Pure and Applied Mathematics Journal
2021; 110(2): 62-67
http://www.sciencepublishinggroup.com/j/pamj
doi: 10.11648/j.pamj.20211002.13
ISSN: 2326-9790 (Print); ISSN: 2326-9812 (Online)

A Speedy New Proof of the Riemann's Hypothesis

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To cite this article:
Jean-Max Coranson-Beaudu. A Speedy New Proof of the Riemann's Hypothesis. *Pure and Applied Mathematics Journal.* Vol. 10, No. 2, 2021, pp. 62-67. doi: 10.11648/j.pamj.20211002.13

Received: March 20, 2021; **Accepted**: April 24, 2021; **Published**: May 14, 2021

Abstract: In this paper we show that Riemann's function *(xi)*, involving the Riemann's (*zeta)* function, is holomorphic and is expressed as a convergent infinite polynomial product in relation to their zeros and their conjugates. Our work will be done on the critical band in which non-trivial zeros exist. Our approach is to use the properties of power series and infinite product decomposition of holomorphic functions. We take inspiration from the Weierstrass method to construct an infinite product model which is convergent and whose zeros are the zeros of the zeta function. By applying the symetric functional equation of the *xi* function we deduce a relation between each zero of the function *xi* and its conjugate. Because of the convergence of the infinite product, and that the elementary polynomials of the second degree of this same product are irreducible into the complex set, then this relation is well determined. The apparent simplicity of the reasoning is based on the fundamental theorems of Hadamard and Mittag-Leffler. We obtain the sought result: the real part of all zeros is equal to ½. This article proves that the Riemann' hypothesis is true. Our perspectives for a next article are to apply this method to Dirichlet series, as a generalization of the Riemann function.

Keywords: Riemann Hypothesis, Weierstrass, Zeta Function, Holomorphic Function, Infinite Product, Dirichlet Series

1. Introduction

Riemann's Hypothesis ([1, 3, 6]), formulated in 1859, concerns the location of the zeros of Riemann's Zeta function. The history of the Riemann Hypothesis is well known. In 1859, the German mathematician B. Riemann presented a paper to the Berlin Academy of Mathematic. In this paper, he proposed that this function, called Riemann-zeta function takes values 0 on the complex plane when s=0.5+it. This hypothesis has great significance for the world of mathematics and physics ([4]).

This solutions would lead to innumerable completions of theorems that rely upon its truth. Over a billion zeros of the function have been calculated by computers and shown that all are on this line s = 0.5+it. The demonstration of this conjecture would improve in particulary the knowledge of the distribution of prime numbers. This is one of the most important unsolved problems in 21st century mathematics: it is one of the twenty-three famous Hilbert problems proposed in 1900. At the origin, the Riemann zeta function is a complex function defined for any complex number s such that R(s) > 1, by the Riemann series:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} \ldots$$

The link between the function ζ and the prime numbers had already been established by Leonard Euler with the formula, valid for

$$\Re(s) > 1$$

$$\zeta(s) = \prod_{k \in P}^{\infty} \frac{1}{1 - p^{-s}}$$

We can extend the function ζ on $\mathbb{C} - \{0; 1\}$ from the definition of the functional equation

$$\zeta(s) = 2^s \pi^{s-1} sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s)$$

Many attempts to demonstrate the zeros of ζ are in the line 0.5+it have already been made:

V. Garcia-Morales [8] has tried by the asymptotic properties of complex functions;

H. Kobayashi [9] shows that the zeta function on the critical axis admits local minima;

F. Stenger([10] combines novel methods about indefinite integration, indefinite convolutions and inversions of Fourier transforms with numerical ranges to prove the Riemann hypothesis;

G. Mussardo and A. Leclair [11] have used the Dirichlet theorem on the equidistribution of residue classes modulo q and the Lemke Oliver-Soundararajan conjecture on the distribution of pairs of residues on consecutive primes to try to show the conjecture;

M. Atiyah [12] has used the properties of the Todd functions;

A. Connes [13] has reviewed the impact in of the Riemann's Hypothesis on the development of algebraic geometry. He has discussed on three axes, working concretely at the level of the explicit formulas. He has based on Riemannian spaces and Selberg's work, on algebraic geometry and the Riemann-Roch theorem, and on the development of a suitable "Weil coho- mology";

