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Application of the Reduced Differential Transform Method to Solve the Navier-Stokes Equations

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Abstract: In fluid mechanics, the Navier-Stokes equations are non-linear partial differential equations that describe the motion of Newtonian fluids. A fluid can be a liquid or a gas. Therefore, the Navier-Stokes equation concerns many phenomena that surround us. The analytical resolution, the search for exact solutions of these equations modeling a fluid is difficult. But they often allow, by an approximate resolution, to propose a model of many phenomena, such as ocean currents and air mass movements in the atmosphere for meteorologists, the behavior of skyscrapers or bridges under the action of wind for architects and engineers, or that of airplanes, trains or high-speed cars for their design offices, as well as the flow of water in a pipe and many other flow phenomena of various fluids. In mathematics, nonlinearity complicates things. In physics, too, the difficulty arises. For this term nonlinearity has its translation in the complexity of the physical phenomena described. This difficulty of resolution partly affects the analyses or descriptions of the modeled phenomena. The objective of this work is the search for exact solutions of the Navier-Stokes equations in dimension 2 and in dimension 3. The method of the reduced differential transform is used to find the exact solutions of these Navier-Stokes equations in 2D and 3D. This method gives an algorithm that favors the rapid convergence of the problem to the exact solution sought. Besides the introduction, this article is structured as follows: the presentation of the method, its application on the two selected Navier-Stokes problems whose exact solutions are obtained with ease, then intervenes the conclusion of the whole work.

Keywords: Navier-Stokes Equation 2D, Navier-Stokes Equation 3D, Exact Solution, Reduced Differential Transform Method

1. Introduction

In the context of climate change, population growth and lack of appropriate urbanization policy, several serious problems have become recurrent; especially in developing countries with considerable population growth. Among these problems there are: the deforestation, the presence of several untreated garbage dumps, uncontrolled construction of houses or uncontrolled urbanization; this untreated household waste pollutes the atmosphere because of the inappropriate gases that it can generate. Let us take the case of methane. Where does methane come from? Agriculture is the predominant source. Emissions generated by livestock, from manure and the gastrointestinal waste. These mixtures of gases that rise into the atmosphere cause turbulence, and other situations. The phenomena of silting of rivers, roads, plantations, transport of waste and pollutants are worrying. Here, we are in the flows, the transport. We are interested Reduced Differential Transform Method in the Navier-Stokes equations [12-15]. In the fluid mechanics, the Navier-Stokes equations are nonlinear partial differential equations that describe the motion of Newtonian fluids (i.e. gases and most liquids) [3]. The resolution of these equations modeling a fluid as continuous medium with a single phase is difficult.

Our work is motivated by the search for the exact solution of these nonlinear partial differential equations and the problematic that arises, namely the manipulation of the nonlinear terms.

The general objective is determining the exact solutions of the partial differential equations when they exist. The specific objective is determining the exact solutions of the Navier-Stokes equations by the reduced differential transform method (RDTM).

Before determining the exact solutions of the chosen problems, a presentation of the method will be made. The whole work will end with a conclusion.

2. Description of the RDTM

The Reduced Differential Transformation Method (RDTM) was first proposed by the Turkish mathematician Yildiray Keskin [7, 8]. This method is applicable to a large class if it exists. After Yildiray Keskin and Oturanc [9], The RDTM has also been used by many authors to obtain analytical approximate and in some cases exact solutions to nonlinear equations. Several types of nonlinear equations have had their different exact solutions easily obtained. We can quote the nonlinear Voltera partial integro-differential equation, the Telegraph equation The inhomogeneous nonlinear wave equation. For more details, we can refer [1, 2, 4-14]. Nevertheless, now suppose that function of two variables $u(x,t)$ which is analytic and k-times continuously differentiable with respect to space x in the domain of our interest [2-6, 16]. Suppose that we can consider this function in this form: $u(x,t) = f(x).g(t)$. Based on the properties of differential transform, function can be represented as:

$$u(x,t) = \left(\sum_{i=0}^{\infty} F(i)x^i\right)\left(\sum_{j=0}^{\infty} G(j)t^j\right) = \sum_{k=0}^{\infty} U_k(x)t^k \quad (1)$$

