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On Norm of Elementary Operator: An Application of Stampfli's Maximal Numerical Range

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Abstract: Many researchers in operator theory have attempted to determine the relationship between the norm of an elementary operator of finite length and the norms of its coefficient operators. Various results have been obtained using varied approaches. In this paper, we attempt this problem by the use of the Stampfli's maximal numerical range.

Keywords: Elementary Operator, Maximal Numerical Range, Rank-One Operator

1. Introduction

Properties of elementary operators have been investigated in the resent past under varied aspects. Their norms have been a subject of interest for research in operator theory. Deriving a formula to express the norm of an arbitrary elementary operator in terms of its coefficient operators remains a topic of research in operator theory. In the current paper, the concept of the maximal numerical range is applied in determining the lower bound of the norm of an elementary operator consisting of two terms, and also to determine the conditions under which the norm of this operator is expressible in terms of its coefficient operators in $B(H)$. Specifically, the Stampfli's maximal numerical range is employed in arriving at our results.

Let H be a complex Hilbert space and $B(H)$ be the set of bounded linear operators on H. We define an *elementary operator*,

$$E_n: B(H) \longrightarrow B(H), W \longmapsto \sum_{i=1}^{n} T_i W S_i,$$

for all $W \in B(H)$ where T_i, S_i are fixed elements of $B(H)$. When $n = 1$, then we obtain a *basic elementary operator*,

$$E_n: B(H) \longrightarrow B(H), W \longrightarrow TWS,$$

for all $W \in B(H)$ and T, S fixed in $B(H)$. We denote the basic elementary operator by $M_{T,S}$. When $n = 2$, we obtain the *elementary operator of length two*, whereby,

$$E_2(W) = T_1 W S_1 + T_2 W S_2,$$

for all $W \in B(H)$ and T_i, S_i fixed in $B(H)$ for $i = 1,2$.

The *Jordan elementary operator*, $U_{T,S}$, is defined as;

$$U_{T,S}: B(H) \longrightarrow B(H), W \longrightarrow TWS + SWT,$$

for all $W \in B(H)$ and T, S fixed in $B(H)$, is an example of elementary operators.

Let $\mathcal{A}$ be an algebra. A *derivation* is a function $\Delta: \mathcal{A} \longrightarrow \mathcal{A}$ for which $\Delta(x,y) = x(\Delta y) + (\Delta x)y$ for all $x, y \in \mathcal{A}$. If there is an $a \in \mathcal{A}$ such that $\Delta x = xa - ax$ for all $x \in \mathcal{A}$, then $a??$ is called an *inner derivation*. A derivation is another example of elementary operators.

The Stampfli's maximal numerical range of $T \in B(H)$ is the set,

$$W_0(T) = \{\lambda \in \mathbb{C}: \langle Tx_n, x_n\rangle \longrightarrow \lambda, \|x_n\| = 1, \|Tx_n\| \longrightarrow \|T\|\},$$

while the maximal numerical range $W_S(T^*S)$ of T^*S relative to S is defined as,

$$W_S(T^*S) = \{\lambda \in \mathbb{C}: \exists\{x_n\} \subseteq H, \|x_n\| = 1, \lim_{n\to\infty}\langle T^*Sx_n, x_n\rangle = \lambda, \lim_{n\to\infty}\|Sx_n\| = \|S\|\},$$

where T^* is the Hilbert adjoint of T.

For any $x, y \in H$, *the rank one operator*, $x \otimes y \in B(H)$, is defined by $(x \otimes y)(z) = \langle z, y\rangle x$, for all $z \in H$.

This paper has determined the norm of the elementary operator of length two.

Given and elementary operator E_2 on $B(H)$ with fixed operators T_i, S_i on $B(H)$ for $i = 1,2$, does the relationship

$\|E_2\| = \sum_{i=1}^{2}\|T_i\|\|S_i\|$ hold? King'ang'i [1] attempted this problem using the maximal numerical range of T^*S relative to S. The current paper employs the concept of the Stampfli's maximal numerical range to determine the lower bound of the norm of elementary operator E_2, and also to determine the conditions under which the norm of this operator is expressible in terms of the norms of its coefficient operators in $B(H)$. The approach used by Barraa and Boumazguor [2], and also by King'ang'i [1] is employed in obtaining our results.