C. Chuanmiao [14] has analysed of the local geometry of the Riemann function, in particular its symmetry properties;

A. Dobner [15] has making one approach by the Bruijn-Newman constant theorem applied to the study of Dirichlet functions in the Selberg class;

L. Tao and W. Juhao [16] has used the symmetry and convergence property of the meromorphic function of the functional equation for the zeta function;

F. Alhargan [17] has recently tried by a Hadamard's factorization. It seems that difficulties in the convergence of the infinite product did not allow to confirm his demonstration.

In a previous article [18] we considered studying the integral expression of the function zeta by establishing necessary conditions for the existence of zeros on the critical line. Then by pure reasoning we showed that there can be no roots outside the critical line.

In this article, our method is based on a smart factorization of the ξ function inspired by a convergent Weierstrass factorization.

We use the symmetric and convergence properties of an infinite product to establish a relationship between the zeros from the symmetry conditions of the function.

$$\xi(s) = \xi(1-s)$$

2. The Riemann's Functional Equation

The zeta function satisfies the functional equation was established by Riemann in 1859([4, 6]).

For all complex numbers except 0 and 1

$$\zeta(s) = 2^s \pi^{s-1} sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)$$

Riemann also found a symmetric version of the functional equation applying to the ξ function:

$$\xi(s) = \frac{1}{2}\pi^{-\frac{s}{2}} s(s-1)\Gamma\left(\frac{s}{2}\right)\zeta(s) =$$

$$s(s-1)\int_1^\infty \left(u^{\frac{s}{2}-1} + u^{\frac{-s-1}{2}}\right)\psi(u)du + 1$$

With

$$\psi(u) = \sum_{n=1}^{\infty} e^{-\pi u n^2}$$

This function satisfies:

$$\xi(1-s) = \xi(s)$$

Also,

ξ is a holomorphic function on $\mathbb{C}$ because of expression (1), then the conjugate expression $\overline{\xi(s)} = \xi(\overline{s})$

If s_k are zeros of ξ, then $\overline{\xi(s_k)} = \xi(\overline{s_k})$ and $\overline{s_k}$ are zeros of ξ.

3. Infinite Product

We denote by Ω an open set in $\mathbb{C}$, $\mathcal{H}(\Omega)$ the set of holomorphic functions on Ω and Z(*f*) the set of zeros of an element $f \in \mathcal{H}(\Omega)$.

The isolated zeros theorem states that Z(*f*) has no accumulation points.

Reciprocally, $\forall$ A $\subset \Omega$ there exists $g \in \mathcal{H}(\Omega)$ such that Z(g) = A. The set A has no accumulation points.

Let u_n be a sequence of complex numbers and the sequence defined by:

$$p_N = (u_1)(u_2)\ldots(u_N)$$

It is assumed that the limit of p_N exists and is equal to p, then:

$$p = \prod_{n=1}^{\infty}(u_n)$$

and we say that p is the infinite product of the partial products p_N.

Theorem 1.

In an open set Ω of $\mathbb{C}$, let be the holomorphic function sequence $F_n(z)$. If there are positive constants c_n such that

$$\sum_{n=1}^{\infty} c_n < \infty \; and \; |F_n(z) - 1| < c_n$$

Then the infinite product of holomorphic functions

$$\prod_{n=1}^{\infty} F_n(z)$$

converges uniformly in Ω to a holomorphic function $F(z)$, ([7])

Proof:

As c_n converges to 0, there is a sufficiently large n for $c_n < \frac{1}{2}$.

Let $F_n(z) = 1 + a_n(z)$ then $|F_n(z) - 1| < |a_n(z)|$

$$|a_n(z)| < c_n < \frac{1}{2}$$

$$ln\left(\prod_{n=1}^{\infty}\big(1 + a_n(z)\big)\right) = \sum_{n=1}^{\infty} ln\big(1 + a_n(z)\big)$$

and

$$\left|\sum_{n=1}^{\infty} ln\big(1 + a_n(z)\big)\right| \leq \sum_{n=1}^{\infty}\left|ln\big(1 + a_n(z)\big)\right|$$

i.e

$$\sum_{n=1}^{\infty}\left|ln\big(1 + a_n(z)\big)\right| \leq 2\sum_{n=1}^{\infty}|a_n(z)|$$

This demonstrates the absolute convergence of the infinite product.