Where the function $U_k(x)$ is called t-dimensional spectrum function the of $u(x,t)$. If the function $u(x,t)$ is analytic and differentiated continuously with respect to time t and space x in the domain of interest, then let:

$$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} u(x,t)\right]_{t=0} \quad (2)$$

Where the t-dimensional spectrum function $U_k(x)$ is the transformed function. The differential inverse transform of $U_k(x)$ is determined as follows:

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} u(x,t)\right]_{t=0} t^k \quad (3)$$

In fact, the function $u(x,t)$ can written in a finite series as follows,

$$\tilde{u}_n(x,t) = \sum_{k=0}^{n} U_k(x)t^k \quad (4)$$

n is order of approximate where solution.

Therefore, the exact solution of the problem is given by

$$u(x,t) = \lim_{n\to\infty} \tilde{u}_n(x,t) \quad (5)$$

The details for the proper understanding of the reduced differential transformation method are well explained by Keskin who is the author [12]. Many researchers have also contributed to facilitate the understanding and use of this rich method [1-3, 6, 7, 9].

To illustrate the basic concepts of the RDM, consider the following nonlinear partial differential equation written in an operator form:

$$Lu(x,t) + Ru(x,t) + Nu(x,t) = h(x,t) \quad (6)$$

with initial condition:

$$u(x,0) = f(x) \quad (7)$$

According to the RDTM, the iteration formula can be constructed as follows [1, 3, 5, 7]:

$$(k+1)U_{k+1}(x) = G_k(x) - RU_k(x) - NU_k(x) \quad (8)$$

Some basic essential properties of the two-dimensional reduced differential transform are presented in Table below [1, 2, 4-9, 16].

Table 1. *The fundamental operations of RDTM.*

Functional Form	Transformed Form
$u(x,t)$	$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} u(x,t)\right]_{t=0}$
$w(x,t) = u(x,t) \pm v(x,t)$	$W_k(x) = U_k(x) \pm V_k(x)$
$w(x,t) = \alpha u(x,t)$	$W_k(x) = \alpha U_k(x)$ (α *is a constant*)
$w(x,t) = x^m t^n$	$W_k(x) = x^m \delta(k-n)$
$w(x,t) = x^m t^n u(x,t)$	$W_k(x) = x^m U_{k-n}(x)$
$w(x,t) = u(x,t)v(x,t)$	$lW_k(x) = \sum_{r=0}^{k} V_r(x)U_{k-r}(x) = \sum_{r=0}^{k} U_r(x)V_{k-r}(x)$
$w(x,t) = \frac{\partial^r}{\partial t^r} u(x,t)$	$W_k(x) = (k+1)\cdots(k+r)U_{k+1}(x) = \frac{(k+r)!}{k!} U_{k+r}(x)$
$w(x,t) = \frac{\partial}{\partial x} u(x,t)$	$W_k(x) = \frac{\partial}{\partial x} U_k(x)$
Functional form: $\frac{\partial^{r+s} u(x,t)}{\partial x^r \partial t^s}$	Transformed form: $\frac{(k+s)!}{k!}\frac{\partial^r}{\partial t^r} U_{k+s}(x)$

3. Application

Two problems have been selected to test the method; one is in dimension 2, the other in dimension 3 [11-13].

3.1. Example 1

Consider the two-dimensional Navier-Stokes equation

$$\begin{cases} u_t + uu_x + vu_y = \rho_0\left(u_{xx} + u_{yy}\right) \\ v_t + uv_x + vv_y = \rho_0\left(v_{xx} + v_{yy}\right) \end{cases} \tag{9}$$

With the initial conditions

$$u(x, y, 0) = -e^{x+y};\ v(x, y, 0) = e^{x+y} \tag{10}$$

In equations (9), ρ_0 denotes the kinematic viscosity of the flow. ρ_0 is the ratio η/ρ, where η denotes dynamic vis-cosity of flow, and ρ the density of flow.