2. The Norm of the Jordan Elementary Operator

Let H be a complex Hilbert space, $B(H)$ be the algebra of bounded linear operators on H, and $T, S \in B(H)$ be fixed. For a Jordan elementary operator $U_{T,S}$, Mathieu [5], in 1990 proved that in the case of prime C*-algebras, the lower bound of the norm of $U_{T,S}$ can be estimated by

$$\|U_{T,S}\| \geq \frac{2}{3}\|T\|\|S\|.$$

In 1994, Cabrera and Rodriguez [3], proved that

$$\|U_{T,S}\| \geq \frac{1}{20412}\|T\|\|S\|,$$

for prime JB*-algebras.

On their part, Stacho and Zalar [6], in 1996, worked on the standard operator algebra (which is a sub-algebra of $B(H)$ that contains all finite rank operators). They first showed that the operator $U_{T,S}$ actually represents a Jordan triple structure of a C*-algebra. They also showed that if A is a standard operator algebra acting on a Hilbert space H, and $T, S \in A$, then

$$\|U_{T,S}\| \geq 2(\sqrt{2}-1)\|T\|\|S\|.$$

They later (1998), proved that

$$\|U_{T,S}\| \geq \|T\|\|S\|$$

for the algebra of symmetric operators acting on a Hilbert space. They attached a family of Hilbert spaces to standard operator algebra and used the inner products in them to obtain their results.

Barraa and Boumazguor [2], in the year 2001 used the concept of the numerical range of T relative to S, denoted by $W_S(T^*S)$, to obtain their results. They employed the idea of finite rank operators to prove the following theorem;

Theorem 2.1 *Let H be a complex Hilbert space and $B(H)$ be the algebra of all bounded linear operators on H. If $T, S \in B(H)$ with $S \neq 0$, then;*

$$\|U_{T,S}\| \geq \sup_{\lambda \in W_S(T^*S)}\left\{\left\|\|S\|T + \frac{\bar{\lambda}}{\|S\|}S\right\|\right\}.$$

As a consequent of this, they proved the following corollary;

Corollary 2.2 *Let H be a complex Hilbert space and T, S be bounded linear operators on H. If $0 \in W_S(T^*S) \cup W_T(S^*T)$, then;*

$$\|U_{T,S}\| \geq \|T\|\|S\|.$$

They also proved the following proposition;

Proposition 2.3 *Let H be a complex Hilbert space and T, S be bounded linear operators on H. If $\|T\|\|S\| \in W_T(S^*T) \cap \in W_{T^*}(ST^*)$*, then;

$$\|U_{T,S}\| = 2\|T\|\|S\|.$$

Proves to theorems 2.1, 2.2 and 2.3, can be obtained from [2].

3. Norm of Elementary Operator of Length Two

Kingangi et al [4] in 2014 used finite rank operators to determine the norm of the elementary operator E_2. Below is the theorem they proved (see theorem 2.5):

Theorem 3.1 *Let H be a complex Hilbert space and $B(H)$ be the algebra of all bounded linear operators on H. Let E_2 be the elementary operator on $B(H)$ defined above. If for an operator $W \in B(H)$ with $\|W\| = 1$ we have $W(x) = x$ for all unit vectors $x \in H$, then;*

$$\|E_2\| = \sum_{i=1}^{2}\|T_i\|\|S_i\|,$$

for $i = 1,2$.

For the proof of theorem 3.1, see King' ang'i [4], theorem 2.5.

Odero *et al* [7], determined the norm of tensor product elementary operator. They showed that for an inner derivation Δ_T, we have $\|\Delta_T\| = 2\|T\|$ if and only if $0 \in W(T)$. They also proved that if $S, T \in B(H)$, then we have;

$$\|E_2\| = sup\{\|TX - XS\| : X \in B(H), \|X\| = 1\}.$$

Jocic *et al* [8], proved that if $\{T_\alpha\}_{\alpha\in\Lambda}$ and $\{S_\alpha\}_{\alpha\in\Lambda}$ are weakly*-measurable families of bounded Hilbert space operators such that transfers $X \longmapsto \int_\Lambda T_\alpha^* X T_\alpha d\mu(\alpha)$ and $X \longmapsto \int_\Lambda S_\alpha^* X S_\alpha d\mu(\alpha)$ on $B(H)$ have their spectra contained in the unit disc, then for all bounded operators X, we have;

$$\|\Delta_T X \Delta_S\| \leq \left\|X - \int_\Lambda T_\alpha^* X S_\alpha d\mu(\alpha)\right\|,$$

where

$$\Delta_T = S - \lim_{r \nearrow 1}\left(I + \sum_{n=1}^{\infty} r^{2n} \int_\Lambda \dots \int_\Lambda \left|T_{\alpha_1} \dots T_{\alpha_n}\right|^2 d\mu^n(\alpha_1, \dots, \alpha_n)\right)^{-\frac{1}{2}}$$

and Δ_S by analogy.