And the theorem of Cauchy proof that if $ln\big(1 + a_n(z)\big)$ is a holomorphic function then

$$F(z) = \prod_{n=1}^{\infty}\big(1 + a_n(z)\big) = e^{\sum_{n=1}^{\infty} ln(1+a_n(z))}$$

is a holomorphic function.

4. Factorization of the Zeta Function

In this section we will consider the problem of constructing an analytic function ξ with an infinite product of sequences of complex function.

Let s_n and $\overline{s_n}$ be the zeros of the function zeta.

Let $\Omega_n = \left\{s \in \mathbb{C} \,/\, \left|\frac{s}{s_n}\right| < 1\right\}$

And $\Omega = \bigcup_n^{\infty} \Omega_n$

All holomorphic functions in Ω can be represented as an infinite product involving its zeros [7].

$$\xi(s) = H\big(s(1-s)\big)\prod_{n=1}^{\infty}\left(1 - \frac{s}{s_n}\right)\left(1 - \frac{s}{\overline{s_n}}\right) =$$

$$H\big(s(1-s)\big)\prod_{n=1}^{\infty}\left(1 - s\left(\frac{s_n + \overline{s_n} - s}{s_n\overline{s_n}}\right)\right)$$

$\xi(1-s) = \xi(s)$ then $\xi(1-s_n) = \xi(s_n) = 0$

H is a holomorphic function whose zeros are not s_n or $\overline{s_n}$ and

$$H(0) = 1 \text{ since } \xi(0) = \xi(1) = 1$$

A more subtle approach is required. One that achieves convergence by using factors more elaborate than

$$\left(1 - \frac{s}{s_n}\right)\left(1 - \frac{s}{\overline{s_n}}\right).$$

We take inspiration from the Weierstrass method to construct an infinite product model which is convergent and whose zeros are the zeros of the zeta function.

Note: For the Weierstrass factorization we have:

$E_n(s) = \left(1 - \frac{s}{s_n}\right)e^{\varpi_n(s)}$ *with*

$$\varpi_n(s) = \sum_{k=1}^{\lambda(n)}\frac{1}{k}\left(\frac{s}{s_n}\right)^k$$

$$F(s) = s^p e^{g(s)}\prod_{n=1}^{\infty}E_n(s)$$

We propose as a factor

$F_n(s) = \left(1 - s\left(\frac{s_n + \overline{s_n} - s}{s_n\overline{s_n}}\right)\right)e^{w_n(s)}$ with

$$w_n(s) = \sum_{k=1}^{\lambda(n)}\frac{1}{k}\left(s\left(\frac{s_n + \overline{s_n} - s}{s_n\overline{s_n}}\right)\right)^k$$

Proposition :

Our model for the zeta function is

$$\xi(s) = \prod_{n=1}^{\infty}F_n(s)$$

At this point, we show that the infinite product converges to a holomorphic function using *Theorem 1.*

$|F_n(s) - 1| < c_n$ with c_n is a convergent sequence.

Proof: Let

$$u_n(s) = s\left(\frac{s_n + \overline{s_n} - s}{s_n\overline{s_n}}\right)$$

$$\frac{du_n(s)}{ds} = \left(\frac{s_n + \overline{s_n} - 2s}{s_n\overline{s_n}}\right)$$

