Homotopy Perturbation Transform Method for Solving Korteweg-DeVries (KDV) Equation

Mohannad H. Eljaily[1], Tarig M. Elzaki[2]

[1]Department of Mathematic, Faculty of Sciences, Sudan University of Sciences and Technology, Khartoum, Sudan
[2]Mathematics Department, Faculty of Sciences and Arts-Alkamil, University of Jeddah, Jeddah, Saudi Arabia

Email address:

mohannadhamid757@hotmail.com (M. H. Eljaily), Tarig.alzaki@gmail.com (T. M. Elzaki)

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Abstract: In this paper, a combined form of the Laplace transforms method with the homotopy perturbation method is proposed to solve Korteweg-DeVries (KDV) Equation. This method is called the homotopy perturbation transform method (HPTM). The (HPTM) finds the solution without any discretization or restrictive assumptions and avoids the round-off errors. The results reveal that the proposed method is very efficient, simple and can be applied to other nonlinear problems.

Keywords: Laplace Transform, Homotopy Perturbation Method, Korteweg-DeVries (KDV) Equation

1. Introduction

In the recent years, the idea of homotopy was coupled with perturbation. The fundamental work was done by He. In 1992. He [5–18] developed the homotopy perturbation method (HPM) by merging the standard homotopy and perturbation for solving various physical problems. The authors have applied this method successfully to problems arising in mathematics engineering. The KDV equation plays an important role in diverse areas of engineering and scientific applications, and therefore, the enormous amount of research work has been invested in the study of KDV equations [29–33]. The Laplace transform is totally incapable of handling nonlinear equations because of the difficulties that are caused by the nonlinear terms. Various ways have been proposed recently to deal with these nonlinearities such as the Adomian decomposition method [21] and the Laplace decomposition algorithm [24–28]. Furthermore, the homotopy perturbation method is also combined with the well-known Laplace transform method [22] and the variational iteration method [23] to produce a highly effective technique for handling many nonlinear problems. In this paper, we shall deal with the KDV equation in different forms by the homotopy perturbation transform method (HPTM). The KDV equation can be presented in the following form,

$$u_t - 6uu_x + u_{xxx} = 0 \qquad (1)$$

where $u(x,t)$ is the displacement.

The purpose of this paper is to extend the (HPTM) for the solution of Korteweg-DeVries (KDV) Equation. The method has been successfully applied for obtaining exact solutions for nonlinear equations.. In this paper considers the effectiveness of the homotopy perturbation transform method in solving Korteweg-DeVries (KDV) Equation.

2. Basic Idea

To illustrate the basic idea of this method, we consider a general nonlinear non homogeneous partial differential equation with initial conditions of the form:

$$Du(x,t) + Ru(x,t) + Nu(x,t) = g(x,t), \qquad (2)$$

$$u(x,0) = h(x), u_t(x,0) = f(x)$$

Where D is the second order linear differential operator $D = \partial^2/\partial t^2$, is the linear differential operator of less order than D, N represents the general non-linear differential operator and $g(x,t)$ is the source term.

Taking Laplace transform (denoted throughout this paper by L) on both sides of Eq. (2), to get:

$$L[Du(x,t)] + L[Ru(x,t)] + L[Nu(x,t)] = L[g(x,t)] \quad (3)$$

Using the differentiation property of the Laplace transform, we have:

$$L[u(x,t)] = \frac{h(x)}{s} + \frac{f(x)}{s^2} - \frac{1}{s^2} L[Ru(x,t)]$$
$$+ \frac{1}{s^2} L[g(x,t)] - \frac{1}{s^2} L[Nu(x,t)]. \quad (4)$$

Operating with the Laplace inverse on both sides of Eq. (4) gives:

$$u(x,t) = G(x,t) - L^{-1}\left[\frac{1}{s^2} L[Ru(x,t) + Nu(x,t)]\right], \quad (5)$$

where $G(x,t)$ represents the term arising from the source term and the prescribed initial conditions. Now, we apply the homotopy perturbation method,

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \quad (6)$$

and the nonlinear term can be decomposed as:

$$Nu(x,t) = \sum_{n=0}^{\infty} p^n H_n(x,t) \quad (7)$$

for some He's polynomials H_n(see [19,20] that are given by:

$$H_n(u_0,\ldots,u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n} N\left(\sum_{i=0}^{\infty} p^i u_i\right)_{p=0}, n = 0,1,2,3\ldots \quad (8)$$

