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Oscillations of Neutral Difference Equations of Second Order with Positive and Negative Coefficients

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Abstract: In this paper some necessary and sufficient conditions are obtained to guarantee the oscillation for bounded and all solutions of second order nonlinear neutral delay difference equations. In Theorem 5 and Theorem 8, We have studied the oscillation criteria as well as the asymptotic behavior, where was established some sufficient conditions to ensure that every solution are either oscillates or $|y_n| \to \infty$ as $n \to \infty$. Examples are given to illustrate the obtained results.

Keywords: Oscillation, Neutral Difference Equations, Second Order Difference Equations

1. Introduction

In this paper the oscillation for bounded and all solutions of second order neutral delay difference equation with positive and negative coefficients:

$$\Delta^2(y_n - p_n y_{n-m}) + q_n G(y_{n-k}) - r_n G(y_{n-l}) = f_n \qquad (1)$$

will be studied, where Δ is the forward difference operator, q_n, r_n are nonnegative infinite sequences of real numbers and f_n, p_n, are infinite sequences of real numbers. $G \in (R, R)$ is function $y_n G(y_n) > 0$. The purpose of this research is to obtain new sufficient conditions for the oscillation of all solutions of equation (1). The following assumptions are used:

(H$_1$) $\displaystyle\sum_{j=n_1}^{\infty}\sum_{i=j+l-k}^{j-1} r_i < \infty$;

(H$_2$) $\displaystyle\sum_{j=n_1}^{\infty}\sum_{i=j-l+k}^{j-1} q_i < \infty$;

(H$_3$) There exists a sequence $\{F_n\}$ such that $\Delta^2 F_n = f_n$ and $\lim_{n\to\infty} F_n = 0$;

(H$_4$) $G(y) \ge \beta_1 y$;

(H$_5$) $G(y) \le \beta_2 y$.

2. Main Result

The next results provide sufficient conditions for the oscillation of all bounded solutions of Eq. (1). For a simplicity set

$$z_n = y_n - p_n y_{n-m} \qquad (2)$$

Let the sequence w_n be defined as

$$w_n = y_n - p_n y_{n-m} + \sum_{j=n}^{\infty}\sum_{i=j+l-k}^{j-1} r_i G(y_{i-l}) - F_n, \; k > l \qquad (3)$$

and the sequence W_n be defined as

$$W_n = y_n - p_n y_{n-m} - \sum_{j=n}^{\infty}\sum_{i=j-l+k}^{j-1} q_i G(y_{i-k}) - F_n, l > k \qquad (4)$$

The following theorem based on Theorem 7.6.1, [3] pp. 184:

Theorem 1. ([3], pp. 184)

Assume that $\{p_n\}$ is a nonnegative sequence of real numbers and let k be a positive integer. Suppose that

$$\liminf_{n\to\infty} \sum_{i=n-k}^{n-1} p_i > \frac{k^{k+1}}{(k+1)^{k+1}}$$

Then

i. The difference inequality

$$y_{n+1} - y_n + p_n y_{n-k} \le 0, \qquad n = 0,1,2,\cdots$$

cannot have eventually positive solutions.

ii. The difference inequality

$$y_{n+1} - y_n + p_n y_{n-k} \geq 0, n = 0,1,2,\cdots$$

cannot have eventually negative solutions.

Theorem 2. ([12], pp.10) Let $x, y \in R$ then

(i) $x < y + \varepsilon$ for all $\varepsilon > 0$ if and only if $x \leq y$.

(ii) $x > y - \varepsilon$ for all $\varepsilon > 0$ if and only if $x \geq y$.

Theorem 3. Suppose that $p_n \geq 1$ is bounded $(r_{n+l-k} - q_n) \leq 0$, let (H$_1$), (H$_3$), and (H$_4$) hold, in addition to

$$\liminf_{n\to\infty} \sum_{j=n-(k-\rho-m)}^{n-1} \sum_{i=j}^{j+\rho} \frac{|r_{i+l-k} - q_i|}{p_{i-k+m}} > \frac{(k-\rho-m)^{k-\rho-m+1}}{\beta_1 (k-\rho-m+1)^{k-\rho-m+1}},$$

$$k > \rho + m \tag{5}$$

Then every bounded solution of equation (1) oscillates.