The application of RDTM to system's equations (9) gives the algorithm

$$(k+1)U_{k+1}(x, y) + \sum_{r=0}^{k} U_r(x)\frac{\partial V_{k-r}(x,y)}{\partial x} + \sum_{r=0}^{k} V_r(y)\frac{\partial U_{k-r}(x,y)}{\partial y} = \rho_0\left(\frac{\partial^2 U_k}{\partial x^2} + \frac{\partial^2 U_k}{\partial y^2}\right) \tag{11}$$

$$(k+1)V_{k+1}(x, y) + \sum_{r=0}^{k} U_r(x)\frac{\partial V_{k-r}(x)}{\partial x} + \sum_{r=0}^{k} V_r(y)\frac{\partial U_{k-r}(x)}{\partial y} = \rho_0\left(\frac{\partial^2 V_k(x,y)}{\partial x^2} + \frac{\partial^2 V_k(x,y)}{\partial y^2}\right) \tag{12}$$

The algorithm is manipulated by varying k, thus

$$k = 0:\ U_1(x, y) + U_0(x)\frac{\partial V_0(x)}{\partial x} + V_0(y)\frac{\partial U_0(x)}{\partial y} = \rho_0\left(\frac{\partial^2 U_0(x)}{\partial x^2} + \frac{\partial^2 U_0(y)}{\partial y^2}\right)$$

$$U_1(x, y) + e^{2(x+y)} - e^{2(x+y)} = \rho_0(-e^{x+y} - e^{x+y})$$

Either

$$U_1(x, y) = -2\rho_0 e^{x+y}$$

$$k = 0:\ V_1(x, y) + U_0(x)\frac{\partial V_0}{\partial x} + V_0(y)\frac{\partial U_0}{\partial y} = \rho_0\left(\frac{\partial^2 V_0(x,y)}{\partial x^2} + \frac{\partial^2 V_0(x,y)}{\partial y^2}\right)$$

$$V_1(x, y) + e^{2(x+y)} - e^{2(x+y)} = \rho_0(-e^{x+y} - e^{x+y})$$

Either

$$V_1(x, y) = -2\rho_0 - e^{x+y}$$

$$k = 1:\ 2U_2(x, y) + U_0\frac{\partial U_1}{\partial x} + U_1\frac{\partial U_0}{\partial x} + V_0\frac{\partial U_1}{\partial y} + V_1\frac{\partial U_0}{\partial y} = \rho_0\left(\frac{\partial^2 U_1(x,y)}{\partial x^2} + \frac{\partial^2 U_1(x,y)}{\partial y^2}\right)$$

Either

$$U_2(x, y) = -\rho_0^2 e^{x+y}$$

$$k = 1: 2V_2(x, y) + U_0\frac{\partial V_1}{\partial x} + U_1\frac{\partial V_0}{\partial x} + V_0\frac{\partial V_1}{\partial y} + V_1(x)\frac{\partial V_0}{\partial y} = \rho_0\left(\frac{\partial^2 V_1(x,y)}{\partial x^2} + \frac{\partial^2 V_1(x,y)}{\partial y^2}\right)$$

Either

$$V_2(x, y) = \rho_0^2 e^{x+y}$$

By the same principle of iterations, the following expressions can be deduced

For $k = 2$,

$$U_3(x, y) = -\frac{4}{3}\rho_0^3 e^{x+y};\ V_3(x, y) = \frac{4}{3}\rho_0^3 e^{x+y}$$

$$u(x, y, t) = \sum_{k=0}^{\infty} U_k\,(x, y)t^k = U_0 + U_1 t + U_2 t^2 + U_3 t^3 + \cdots$$

$$u(x, y, t) = -e^{x+y} - 2\rho_0 e^{x+y}t - -\rho_0^2 e^{x+y}t^2 - \frac{4}{3}\rho_0^3 e^{x+y}t^3 - \cdots$$

$$u(x, y, t) = -e^{x+y}\left[1 + 2\rho_0 t + \frac{(2\rho_0 t)^2}{2!} + \frac{(2\rho_0 t)^3}{3!} + \frac{(2\rho_0 t)^4}{4!} + \cdots\right]$$