Substituting Eqs. (7) and (6) in Eq. (5) we get:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = G(x,t)$$
$$-p\left(L^{-1}\left[\frac{1}{s^2} L[\sum_{n=0}^{\infty} p^n u_n(x,t) + + \sum_{n=0}^{\infty} p^n H_n(u)]\right]\right) \quad (9)$$

Which is the coupling of the Laplace transform and the homotopy perturbation method using He's polynomials. Comparing the coefficient of like powers of p, the following approximations are obtained:

$$p^0: u_0(x,t) = G(x,t),$$

$$p^1: u_1(x,t) = -\frac{1}{s^2} L[Ru_0(x,t) + H_0(u)],$$

$$p^2: u_2(x,t) = -\frac{1}{s^2} L[Ru_1(x,t) + H_1(u)],$$

$$p^3: u_3(x,t) = -\frac{1}{s^2} L[Ru_2(x,t) + H_2(u)],$$

Then the solution is:

$$u(x,t) = \lim_{p \to 1} u_n(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots \quad (10)$$

3. Applications

In this section, the effectiveness and the usefulness ofhomotopyperturbationtransform method (HPTM) are demonstrated by finding exact solutions of Korteweg-DeVries (KDV) Equation.

Example 3.1. Consider the following homogeneous KDV equation;

$$u_t - 6uu_x + u_{xxx} = 0 \quad (11)$$

With the initial condition;

$$u(x,0) = 6x \quad (12)$$

Taking the Laplace transform on both sides of Eq. (11) subject to the initial condition Eq.(12), we get;

$$u(x,s) = \frac{6x}{s} + \frac{1}{s} L[6uu_x - u_{xxx}] \quad (13)$$

The inverse of Laplace transformimplies that

$$u(x,t) = 6x - L^{-1}\left[\frac{1}{s} L[u_{xxx} - 6uu_x]\right] \quad (14)$$

Now, we apply the homotopy perturbation method, we get:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = 6x$$
$$-PL^{-1}\left[\frac{1}{s} L((\sum_{n=0}^{\infty} p^n u_n(x,t))_{xxx} - \sum_{n=0}^{\infty} p^n H_n(u))\right] \quad (15)$$

Where $H_n(u)$ are He's polynomials that represents the nonlinear terms.

The first few components of He's polynomials, are given by;

$$H_0(u) = u_0 u_{0x}$$
$$H_1(u) = u_0 u_{1x} + u_1 u_{0x} \quad (16)$$
$$H_2(u) = u_0 u_{2x} + u_1 u_{1x} + u_2 u_{0x}$$

Comparing the coefficient of like powers of p, the following approximations are obtained;

$$p^0: u_0(x,t) = 6x$$

$$p^1: u_1(x,t) = -L^{-1}\left[\frac{1}{s} L[(u_0)_{xxx} - 6H_0(u)]\right] = 6^3 xt,$$

$$p^2: u_2(x,t) = -L^{-1}\left[\frac{1}{s} L[(u_1)_{xxx} - 6H_1(u)]\right] = 6^5 xt^2, \quad (17)$$

$$p^3: u_3(x,t) = -L^{-1}\left[\frac{1}{s} L[(u_2)_{xxx} - 6H_2(u)]\right] = 6^7 xt^3,$$

Therefore the solution $u(x,t)$ is given by:

$$u(x,t) = 6x(1 + (36t) + (36t)^2 + (36t)^3 + (36t)^4 + \cdots) \quad (18)$$

In series form, and,

$$u(x,t) = \frac{6x}{1-36t}, |36t| < 1 \quad (19)$$

In closed form.

Example 3.2. Consider the following homogeneous KDV equation;

$$u_t + 6uu_x + u_{xxx} = 0 \quad (20)$$

With the initial condition;

$$u(x, 0) = x \qquad (21)$$

Taking the Laplace transform on both sides of Eq. (20) subject to the initial
Condition Eq.(21), we get;

$$u(x, s) = \frac{x}{s} - \frac{1}{s} L[6uu_x + u_{xxx}] \qquad (22)$$

The inverse of Laplace transformimplies that:

$$u(x, t) = x - L^{-1}\left[\frac{1}{s} L[6uu_x + u_{xxx}]\right] \qquad (23)$$

Now, we apply the homotopy perturbation method, we get:

$$\sum_{n=0}^{\infty} p^n u_n(x, t) = x$$

$$-PL^{-1}\left[\frac{1}{s} L((\sum_{n=0}^{\infty} p^n u_n(x, t))_{xxx} + \sum_{n=0}^{\infty} p^n H_n(u))\right] \quad (24)$$