Proof. Assume for the sake of contradiction that $\{y_n\}$ be positive bounded solution of eq. (1) for $n \geq n_0 \geq 0$, then from equations (1), (2) and (3) we obtain

$$\Delta^2 w_n = (r_{n+l-k} - q_n) G(y_{n-k}) \leq 0 \tag{6}$$

Hence, $\Delta w_n, w_n$ are monotone sequences. We claim that $\Delta w_n > 0$ for $n \geq n_1 \geq n_0$, otherwise, $\Delta w_n < 0$, $n \geq n_1$, thus, $w_n < 0$ and $w_n \to -\infty$ as $n \to \infty$. From (3) we get

$$w_n \geq -p_n y_{n-m} - F_n$$

$$\geq -p y_{n-m} - F_n, p_n \leq p$$

then $y_n \to \infty$ as $n \to \infty$, which is a contradiction. Hence our claim is established. We have two cases for $n \geq n_2 \geq n_1$:

Case 1: $w_n > 0, \Delta w_n > 0, \Delta^2 w_n \leq 0$; Case 2: $w_n < 0, \Delta w_n > 0, \Delta^2 w_n \leq 0$

Case 1: $w_n > 0$, $\Delta w_n > 0, \Delta^2 w_n \leq 0$, then there exists $\gamma > 0$ such that $w_n \geq \gamma > 0$ for $n \geq n_2 \geq n_1$. Since y_n is bounded, let $\liminf_{n\to\infty} y_n = h_*$ $h_* \geq 0$, so there exists a subsequence $\{n_j\}$ of $\{n\}$ such that $\lim_{j\to\infty} n_j = \infty$, $\lim_{j\to\infty} y_{n_j} = h_*$. From (3) we get

$$y_{n_j} = w_{n_j} + p_{n_j} y_{n_j-m} - \sum_{s=n_j}^{\infty} \sum_{i=s+l-k}^{s-1} r_i G(y_{i-l}) + F_{n_j}$$

$$y_{n_j} \geq \gamma + p_{n_j} y_{n_j-m} - \delta_2 \sum_{s=n_j}^{\infty} \sum_{i=s+l-k}^{s-1} r_i + F_{n_j}, G(y_n) \leq \delta_2$$

$$y_{n_j} > \gamma + y_{n_j-m} - \varepsilon$$

Since ε is arbitrary, by Theorem 2, it follows that for sufficiently large j we get

$$y_{n_j} \geq \gamma + y_{n_j-m}$$

As $j \to \infty$, it follows that $h_* \geq \gamma + h_*$ which is a contradiction.

Case 2: $w_n < 0$, $\Delta w_n > 0$, $\Delta^2 w_n \leq 0$. By taking the summation of both sides of (6) from n to $n+\rho$, $0 < \rho < k - m$, we get

$$\Delta w_{n+1} - \Delta w_n = \sum_{i=n}^{n+\rho} (r_{i+l-k} - q_i) G(y_{i-k})$$

$$-\Delta w_n \leq \beta_1 \sum_{i=n}^{n+\rho} (r_{i+l-k} - q_i) y_{i-k}$$

$$\Delta w_n \geq \beta_1 \sum_{i=n}^{n+\rho} |r_{i+l-k} - q_i| \, y_{i-k} \tag{7}$$

