $x(j)=a$, and thus $\bigcup C\,(j)=x(j)$ belongs to X_j.

Axiom of choice. *For an arbitrary family of non-empty sets*$\{X_i\}_{i\in I}$, *the product set* $\prod_{i\in I}X_i$ *is not empty.*

Proof: Let P be the set of partial choice maps of $\{X_i\}_{i\in I}$. If we introduce a partial order on P by the inclusion, then P satisfies the assumption of Zorn's lemma. Namely, the empty map ϕ belongs to P, and for any non-empty chain $C\subset P$, the union $\bigcup C$ is in P, by Lemma 5.1. Therefore a maximal element x of P exists. If $x\colon J\to\bigcup_{i\in I}X_i$, with $J\neq I$, then there is an element $i\in I$ with $i\notin J$. Since X_i is not empty, there exists an element p of X_i. This implies that $x\cup\{(i,p)\}$ in P is strictly bigger than x, contradicting its maximality. Thus $J=I$, and therefore, x is an element of the product set $\prod_{i\in I}X_i$.

Lemma 5.2. *For any non-empty chain* C *of filters on a set* X*, the union* $\bigcup C$ *is also a filter.*

Proof: We show that the union satisfies the three conditions in Definition 2.1. (1) For all filters $F \in \bigcup C$, we have $\phi \notin F$, $X \in F$, and thererfore $\phi \notin \bigcup C$, $X \in \bigcup C$. (2) If $U \in \bigcup C$ and $U \subset V \subset X$, then there exists a filter F such that $U \in F \in C$, and thus $V \in F \in C$ and $V \in \bigcup C$. (3) Suppose that $U, V \in \bigcup C$, then there exist two filters F, G such that $U \in F \in C$, and $V \in G \in C$. Since $F \cup G = F$ or $F \cup G = G$, we have $U, V \in F \cup G \in C$, which implies $U \cap V \in F \cup G \in C$, and thus we have $U \cap V \in \bigcup C$.

Ultrafilter Theorem. *For any filter* F *on a set* X*, there exists an ultrafilter (i.e. a maximal filter) including* F.

Proof: Let P be the set of filters on X including F. If we introduce a partial order on P by the inclusion, then P satisfies the assumption of Zorn's lemma. Namely, $F \in P$, and for any non-empty chain $C \subset P$, the union $\bigcup C$ is a filter and is in P, by Lemma 5.2. Therefore a maximal element G of P exists. G is an ultrafilter including F.

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