Comparing the coefficient of like powers of p, the following approximations are obtained;

$$p^0: u_0(x, t) = x$$

$$p^1: u_1(x, t) = -L^{-1}\left[\frac{1}{s} L[(u_0)_{xxx} - 6H_0(u)]\right] = -x(6t),$$

$$p^2: u_2(x, t) = -L^{-1}\left[\frac{1}{s} L[(u_1)_{xxx} - 6H_1(u)]\right] = x(6t)^2, \qquad (25)$$

$$p^3: u_3(x, t) = -L^{-1}\left[\frac{1}{s} L[(u_2)_{xxx} - 6H_2(u)]\right] = -x(6t)^3,$$

Therefore the solution $u(x, t)$ is given by:

$$u(x, t) = x(1 - (6t) + (6t)^2 - (6t)^3 + (6t)^4 - (6t)^5 + \cdots) \quad (26)$$

In series form, and,

$$u(x, t) = \frac{x}{1+6t} \qquad (27)$$

In closed form.

Example 3.3. Consider the following homogeneous KDV equation;

$$u_t - 6uu_x + u_{xxx} = 0 \qquad (28)$$

With the initial condition;

$$u(x, 0) = -2\frac{k^2 e^{kx}}{(1+e^{kx})^2} \qquad (29)$$

Taking the Laplace transform of both sides of Eq. (28) subject to the initial
Condition Eq. (29), we get;

$$u(x, s) = \frac{-2\frac{k^2 e^{kx}}{(1+e^{kx})^2}}{s} + \frac{1}{s} L[6uu_x - u_{xxx}] \qquad (30)$$

The inverse of Laplace transform implies that:

$$u(x, t) = -2\frac{k^2 e^{kx}}{(1+e^{kx})^2} + L^{-1}\left[\frac{1}{s} L[6uu_x - u_{xxx}]\right] \qquad (31)$$

Now, we apply the homotopy perturbation method, we get:

$$\sum_{n=0}^{\infty} p^n u_n(x, t) = -2\frac{k^2 e^{kx}}{(1+e^{kx})^2}$$

$$-PL^{-1}\left[\frac{1}{s} L((\sum_{n=0}^{\infty} p^n u_n(x, t))_{xxx} - \sum_{n=0}^{\infty} p^n H_n(u))\right] \quad (32)$$

Comparing the coefficient of like powers of p, the following approximations are obtained;

$$p^0: u_0(x, t) = -2\frac{k^2 e^{kx}}{(1+e^{kx})^2}$$

$$p^1: u_1(x, t) = -L^{-1}\left[\frac{1}{s} L[(u_0)_{xxx} - 6H_0(u)]\right] =$$

$$-2\frac{k^5 e^{kx}(e^{kx}-1)}{(1+e^{kx})^3} t, \qquad (33)$$

$$p^2: u_2(x, t) = -L^{-1}\left[\frac{1}{s} L[(u_1)_{xxx} - 6H_1(u)]\right] =$$

$$-\frac{k^8 e^{kx}(e^{2kx} - 4e^{kx} + 1)}{(1+e^{kx})^4} t^2,$$

Therefore, the solution of Eq. (8), when $p \to 1$ will be as:

$$u(x, t) = -2\frac{k^2 e^{kx}}{(1+e^{kx})^2} - 2\frac{k^5 e^{kx}(e^{kx}-1)}{(1+e^{kx})^3} t$$

$$-\frac{k^8 e^{kx}(e^{2kx}-4e^{kx}+1)}{(1+e^{kx})^4} t^2 + \cdots \qquad (34)$$

Using Taylor series, the closed form solution will be as follows:

$$u(x, t) = -2\frac{k^2 e^{k(x-k^2 t)}}{\left(1+e^{k(x-k^2 t)}\right)^2} \qquad (35)$$

4. Conclusions

In this paper, we have applied thehomotopy perturbation transform method toKorteweg-DeVries (KDV) Equation. It can be concluded that the HPTM is a very powerful and efficient technique in findingexact and approximatesolutions for nonlinear problems.Byusing this method we obtain a new efficient recurrent relation to solve (KDV) Equation. In conclusion, HPTM provide highly accurate numerical solutions for nonlinear problems in comparison with other method.The results show that the HPTM is a powerful mathematical tool for solving the KDV having wide applications in engineering and applied mathematics.

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