From (3) we get

$$w_n = y_n - p_n y_{n-m} + \sum_{j=n}^{\infty} \sum_{i=j+l-k}^{j-1} r_i G(y_{i-l}) - F_n$$

$$\geq -p_n y_{n-m} - F_n$$

$$w_n > -p_n y_{n-m} - \varepsilon, \varepsilon > 0$$

Since ε is arbitrary, it follows that

$$w_n \geq -p_n y_{n-m}$$

$$y_{n-m} \geq -\frac{1}{p_n} w_n$$

$$y_{n-k} \geq -\frac{1}{p_{n-k+m}} w_{n-k+m} \tag{8}$$

Substituting (8) in (7) to obtain

$$\Delta w_n \geq -\beta_1 \sum_{i=n}^{n+\rho} \frac{|r_{i+l-k} - q_i|}{p_{i-k+m}} w_{i-k+m}$$

$$\Delta w_n + \beta_1 \sum_{i=n}^{n+\rho} \frac{|r_{i+l-k} - q_i|}{p_{i-k+m}} w_{n-(k-\rho-m)} \geq 0$$

By theorem 1-ii and in virtue of (5), it follows that the last inequality cannot have eventually negative solution, which is a contradiction.

Example 4. Consider the difference equations

$$\Delta^2 \left(y_n - \left(1 + \left(\frac{1}{2}\right)^n \right) y_{n-1} \right) + \frac{27}{32} y_{n-3} - \frac{25}{32} \left(\frac{1}{2}\right)^n y_{n-2}$$

$$= \left(\frac{1}{2}\right)^{n+3} \left(-\frac{1}{2}\right)^n \tag{E1}$$

where m =1, k=3, l= 2, $\rho = 1$, $p_n = 1 + \left(\frac{1}{2}\right)^n, q_n = \frac{27}{32}$, $r_n = \frac{25}{32}\left(\frac{1}{2}\right)^n$, $G(y_n) = y_n, \beta_1 = 1, f_n = \left(\frac{1}{2}\right)^{n+3}\left(-\frac{1}{2}\right)^n$

- $\sum_{j=n_1}^{\infty} \sum_{i=j+l-k}^{j-1} r_i = \frac{25}{32} \sum_{j=n_1}^{\infty} \left(\frac{1}{2}\right)^{i-1} = \frac{25}{16} < \infty$
- $p_n = 1 + \left(\frac{1}{2}\right)^n \geq 1.$
- $r_{n+l-k} - q_n = 25\left(\frac{1}{2}\right)^{n+3} - \frac{27}{32} < 0, n \geq 1.$
- $\liminf_{n\to\infty} \sum_{j=n-1}^{n-1} \sum_{i=j}^{j+1} \frac{|r_{i+l-k} - q_i|}{p_{i-k+m}} =$

  $\lim_{n\to\infty} \sum_{n-1}^{n-1} \sum_{i=j}^{j+1} \frac{\frac{27}{32} - 25\left(\frac{1}{2}\right)^{i+3}}{1 + \left(\frac{1}{2}\right)^{i-2}} = \frac{27}{16} > \frac{1}{4}$

By Theorem 3, it follows that every bounded solution of $(E1)$ oscillates, for instance $y_n = \left(-\frac{1}{2}\right)^n$ is such a solution.

Theorem 5. Suppose that $p_n \leq p < 1, (r_{n+l-k} - q_n) \leq 0$, (H_1), $(\mathrm{H}_3) - (\mathrm{H}_5)$ hold, in addition to

$$\sum_{i=n_0}^{\infty} q_i = \infty, n_0 \geq 0 \tag{9}$$

$$\liminf_{n\to\infty} \sum_{j=n-(k-\rho-m)}^{n-1} \sum_{i=j}^{j+\rho} |r_{i+l-k} - q_i|$$

$$> \frac{p(k-\rho-m)^{k-\rho-m+1}}{\beta_1 (k-\rho-m+1)^{k-\rho-m+1}}, k > \rho + m \tag{10}$$

Then every solution $\{y_n\}$ of equation (1) either oscillates or $|y_n| \to \infty$ as $n \to \infty$.