As $\left|\frac{s}{s_n}\right| \leq 1$ then $|u_n(s)| \leq 1$

$$F_n(s) = E_n(u_n(s))$$

with

$$w_n(s) = \varpi(u_n(s))$$

Let

$$E_n(u_n(s)) = (1 - u_n(s))e^{\varpi_n(u_n(s))}$$

$$\varpi_n(u_n(s)) = \sum_{k}^{n}\frac{1}{k}(u_n(s))^k$$

then

$$\frac{\partial F_n(s)}{\partial s} = -\frac{du_n(s)}{ds}(u_n(s))^n e^{\varpi_n(u_n(s)}$$

as

$(u_n(s))^n$ has a pole of order n in zero, and

$$\int_0^s \frac{\partial F_n(z)}{\partial z}\,dz = F_n(s) - F_n(0)$$

As $F_n(0) = 1$
then

$$|F_n(s) - 1| \le |u_n(s)|^{n+1} \le \left|\frac{s}{s_n}\right|^{n+1}$$

For $\left|\frac{s}{s_n}\right| < 1$
i.e
$|F_n(s) - 1|$ is convergent which demonstrate the proposition.
Let,

$$\xi(s) = \prod_{n=1}^{\infty} F_n(s)$$

a holomorphic function with zeros s_p *and* $\overline{s_p}$.

$$\xi(s_p) = \xi(\overline{s_p}) = 0 \implies \exists\, p \in \mathbb{N} \,/\, \begin{array}{l} F_p(s_p) = 0 \\ F_p(\overline{s_p}) = 0 \end{array}$$

$$F_n(0) = F_n(1) = 1$$

Then

$$\xi(0) = \xi(1) = 1$$

5. Proof of the Riemann's Hypothesis

Lemma 1
Let's take two convergent infinite products

$$\prod_k^{\infty} Q_k(z) \neq 0 \;\; and \;\; \prod_k^{\infty} P_k(z) \neq 0$$

If

$$\prod_k^{\infty} P_k(z) = \prod_k^{\infty} Q_k(z) \Rightarrow \lim_{k\to+\infty} \frac{P_k(z)}{Q_k(z)} = 1$$

Let

$$\xi(s) = \prod_{n=1}^{\infty} F_n(s)$$

And

$$\xi(1-s) = \prod_{n=1}^{\infty} F_n(1-s)$$

As $\xi(s) = \xi(1-s)$ then

$$\forall\, s \in \Omega, \prod_{n=1}^{\infty} F_n(s) = \prod_{n=1}^{\infty} F_n(1-s)$$

By applying the Lemma 1 for the expression

$$\frac{\xi(1-s)}{\xi(s)} = \frac{\prod_{n=1}^{\infty} F_n(1-s)}{\prod_{n=1}^{\infty} F_n(1-s)} = 1$$

we show that,

$$\lim_{n\to+\infty} \frac{F_n(1-s)}{F_n(s)} = 1$$

$$\lim_{n\to\infty}\left(1 - \frac{s_n + \overline{s_n} - 1}{s_n\overline{s_n}} - s\left(\frac{2 - s_n - \overline{s_n} - s}{s_n\overline{s_n}}\right)\right)e^{w_n(1-s)} = \lim_{n\to\infty}\left(1 - s\left(\frac{s_n + \overline{s_n} - s}{s_n\overline{s_n}}\right)\right)e^{w_n(s)}$$

As the functions ($e^{w_n(s)}$) are continuous, the transition to the limit is continuous.
This equality is true $\forall\, s \in \Omega \subset \mathbb{C}$ if and only if

$$\lim_{n\to\infty} w_n(s) = \lim_{n\to\infty} w_n(1-s)$$

And $\forall\, s \in \Omega \subset \mathbb{C}$

$$\lim_{n\to\infty}\left(\frac{s_n + \overline{s_n} - 1}{s_n\overline{s_n}} + s\left(\frac{2 - s_n - \overline{s_n} - s}{s_n\overline{s_n}}\right)\right) = \lim_{n\to\infty}\left(s\left(\frac{s_n + \overline{s_n} - s}{s_n\overline{s_n}}\right)\right)$$

i.e

$$\lim_{n\to\infty}(s_n + \overline{s_n} - 1) = 0$$

And if $\lim_{n\to\infty}|s_n| = +\infty$ then
The zeros of the function ξ converge asymptotically to the line of the complex plane $\frac{1}{2} + iy$
If

$$\xi(1-s_p) = \prod_{n=1}^{\infty} F_n(1-s_p) = 0$$

For all zeros s_p
Then there exists p such that
$F_p(1-s_p) = 0$ for $p \in \mathbb{N}^*$