Wafula, *et al* [9], considered normally represented elementary operators. They proved that the norm of an elementary operator is equal to the largest singular value of the operator itself. They also proved that, if the Jordan elementary operator $U_{T,S}$, we have;

$$\|U_{T,S}\|_{inj} \geq 2\sqrt{2-l}\|T\|\|S\|,$$

where $S, T \in B(H)$.

In 2017, King'ang'i [1] employed the concept of the maximal numerical range of T^*S relative to S to determine the lower norm of an elementary operator of length two. He proved the following theorem (see theorem 3.1):

Theorem 3.2. *Let E_2 be an elementary operator of length two on $B(H)$.* Then,

$$\|E_2\| \geq \sup_{\lambda \in W_{S_1}(S_2{}^*S_1)} \left\{ \left\| \|S_1\|T_1 + \frac{\bar{\lambda}}{\|S_1\|}T_2 \right\| \right\},$$

where S_i, T_i are fixed elements of $B(H)$ for $i = 1,2$.

He also determined the conditions on which the norm of E_2 is expressible in terms of the norms of its coefficient operators by proving the following theorems (see corollary 3.2 and theorem 3.3):

Corollary 3.3 *Let H be a complex Hilbert space and T_i, S_i be bounded linear operators on H* for $i = 1,2$. Let $0 \in W_{S_1}(S_2^*S_1) \cup W_{S_2}(S_1^*S_2)$. Then, $\|E_2\| \geq \|T_1\|\|S_1\|$, *where E_2 is as defined earlier.*

Theorem 3.4 *Let H be a complex Hilbert space and T_i, S_i be bounded linear operators on H for $i = 1,2$. Let E_2 be an elementary operator of length two.* If $\|T_1\|\|T_2\| \in W_{T_1^*}(T_2T_1^*)$ and $\|S_1\|\|S_2\| \in W_{S_2}(S_1^*S_2)$, then, $\|E_2\| = \sum_{i=1}^{2}\|T_i\|\|S_i\|$.

Below, we present more results on the norm of this operator by employing the concept of theStampfli's maximal numerical range. In theorem 3.5, we determine the lower bound of the norm of the operator E_2 while in theorem 3.6 we determine the conditions necessary to express the norm of E_2 in the form $\|E_2\| = \sum_{i=1}^{2}\|T_i\|\|S_i\|$.

Theorem 3.5 *Let E_2 be an elementary operator on $B(H)$ and $S_1, S_2 \in B(H)$. If $\lambda_i \in W_0(S_i)$ for each $\lambda_i \in \mathbb{C}$, $i = 1,2$, then we have* $\|E_2\| \geq \sup_{\lambda_i \in W_0(S_i)}\{\|\sum_{i=1}^{2}\lambda_i T_i\| : T_i \in B(H), i = 1,2\}$.

Proof. Let $\{x_n\}_{n\geq 1}$ be a sequence of unit vectors in a Hilbert space H and $y \otimes x_n \in B(H)$ be a rank-one operator on H for a unit vector in $y \in H$, defined by $(y \otimes x_n)(x) = \langle x, x_n \rangle y$ for all $x \in H$. Recall the Stampfli' smaximal numerical range of $T \in B(H)$ is the set,

$$W_0(T) = \{\lambda \in \mathbb{C} : \langle Tx_n, x_n \rangle \longrightarrow \lambda, \|x_n\| = 1, \|Tx_n\| \longrightarrow \|T\|\},$$

If $\lambda_1 \in W_0(S_1)$ and $\lambda_2 \in W_0(S_2)$, then there are sequences $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ of unit vectors in H such that $\lim_{n\to\infty}\langle S_1x_n, x_n\rangle = \lambda_1$, $\lim_{n\to\infty}\|S_1x_n\| = \|S_1\|$ and $\lim_{n\to\infty}\langle S_2x_n, x_n\rangle = \lambda_2$, $\lim_{n\to\infty}\|S_2x_n\| = \|S_2\|$.