Proof. For the sake of contradiction, assume that $\{y_n\}$ be an eventually positive solution of eq. (1), then from equations (1), (2) and (3) it follows that (6) hold, that is

$$\Delta^2 w_n = (r_{n+l-k} - q_n) G(y_{n-k}) \leq 0$$

Hence Δw_n, w_n are monotone sequences. If $\Delta w_n < 0$ for $n \geq n_1 \geq n_0$, thus $w_n < 0$ and $w_n \to -\infty$ as $n \to \infty$. From (3) we obtain

$$w_n = y_n - p_n y_{n-m} + \sum_{j=n}^{\infty} \sum_{i=j+l-k}^{j-1} r_i G(y_{i-l}) - F_n,$$

$$w_n \geq -p_n y_{n-m} - F_n \geq -p y_{n-m} - F_n$$

which implies that $y_n \to \infty$ as $n \to \infty$.

If $\Delta w_n > 0$ for $n \geq n_1 \geq n_0$, we have two cases to consider for $n \geq n_2 \geq n_1$:

Case 1: $w_n > 0$, $\Delta w_n > 0, \Delta^2 w_n \leq 0$; Case 2: $w_n < 0$, $\Delta w_n > 0, \Delta^2 w_n \leq 0$

Case 1: $w_n > 0$, $\Delta w_n > 0, \Delta^2 w_n \leq 0$. Then $\lim_{n\to\infty} w_n = L$, where $0 < L \leq \infty$.

If $L = \infty$, From (3) we get

$$w_n = y_n - p_n y_{n-m} + \sum_{j=n}^{\infty} \sum_{i=j+l-k}^{j-1} r_i G(y_{i-l}) - F_n,$$

$$w_n \leq y_n + \sum_{j=n}^{\infty} \sum_{i=j+l-k}^{j-1} r_i G(y_{i-l}) - F_n$$

which implies that $\lim_{n\to\infty} y_n = \infty$, otherwise if y_n is bounded it follows from the last inequality $w_n < y_n + \varepsilon$, which is a contradiction.

If $0 < L < \infty$, then there exists $\gamma > 0$ such that $w_n \geq \gamma > 0$, for $n \geq n_2$. If $\lim_{n\to\infty} y_n < \infty$, then from (3) we get

$$w_n \leq y_n + \beta_2 \sum_{j=n}^{\infty} \sum_{i=j+l-k}^{j-1} r_i y_{i-k} - F_n$$

$$\leq y_n + \beta_2 \delta_1 \sum_{j=n}^{\infty} \sum_{i=j+l-k}^{j-1} r_i - F_n, y_n \leq \delta_1$$

$$w_n < y_n + \varepsilon, \varepsilon > 0$$

ε is arbitrary, so for sufficiently large n we get

$$y_n \geq w_n \geq \gamma > 0$$

By taking the summation of both sides of (6) from n_2 to $n-1$, it follows that

$$\sum_{i=n_2}^{n-1} \Delta w_{i+1} - \Delta w_i = \sum_{i=n_2}^{n-1} (r_{i+l-k} - q_i) G(y_{i-k})$$

$$\Delta w_n - \Delta w_{n_2} \leq \beta_1 \sum_{i=n_2}^{n-1} (r_{i+l-k} - q_i) y_{i-k}$$

$$\leq \gamma \beta_1 \sum_{i=n_2}^{n-1} (r_{i+l-k} - q_i)$$

$$-w_n - \Delta w_{n_2} \leq \gamma \beta_1 \sum_{i=n_2}^{n-1} (r_{i+l-k} - q_i)$$

In virtue of (9) the last inequality implies that $\lim_{n\to\infty} w_n = \infty$. Leads to a contradiction.