$$\Rightarrow \left(1 - (1-s_p)\left(\frac{s_p + \overline{s_p} - (1-s_p)}{s_p\overline{s_p}}\right)\right) = 0$$

i.e

$$s_p\overline{s_p} - (1-s_p)(s_p + \overline{s_p}) + (1-s_p)^2 = 0$$

i.e

$$s_p\overline{s_p} - (s_p + \overline{s_p}) + s_p(s_p + \overline{s_p}) + 1 - 2s_p + s_p{}^2 = 0$$

then

$$2s_p\left(\overline{s_p}+s_p-1\right) - \left(\overline{s_p} + s_p - 1\right) = 0$$

i.e

$$\left(\overline{s_p}+s_p-1\right)\left(2s_p-1\right) = 0$$

Then $\forall\, p \in \mathbb{N}^* \,/\, \overline{s_p}+s_p = 1$

$\Re\left(s_p\right) = \frac{1}{2}$ for all zeros of ξ

The function ξ can be written as

$$\xi(s) = \prod_{n=1}^{\infty}\left(1 - s\left(\frac{1-s}{s_n\overline{s_n}}\right)\right)e^{w_n(s)}$$

$$w_n(s) = \sum_{k=1}^{\lambda(n)}\frac{1}{k}\left(s\left(\frac{1-s}{s_n\overline{s_n}}\right)\right)^k$$

This factorization is not unique and depends on the choice of $\lambda(n)$.

6. Conclusion

We have demonstrated:
that the holomorphic function $\xi(s)$ had the same zeros as the function $\zeta(s)$ which is a functional equation.

$$\xi(s) = \frac{1}{2}\pi^{-\frac{s}{2}}s(s-1)\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

ξ verifies the following symmetric relation also

$$\xi(s) = \xi(1-s)$$

in the critical band $]0;1[\cup]-i\infty; i\infty[$.

As

$$\xi\left(s_p\right) = \xi\left(1-s_p\right) = \xi\left(\overline{s_p}\right) = \xi\left(1-\overline{s_p}\right) = 0$$

and we used a factorization inspired by Weierstrass's factorization ([5, 7]) to establish a relationship between each zero and its conjugate.

$$s_p + \overline{s_p} - 1 = 0, \forall\, p \in \mathbb{N}^*$$

Then
We have demonstrated that all non-trivial zeros s_p of the function ζ have their real part equal to ½.

Our perspectives for a next article are to apply this method to Dirichlet series, as a generalization of the Riemann function.

$$f(s) = \sum_{n=1}^{\infty}\frac{C(n)}{n^s}$$

Which satisfies a functional equation similar to that of the Riemann zeta function, typically of the form

$$f(s) = 2^s q\kappa^{1/2-s}\pi^{s-1}\Gamma(1-s)sin\left(\frac{\pi}{2}(s+\kappa)\right)f(1-s)$$

but for which the Riemann hypothesis is false. Here κ is 0 or 1 and q is a prime number. For all the examples investigated by mathematicians the hypothesis is fundamentally false, for the function f(s) have zeros in their regions of absolute convergence, and examples are also given which can have zeros arbitrarily close to any point in the complex plane.

However, under the assumption of certain hypothesis concerning the density of zeros on average, the sequence formed by the imaginary parts of the zeros of a Dirichlet series is uniformly distributed in the complex plane.

Working with infinite products for analytical extensions of the Dirichlet series could have some advantages.

Infinite products play an important role in many branches of mathematics. In number theory, they for instance provide an elegant way of encoding and manipulating combinatorial identities. The product expansion of the generating function of the partition function is a well-known example. On the other hand, infinite products are a fundamental tools in complex analysis to construct meromorphic functions with prescribed zeros and poles, this present article has proved theorem being a prominent example. In that way, they become interesting for the study of geometric problems.

They are modular forms for the orthogonal group of a suitable rational quadratic space. Although some very classical modular forms appear here, as for instance certain Eisenstein series , most of these product expansions were only discovered rather recently.

The properties of Borcherds' products on Hilbert modular surfaces can be used to recover important classical results on the geometry of such surfaces.

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