Now, we have;

$$\|(E_2(y\otimes x_n))x_n\| = \left\|\left(\sum_{i=1}^{2} M_{T_i,S_i}(y\otimes x_n)\right)x_n\right\|$$

$$\leq \left\|\sum_{i=1}^{2} M_{T_i,S_i}(y\otimes x_n)\right\| \|x_n\|$$

$$\leq \left\|\sum_{i=1}^{2} M_{T_i,S_i}\right\| \|y\otimes x_n\|$$

$$\leq \left\|\sum_{i=1}^{2} M_{T_i,S_i}\right\| \|y\|\|x_n\|$$

$$= \left\|\sum_{i=1}^{2} M_{T_i,S_i}\right\|$$

Therefore,

$$\left\|\sum_{i=1}^{2} M_{T_i,S_i}\right\| \geq \left\|\left(\sum_{i=1}^{2} M_{T_i,S_i}(y\otimes x_n)\right)x_n\right\|$$

$$= \|(T_1(y\otimes x_n)S_1 + T_2(y\otimes x_n)S_2)x_n\|$$

$$= \|T_1(y\otimes x_n)S_1x_n + T_2(y\otimes x_n)S_2x_n\|$$

$$= \|\langle S_1x_n, x_n\rangle T_1y + \langle S_2x_n, x_n\rangle T_2y\|$$

Taking limits as $n \longrightarrow \infty$, we obtain;

$$\left\|\sum_{i=1}^{2} M_{T_i,S_i}\right\| \geq \|\lambda_1T_1y + \lambda_2T_2y\| = \left\|\sum_{i=1}^{2}\lambda_iT_iy\right\|,$$

and this is true for any $\lambda_1 \in W_0(S_1)$ and $\lambda_2 \in W_0(S_2)$, and for any unit vector $y \in H$.

Since $\lambda_i \in W_0(S_i)$ for $i = 1,2$, and $y \in H$ are arbitrarily chosen, then we obtain;

$$\left\|\sum_{i=1}^{2} M_{T_i,S_i}\right\| \geq \sup_{\lambda_i \in W_0(S_i)}\{\sup_{\|y\|=1}\{\|\sum_{i=1}^{2}\lambda_iT_iy\|\}\} = \sup_{\lambda_i \in W_0(S_i)}\{\|\sum_{i=1}^{2}\lambda_iT_i\|\}.$$

Thus, $\left\|\sum_{i=1}^{2} M_{T_i,S_i}\right\| \geq \sup_{\lambda_i \in W_0(S_i)}\{\|\sum_{i=1}^{2}\lambda_iT_i\|\}$, or $\|E_2\| \geq \sup_{\lambda_i \in W_0(S_i)}\{\|\sum_{i=1}^{2}\lambda_iT_i\|\}$, and this completes the proof.

In the next theorem, the condition necessary for the norm of the elementary operator E_2 to be equal to the sum of the product of the norms of the corresponding coefficient operators in its definition is given.

Theorem 3.6 Let E_2 be an elementary operator on $B(H)$ and $S_i, T_i \in B(H)$ for $i = 1,2$. If $\|S_i\| \in W_0(S_i)$ and $\|T_i\| \in W_0(T_i)$ for $i = 1,2$, then;

$$\|E_2\| = \sum_{i=1}^{2}\|T_i\|\|S_i\|.$$

Proof. Let $\{x_n\}_{n>1}$ be a sequence of unit vectors on H and $x_n \otimes x_n \in B(H)$ be a rank one operator on H for a unit vector $x_n \in H$, defined by $(x_n\otimes x_n)(y) = \langle y, x_n\rangle x_n$, for all y$\in H$.

If $S_i \in W_0(S_i)$ and $T_i \in W_0(T_i)$, for $i = 1,2$, then there is a sequence $\{x_n\}_{n\geq 1}$ of unit vectors in H such that $\lim_{n\to\infty}\langle S_i x_n, x_n\rangle = S_i$, $\lim_{n\to\infty}\|S_i x_n\| = \|i\|$ for $i = 1,2$, and there is a sequence $\{y_n\}_{n>1}$ of unit vectors in H such that $\lim_{n\to\infty}\langle T_i y_n, y_n\rangle = T_i$, $\lim_{n\to\infty}\|T_i y_n\| = \|T_i\|$, for $i = 1,2$.