Case 2: $w_n < 0$, $\Delta w_n > 0, \Delta^2 w_n \leq 0$. In this case w_n is bounded, we claim that y_n is bounded, otherwise there exists a subsequence $\{n_j\}$ of $\{n\}$ such that $\lim_{j\to\infty} n_j = \infty$, $\lim_{j\to\infty} y_{n_j} = \infty$ and $y_{n_j} = \max\{y_n : n_2 \leq n \leq n_j\}$, from (3) we get

$$w_{n_j} \geq y_{n_j} - p_{n_j} y_{n_j - m} - F_{n_j}$$

$$w_{n_j} \geq (1-p) y_{n_j} - F_{n_j}$$

which implies that $\lim_{j\to\infty} w_{n_j} = \infty$, a contradiction.

By taking the summation of both sides of (6) from n to $n+\rho$, $\rho < k - m$, it follows that

$$\sum_{i=n}^{n+\rho} \Delta w_{i+1} - \Delta w_i = \sum_{i=n}^{n+\rho} (r_{i+l-k} - q_i) G(y_{i-k}) - \Delta w_n \leq$$

$$\beta_1 \sum_{i=n}^{n+\rho} (r_{i+l-k} - q_i) y_{i-k} \tag{11}$$

From (3) we get

$$w_n = y_n - p_n y_{n-m} + \sum_{j=n}^{\infty} \sum_{i=j+l-k}^{j-1} q_i G(y_{i-k}) - F_n$$

$$\geq -p_n y_{n-m} - F_n$$

$$w_n > -p y_{n-m} - \varepsilon, \varepsilon > 0$$

Since ε is arbitrary, it follows that for sufficiently large n:

$$y_n \geq \frac{-1}{p} w_{n+m}$$

Substituting the last inequality in (11) we obtain

$$-\Delta w_n \leq \frac{-\beta_1}{p} \sum_{i=n}^{n+\rho} (r_{i+l-k} - q_i) w_{i+m-k}$$

$$\leq \frac{\beta_1}{p}\left(\sum_{i=n}^{n+\rho}|r_{i+l-k}-q_i|\right)w_{n+\rho+m-k}$$

$$\Delta w_n+\frac{\beta_1}{p}\left(\sum_{i=n}^{n+\rho}|r_{i+l-k}-q_i|\right)w_{n-(k-\rho-m)}\geq 0$$

By theorem 1-ii and in virtue of (10), it follows that the last inequality cannot have eventually negative solution, which is a contradiction.

In the next theorem we will use the sequence W_n already defined in (4).

Theorem 6. Suppose that $p_n\leq 1$, $(r_n-q_{n-l+k})\geq 0$, $(\mathrm{H}_2)-(\mathrm{H}_3)$, and (H_5) hold, in addition to

$$\liminf_{n\to\infty}\sum_{j=n-(l-\rho)}^{n-1}\sum_{i=j}^{j+\rho}(r_i-q_{i-l+k})>\frac{(l-\rho)^{l-\rho+1}}{\beta_1\,(l-\rho+1)^{l-\rho+1}},\quad l>\rho \tag{12}$$

Then every bounded solution of equation (1) oscillates.

Proof. For the sake of contradiction, assume that $\{y_n\}$ be an eventually positive bounded solution of eq. (1), then from equations (1), (2) and (4) we obtain

$$\Delta^2W_n=(r_n-q_{n-l+k})G(y_{n-l})\geq 0 \tag{13}$$

Hence, $\Delta W_n, W_n$ are monotone sequences, we claim that $\Delta W_n<0$ for $n\geq n_1\geq n_0$, otherwise $\Delta W_n>0$ for $n\geq n_1$, hence $W_n>0$ and $W_n\to\infty$ as $n\to\infty$. Let $y_n\leq\delta_1$, then $G(y_n)\leq\beta_2\delta_1=\delta_2$, where δ_1,δ_2 are positive constants. From (4) we obtain

$$W_n=y_n-p_ny_{n-m}-\sum_{j=n}^{\infty}\sum_{i=j-l+k}^{j-1}q_iG(y_{i-k})-F_n$$

$$W_n\leq y_n-F_n$$

which implies that $y_n\to\infty$ as $n\to\infty$, which is a contradiction. Our claim has been established, then it remains to consider two possible cases for the existence of nonoscillatory solution of eq. (1) for $n\geq n_2\geq n_1$:

Case 1: $W_n<0,\ \Delta W_n<0, \Delta^2W_n\geq 0$; Case 2: $W_n>0,\ \Delta W_n<0,\Delta^2W_n\geq 0$;

Case 1: $W_n<0,\ \Delta W_n<0,\ \Delta^2W_n\geq 0$. Then there exists $\gamma<0$ such that, $W_n\leq\gamma<0$, for $n\geq n_2$. Since y_n is bounded, let $\limsup_{n\to\infty}y_n=h^*$, $h^*\geq 0$ so there exists a sequence $\{n_j\}$ such that $\lim_{j\to\infty}n_j=\infty$, $\lim_{j\to\infty}y_{n_j}=h^*$. From (4) we get

$$y_n=W_n+p_ny_{n-m}+\sum_{j=n}^{\infty}\sum_{i=j+k-l}^{j-1}q_iG(y_{i-k})+F_n$$

$$y_n\leq\gamma+p_ny_{n-m}+\delta_2\sum_{j=n}^{\infty}\sum_{i=j-l+k}^{j-1}q_i+F_n$$

$$y_n<\gamma+y_{n-m}+\varepsilon,\varepsilon>0$$

Since ε is arbitrary, then by theorem 2.2, it follows for sufficiently large j that:

$$y_{n_j}\leq\gamma+y_{n_j-m}$$

as $j\to\infty$, we get from the last inequality $h^*\leq\gamma+h^*$ which is a contradiction.

Case 2: $W_n>0,\ \Delta W_n<0,\Delta^2W_n\geq 0$. By taking the summation of both sides of (13) from n to $n+\rho$, $\rho<l$ it follows that

$$\Delta W_{n+1}-\Delta W_n=\sum_{i=n}^{n+\rho}(r_i-q_{i-l+k})G(y_{n-l})-\Delta W_n$$

$$\geq\beta_1\textstyle\sum_{i=n}^{n+\rho}(r_i-q_{i-l+k})\,y_{i-l} \tag{14}$$

From (4) we get

$$y_n=W_n+p_ny_{n-m}+\sum_{j=n}^{\infty}\sum_{i=j-l+k}^{j-1}q_iG(y_{i-k})+F_n$$

$$\geq W_n+F_n$$

$$y_n>W_n-\varepsilon,\varepsilon>0$$

Since ε is arbitrary, it follows that

$$y_n\geq W_n$$

Substituting the last inequality in (14) we obtain

$$-\Delta W_n\geq\beta_1\sum_{i=n}^{n+\rho}(r_i-q_{i-l+k})\,W_{i-l}$$

$$-\Delta W_n\geq\beta_1\left(\sum_{i=n}^{n+\rho}(r_i-q_{i-l+k})\right)W_{n+\rho-l}$$

$$\Delta W_n+\beta_1\left(\sum_{i=n}^{n+\rho}(r_i-q_{i-l+k})\right)W_{n-(l-\rho)}\leq 0$$

By Theorem 1-i, and in virtue of (12), it follows that the last inequality cannot have eventually positive solution, which is a contradiction.

Example 7. Consider the difference equation

$$\Delta^2\left(y_n-\left(1+\left(\frac{1}{e}\right)^n\right)y_{n-m}\right)+\left(\frac{1}{e}\right)^n\left(-\frac{1}{e}\right)^{n-1}$$

$$-[e^{-4}(1+2e+e^2)]\left(-\frac{1}{e}\right)^{n-2}$$

$$=(e^{-3}+2e^{-1})\left[\left(\tfrac{1}{e}\right)^n\left(-\tfrac{1}{e}\right)^n\right] \tag{E2}$$

where *k=1, m=1, l = 2*, $\rho=1$, $\beta_1=1$, $p_n=1+\left(\frac{1}{e}\right)^n$, $q_n=\left(\frac{1}{e}\right)^n$,

$$r_n=e^{-4}(1+2e+e^2),$$

$$f_n = (e^{-3} + 2e^{-1})\left[\left(\frac{1}{e}\right)^n\left(-\frac{1}{e}\right)^n\right], G(y_n) = y_n$$