Either

$$u(x, y, t) = -e^{x+y}\sum_{r=0}^{\infty}\frac{1}{k!}(2\rho_0 t)^k = -\,e^{x+y}e^{2\rho_0 t}$$

The exact solution is:

$$u(x,y,t) = -e^{x+y+2\rho_0 t} \tag{13}$$

3.2. Example 2

In this example, the problem is the system of Navier-Stokes equations in space dimension [15]. The problem is the following:

$$\begin{cases} u_t + uu_x + vu_y + wu_z = \rho_0\left(u_{xx} + u_{yy} + u_{zz}\right) \\ v_t + uv_x + vv_y + wv_z = \rho_0\left(v_{xx} + v_{yy} + v_{zz}\right) \\ w_t + uw_x + vw_y + ww_z = \rho_0(w_{xx} + w + w_{zz}) \end{cases} \tag{14}$$

The initial conditions are:

$$\begin{cases} u(x,y,z,0) = u_0(x,y,z,t) = -\frac{1}{2}x + y + z \\ v(x,y,z,0) = v_0(x,y,z,t) = x - \frac{1}{2}y + z \\ w(x,y,z,0) = w_0(x,y,z,t) = x + y - \frac{1}{2}z \end{cases} \tag{15}$$

In equations (14), ρ_0 denotes the kinematic viscosity of the flow. ρ_0 is the ratio η/ρ, where η denotes dynamic vis-cosity of flow, and ρ the density of flow.

The application of the RDTM to each equation of the 3D system of Navier-Stokes equations gives the following algorithms:

$$(k+1)U_{k+1} + \sum_{r=0}^{k} U_r \frac{\partial U_{k-r}}{\partial x} + \sum_{r=0}^{k} V_r \frac{\partial U_{k-r}}{\partial y} + \sum_{r=0}^{k} W_r \frac{\partial U_{k-r}}{\partial z} = \rho_0\left(\frac{\partial^2}{\partial x^2}U_k + \frac{\partial^2}{\partial y^2}U_k + \frac{\partial^2}{\partial z^2}U_k\right) \tag{16}$$

$$(k+1)V_{k+1} + \sum_{r=0}^{k} U_r \frac{\partial V_{k-r}}{\partial x} + \sum_{r=0}^{k} V_r \frac{\partial V_{k-r}}{\partial y} + \sum_{r=0}^{k} W_r \frac{\partial V_{k-r}}{\partial z} = \rho_0\left(\frac{\partial^2}{\partial x^2}V_k + \frac{\partial^2}{\partial y^2}V_k + \frac{\partial^2}{\partial z^2}V_k\right) \tag{17}$$

$$(k+1)W_{k+1} + \sum_{r=0}^{k} U_r \frac{\partial W_{k-r}}{\partial x} + \sum_{r=0}^{k} V_r \frac{\partial W_{k-r}}{\partial y} + \sum_{r=0}^{k} W_r \frac{\partial W_{k-r}}{\partial z} = \rho_0\left(\frac{\partial^2}{\partial x^2}W_k + \frac{\partial^2}{\partial y^2}W_k + \frac{\partial^2}{\partial z^2}W_k\right) \tag{18}$$

The manipulation of the values of k, gives:

For $k = 0$;

$$U_1 + U_0\frac{\partial U_0}{\partial x} + V_0\frac{\partial U_0}{\partial y} + W_0\frac{\partial U_0}{\partial z} = \rho_0\left(\frac{\partial^2}{\partial x^2}U_0 + \frac{\partial^2}{\partial y^2}U_0 + \frac{\partial^2}{\partial z^2}U_0\right)$$

$$V_1 + U_0\frac{\partial V_0}{\partial x} + V_0\frac{\partial V_0}{\partial y} + W_0\frac{\partial V_0}{\partial z} = \rho_0\left(\frac{\partial^2}{\partial x^2}V_0 + \frac{\partial^2}{\partial y^2}V_0 + \frac{\partial^2}{\partial z^2}V_0\right)$$

$$W_1 + U_0\frac{\partial W_0}{\partial x} + V_0\frac{\partial W_0}{\partial y} + W_0\frac{\partial W_0}{\partial z} = \rho_0\left(\frac{\partial^2}{\partial x^2}W_0 + \frac{\partial^2}{\partial y^2}W_0 + \frac{\partial^2}{\partial z^2}W_0\right)$$

After all the substitutions and calculations, the following results are obtained:

$$U_1 + \frac{1}{4}x - \frac{1}{2}y - \frac{1}{2}z + x - \frac{1}{2}y + z + x + y - \frac{1}{2}z = 0$$

Let $U_1 = -\frac{9}{4}x$

$$V_1 - \frac{1}{2}x + y + z - \frac{1}{2}x + \frac{1}{4}y - \frac{1}{2}z + x + y - \frac{1}{2}z = 0$$

Let $V_1 = -\frac{9}{4}y$

$$W_1 - \frac{1}{2}x + y + z + x - \frac{1}{2}y + z - \frac{1}{2}x - \frac{1}{2}y + \frac{1}{4}z = 0$$

Let $W_1 = -\frac{9}{4}z$

At the passage of $k = 1$, the same iterative techniques of calculations, will be applied. From these new formulas deduced from the manipulation of the values of k will result the following expressions. After all the substitutions and calculations made, it follows

$$U_2 = \frac{9}{4}\left(-\frac{1}{2}x + y + z\right); \; V_2 = \frac{9}{4}\left(x - \frac{1}{2}y - \frac{1}{2}z\right)$$

$$W_2 = \frac{9}{4}\left(x + y - \frac{1}{2}z\right)$$

The expressions U_3, V_3, W_3 are given by the formulas of

$$3U_3 + U_0\frac{\partial U_2}{\partial x} + U_1\frac{\partial U_1}{\partial x} + U_2\frac{\partial U_0}{\partial x} + V_0\frac{\partial U_2}{\partial y} + V_1\frac{\partial U_1}{\partial y} + V_2\frac{\partial U_0}{\partial y} + W_0\frac{\partial U_2}{\partial z} + W_1\frac{\partial U_1}{\partial z} + W_2\frac{\partial U_0}{\partial z} = 0 \tag{19}$$

$$3V_3 + U_0\frac{\partial V_2}{\partial x} + U_1\frac{\partial V_1}{\partial x} + U_2\frac{\partial V_0}{\partial x} + V_0\frac{\partial V_2}{\partial y} + V_1\frac{\partial V_1}{\partial y} + V_2\frac{\partial V_0}{\partial y} + W_0\frac{\partial V_2}{\partial z} + W_1\frac{\partial V_1}{\partial z} + W_2\frac{\partial V_0}{\partial z} = 0 \tag{20}$$

$$3W_3 + U_0\frac{\partial W_2}{\partial x} + U_1\frac{\partial W_1}{\partial x} + U_2\frac{\partial W_0}{\partial x} + V_0\frac{\partial W_2}{\partial y} + V_1\frac{\partial W_1}{\partial y} + V_2\frac{\partial W_0}{\partial y} + W_0\frac{\partial W_2}{\partial z} + W_1\frac{\partial W_1}{\partial z} + W_2\frac{\partial W_0}{\partial z} = 0 \tag{21}$$

As result:

$$U_3 = -\left(\frac{9}{4}\right)^2 x; \; V_3 = -\left(\frac{9}{4}\right)^2 y; \; W_3 = -\left(\frac{9}{4}\right)^2 z$$

By the same calculations applied to formulas (16), (17) and (18), the sequence of terms follows.