Now, we have;

$$\|(E_2(x_n\otimes x_n))x_n\| = \left\|\left(\sum_{i=1}^{2} M_{T_i,S_i}(x_n\otimes x_n)\right)x_n\right\|$$

$$\leq \left\|\sum_{i=1}^{2} M_{T_i,S_i}(x_n\otimes x_n)\right\| \|x_n\|$$

$$\leq \left\|\sum_{i=1}^{2} M_{T_i,S_i}\right\| \|x_n\otimes x_n\|$$

$$\leq \left\|\sum_{i=1}^{2} M_{T_i,S_i}\right\| \|x_n\|\|x_n\|$$

$$= \left\|\sum_{i=1}^{2} M_{T_i,S_i}\right\|.$$

So, we have;

$$\left\|\sum_{i=1}^{2} M_{T_i,S_i}\right\|^2 \geq \left\|\left(\sum_{i=1}^{2} M_{T_i,S_i}(x_n\otimes x_n)\right)x_n\right\|^2$$

$$= \|(T_1(x_n\otimes x_n)S_1 + T_2(x_n\otimes x_n)S_2)x_n\|^2$$

$$= \|T_1(x_n\otimes x_n)S_1x_n + T_2(x_n\otimes x_n)S_2x_n\|^2$$

$$= \|\langle S_1x_n, x_n\rangle T_1x_n + \langle S_2x_n, x_n\rangle T_2x_n\|^2$$

$$= \langle\langle S_1x_n, x_n\rangle T_1x_n + \langle S_2x_n, x_n\rangle T_2x_n, \langle S_1x_n, x_n\rangle T_1x_n + \langle S_2x_n, x_n\rangle T_2x_n\rangle$$

$$= \|\langle S_1x_n, x_n\rangle T_1x_n\|^2 + \|\langle S_2x_n, x_n\rangle T_2x_n\|^2 + 2Re\langle\langle S_2x_n, x_n\rangle T_2x_n, \langle S_1x_n, x_n\rangle T_1x_n\rangle$$

$$=|\langle S_1x_n, x_n\rangle|^2\|T_1x_n\|^2 + |\langle S_2x_n, x_n\rangle|^2\|T_2x_n\|^2 + 2Re\{\langle S_2x_n, x_n\rangle\langle T_2x_n, T_1x_n\rangle\langle S_1x_n, x_n\rangle\}.$$

Now, by the CBS inequality, we have that;

$$|\langle T_2x_n, T_1x_n\rangle| \leq \|T_2x_n\|\|T_1x_n\| \leq \|T_2\|\|T_1\|,$$

and hence,

$$\lim_{n\to\infty}\langle T_2x_n, T_1x_n\rangle = \|T_2\|\|T_1\|.$$

Therefore, taking limits as $n\to\infty$, we obtain;

$$\left\|\sum_{i=1}^{2} M_{T_i,S_i}\right\|^2 \geq \|S_1\|^2\|T_1\|^2 + \|S_2\|^2\|T_2\|^2 + 2\|S_2\|^2\|T_2\|^2\|S_1\|^2\|T_1\|^2.$$

The right hand side is equal to $(\sum_{i=1}^{2}\|S_i\|\|T_i\|)^2$. Thus we have;

$$\sum_{i=1}^{2} M_{T_i,S_i} \geq \sum_{i=1}^{2}\|S_i\|\|T_i\|.$$

Now, since the inequality $\sum_{i=1}^{2} M_{T_i,S_i} \leq \sum_{i=1}^{2}\|S_i\|\|T_i\|$ always hold (by the triangular inequality), then we obtain $\sum_{i=1}^{2} M_{T_i,S_i} = \sum_{i=1}^{2}\|S_i\|\|T_i\|$.

4. Conclusions

In this paper, we have determined the lower bound of the norm of an elementary operator of length two in a C*-algebra $B(H)$ and using the Stampfli's maximal numerical range. The conditions in which this norm is equal to the sum of the products of the corresponding coefficient operators has also been considered. One may attempt this problem for an elementary operator consisting of more that two terms.

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