- $\sum_{j=n_1}^{\infty}\sum_{i=j-l+k}^{j-1} q_i = \sum_{j=n_1}^{\infty}\sum_{i=j-1}^{j-1}\left(\frac{1}{e}\right)^i = \sum_{j=n_1}^{\infty}\left(\frac{1}{e}\right)^{j-1} = e\sum_{j=n_1}^{\infty}\left(\frac{1}{e}\right)^j < \infty$
- $p_n = \left(\frac{1}{e}\right)^n < 1, n > 0$
- $r_n - q_{n-l+k} = e^{-4}(1 + 2e + e^2) - \left(\frac{1}{e}\right)^{n-1} = e^{-4}(1 + 2e + e^2) - e\left(\frac{1}{e}\right)^n > 0, n \geq 0.$
- $\liminf_{n\to\infty} \sum_{j=n-(l-\rho)}^{n-1}\sum_{i=j}^{j+\rho}(r_i - q_{i-l+k}) = \liminf_{n\to\infty}\sum_{j=n-1}^{n-1}\sum_{i=j}^{j+1} e^{-4}(1 + 2e + e^2) - \left(\frac{1}{e}\right)^{i-1} = 2e^{-4}(1 + 2e + e^2) > \frac{1}{4}$

By theorem 5, every bounded solution of $(E2)$ oscillates, for instance $y_n = \left(-\frac{1}{e}\right)^n$ is such a solution.

Theorem 8. Suppose that $p_n \leq p, (r_n - q_{n-l+k}) \geq 0$, $(\mathrm{H}_2)-(\mathrm{H}_5)$ hold, in addition to (12) and

$$\sum_{i=n_1}^{\infty} r_i = \infty, n_0 \geq 0 \tag{15}$$

Then every solution $\{y_n\}$ of equation (1) either oscillates or $|y_n| \to \infty$ as $n \to \infty$.

proof. For the sake of contradiction, assume that $\{y_n\}$ be an eventually positive solution of eq. (1), then from equations (1), (2) and (4) it follows that (13) hold, that is

$$\Delta^2 W_n = (r_n - q_{n-l+k})G(y_{n-l}) \geq 0$$

Hence, ΔW_n, W_n are monotone sequences. If $\Delta W_n > 0$ for $n \geq n_1 \geq n_0$, thus $W_n > 0$ and $W_n \to \infty$ as $n \to \infty$. From (4) we obtain

$$W_n = y_n - p_n y_{n-m} - \sum_{j=n}^{\infty}\sum_{i=j-l+k}^{j-1} q_i G(y_{i-k}) - F_n$$

$$W_n \leq y_n - F_n$$

which implies that $y_n \to \infty$ as $n \to \infty$.

If $\Delta W_n < 0$ for $n \geq n_1 \geq n_0$ we have two cases to consider for $n \geq n_2 \geq n_1$:

Case 1: $W_n < 0, \Delta W_n < 0, \Delta^2 W_n \geq 0$; Case 2: $W_n > 0, \Delta W_n < 0, \Delta^2 W_n \geq 0$.

Case 1: $W_n < 0, \Delta W_n < 0, \Delta^2 W_n \geq 0$. Then $\lim_{n\to\infty} W_n = L$, where $-\infty \leq L < 0$.

If $L = -\infty$, From (4) we get

$$W_n = y_n - p_n y_{n-m} - \sum_{j=n}^{\infty}\sum_{i=j-l+k}^{j-1} q_i G(y_{i-k}) - F_n$$

$$W_n \geq -p y_{n-m} - \beta_2 \sum_{j=n}^{\infty}\sum_{i=j-l+k}^{j-1} q_i\, y_{i-k} - F_n$$

which implies that $\lim_{n\to\infty} y_n = \infty$.