As before, the exact solution is given by the relation:

$$u(x,y,z,t) = U_0 + U_1 t + U_2 t^2 + U_3 t^3 + \cdots \tag{22}$$

$$u(x,y,z,t) = -\frac{1}{2}x + y + z - \frac{9}{4}xt + \frac{9}{4}\left(-\frac{1}{2}x + y + z\right)t^2 - \left(\frac{9}{4}\right)^2 xt^3 + \left(\frac{9}{4}\right)^2\left(-\frac{1}{2}x + y + z\right)t^4 - \left(\frac{9}{4}\right)^3 xt^5 + \cdots$$

The terms are grouped two by two; U_0 and U_1, U_2 and U_3, U_4 and U_5, U_6 and U_7, and so on; its grouped terms factorize like the sum of U_0 and U_1, then lead to the expression:

$$u(x,y,z,t) = \left(-\frac{1}{2}x + y + z - \frac{9}{4}xt\right) + \frac{9}{4}\left[\left(-\frac{1}{2}x + y + z\right) - \frac{9}{4}xt\right]t^2 + \left(\frac{9}{4}\right)^2\left[\left(-\frac{1}{2}x + y + z\right) - \frac{9}{4}xt\right]t^4 + \cdots \tag{23}$$

$$u(x,y,z,t) = \left(-\frac{1}{2}x + y + z - \frac{9}{4}xt\right)\left[1 + \frac{9}{4}t^2 + \left(\frac{9}{4}\right)^2 t^4 + \cdots\right] \tag{24}$$

Since this expression is not the sum of the terms of a classical numerical series, it requires an appropriate transformation or writing, to bring out an expression easily identifiable with a Taylor development. This is the case of this problem. Thus

$$u(x,y,z,t) = \left(-\frac{1}{2}x + y + z - \frac{9}{4}xt\right)\frac{1}{1-\frac{9}{4}t^2} \tag{25}$$

The calculations for $v(x)$ and $w(x)$ give:

$$v(x,y,z,t) = \left(x - \frac{1}{2}y + z - \frac{9}{4}xt\right)\left[1 + \frac{9}{4}t^2 + \left(\frac{9}{4}\right)^2 t^4 + \cdots\right] \tag{26}$$

$$w(x,y,z,t) = \left(x + y - \frac{1}{2}z - \frac{9}{4}xt\right)\left[1 + \frac{9}{4}t^2 + \left(\frac{9}{4}\right)^2 t^4 + \cdots\right] \tag{27}$$

The exact solution of example 2 is given by the expressions:

$$u(x,y,y,z,t) = \frac{2x-4y-4z+9xt}{9t^2-4} \tag{28}$$

$$v(x,y,y,z,t) = \frac{2y-4x-4z+9xt}{9t^2-4} \tag{29}$$

$$w(x,y,y,z,t) = \frac{2z-4x-4y+9xt}{9t^2-4} \tag{30}$$

On the physical level or in the reality of physical laws, the velocities of flows in classical mechanics cannot be infinite.

This will not reflect the laws of fluid dynamics. Therefore, it is necessary to pose the necessary condition of existence of a physically admissible solution.

These results are valid with the condition

$$|t| \neq \frac{2}{3}$$

4. Conclusion

In general, the search for exact solutions of the Navier-Stokes equations in 2D and 3D has not always been easy. Great difficulties have often existed. However, the RDTM was applied with ease. Good results have been obtained. The expected results are the exact solutions of the Navier-Stokes problems in 2D and 3D.

The exact solutions of the two systems of Navier-Stokes equations, in 2D and 3D, have been obtained.

Of course, the calculations are tedious and the method requires a great mastery of the basic concepts. The results therefore confirm the effectiveness of the method. It should be emphasized that the search for the solution for the 3D case requires a good mastery of the handling of Taylor series developments. From this RDTM method, for future research, it is possible to determine the exact solution of a mathematical model coupled with the Navier-Stokes equations, in the cases of sedimentary basin formation, transport of pollutants and other materials carried, when the problem is well defined.

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