If $-\infty < L < 0$, then there exists $\gamma < 0$ such that $W_n \leq \gamma < 0$, for $n \geq n_2$. If $\lim_{n\to\infty} y_n < \infty$. From (4) we get

$$W_n \geq -p_n y_{n-m} - \beta_2 \sum_{j=n}^{\infty}\sum_{i=j-l+k}^{j-1} q_i\, y_{i-k} - F_n$$

$$W_n > -p_n y_{n-m} - \varepsilon, \varepsilon > 0$$

$$\gamma \geq W_n \geq -p_n y_{n-m}$$

$$y_{n-m} \geq -\frac{\gamma}{p}$$

By taking summation to both sides of (13) from n_2 to $n-1$, it follows that

$$\sum_{i=n_2}^{n-1} \Delta W_{i+1} - \Delta W_i = \sum_{i=n_2}^{n-1}(r_i - q_{i-l+k})G(y_{i-l})$$

$$\Delta W_n - \Delta W_{n_2} \geq \beta_1 \sum_{i=n_2}^{n-1}(r_i - q_{i-l+k})\, y_{i-l}$$

$$\geq -\frac{\gamma\beta_1}{p}\sum_{i=n_2}^{n-1}(r_i - q_{i-l+k})$$

$$-W_n - \Delta W_{n_2} \geq \beta_1 \sum_{i=n_2}^{n-1}(r_i - q_{i-l+k})\, y_{i-l}$$

$$\geq -\frac{\gamma\beta_1}{p}\sum_{i=n_2}^{n-1}(r_i - q_{i-l+k})$$

In virtue of (12) the last inequality implies that $\lim_{n\to\infty} W_n = -\infty$. Leads to a contradiction.

Case 2: $W_n > 0, \Delta W_n < 0, \Delta^2 W_n \geq 0$. By taking the summation of both sides of (13) from n to $n+\rho$, $\rho < l$, it follows that

$$\sum_{i=n}^{n+\rho} \Delta W_{i+1} - \Delta W_i = \sum_{i=n}^{n+\rho}(r_i - q_{i-l+k})G(y_{i-l}) - \Delta W_n$$

$$\geq \beta_1 \sum_{i=n}^{n+\rho}(r_i - q_{i-l+k})\, y_{i-l} \tag{16}$$

From (4) we get

$$y_n = W_n + p_n y_{n-m} + \sum_{j=n}^{\infty}\sum_{i=j-l+k}^{j-1} q_i G(y_{i-k}) + F_n$$

$$\geq W_n + F_n$$

$$y_n > W_n - \varepsilon, \varepsilon > 0$$

Since ε is arbitrary, it follows that

$$y_n \geq W_n$$

Substituting the last inequality in (16) we obtain

$$-\Delta W_n \geq \beta_1 \sum_{i=n}^{n+\rho}(r_i - q_{i-l+k})\, W_{i-l}$$

$$-\Delta W_n \geq \beta_1 \left(\sum_{i=n}^{n+\rho} (r_i - q_{i-l+k}) \right) W_{n+\rho-l}$$

$$\Delta W_n + \beta_1 \left(\sum_{i=n}^{n+\rho} (r_i - q_{i-l+k}) \right) W_{n-(l-\rho)} \leq 0$$

By Theorem 1-i and in virtue of (12), it follows that the last inequality cannot have eventually positive solution, which is a contradiction.

3. Conclusion

1. In this paper we used two series w_n and W_n, and obtained necessary and sufficient conditions for every solution of neutral difference equation of second order with positive and negative coefficients, to be oscillates or tends to infinity as $n \to \infty$.
2. In condition (H_3) we can use $\lim_{n\to\infty} F_n = \mathrm{a}$, where a is constant and the results remain true.
3. The conditions (H_4) and (H_5) can be improved, and established new conditions.

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