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New Modification of Homotopy Perturbation Method and the Fourth - Order Parabolic Equations with Variable Coefficients

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Abstract: In this paper, the exact solution of the fourth - order parabolic equations with variable coefficients is obtained by using a new homotopy perturbation method (NHPM), theoretical consideration are discussed. Finally, three examples are illustrated to show the validity and applicability of the proposed method.

Keywords: New Homotopy Perturbation Method (NHPM), Fourth - Order Parabolic Equations

1. Introduction

In this paper, we consider the fourth – order parabolic partial differential equations with the variable coefficients of the form;

$$\begin{aligned}&\frac{\partial^2 u}{\partial t^2}+\mu(x,y,z)\frac{\partial^4 u}{\partial x^4}+\lambda(x,y,z)\frac{\partial^4 u}{\partial y^4}\\&+\eta(x,y,z)\frac{\partial^4 u}{\partial z^4}=g(x,y,z),\\&a<x,y,z<b \quad ,t>0\end{aligned} \tag{1}$$

Where, $\mu(x,y,z)$, $\lambda(x,y,z)$ and $\eta(x,y,z)$ are positive, Subject to the initial conditions;

$$u(x,y,z,0)=f_0(x,y,z),\ u_t(x,y,z,0)=f_1(x,y,z) \tag{2}$$

And the boundary conditions;

$$u(a,y,z,t)=g_0(y,z,t),\quad u\ (b,y,z,t)=g_1(y,z,t)$$

$$u(x,a,z,t)=k_0(x,z,t),\quad u\ (x,b,z,t)=k_1(x,z,t)$$

$$u(x,y,a,t)=h_0(x,y,t),\quad u(x,y,b,t)=h_1(x,z,t)$$

$$\begin{aligned}&\frac{\partial^2 u}{\partial x^2}(a,y,z,t)=\tilde{g}_0(y,z,t),\\&\frac{\partial^2 u}{\partial x^2}(b,y,z,t)=\tilde{g}_0(y,z,t)\end{aligned} \tag{3}$$

$$\begin{aligned}&\frac{\partial^2 u}{\partial y^2}(x,a,z,t)=\tilde{k}_0(x,z,t),\\&\frac{\partial^2 u}{\partial y^2}(x,b,z,t)=\tilde{k}_1(x,z,t)\end{aligned}$$

$$\begin{aligned}&\frac{\partial^2 u}{\partial x^2}(a,y,z,t)=\tilde{h}_0(x,y,t),\\&\frac{\partial^2 u}{\partial x^2}(x,y,b,t)=\tilde{h}_1(x,y,t)\end{aligned}$$

Where the functions f_i , g_i , k_i , h_i , $\tilde{g}_i$, $\tilde{k}_i$, $\tilde{h}_i$, $i=0$, 1 are continuous functions.it is worth mentioning that the problem equation (1 - 3) arise in the study of the transverse vibrations problem [1]. numerical computations of the transverse vibrations have been carried out by a number of authors for one dimensional space. the main focus of researchers was to obtain numerical solution by using several techniques, such as explicit and implicit finite - difference schemes [2, 3, 4, 5], Evans [6] who expressed the fourth order equation in two space variables as system of two second - order equations to be solved by finite difference method in [7] Khaliq and Twizell solved the variable fourth order parabolic equations by using a family of second order method, Evants etc all [8] investigated the fourth order parabolic equation with constant coefficient by using the AGE method, recently Wazwaz [9, 10] Approached the variable coefficient fourth - order parabolic Equation directly by application Adomian decomposition method. Biazer and Ghazrini [11] applied He's variational iteration method and Deniz Agirseven and Turgut Özis [12] obtained the exact solution of the problem by using homotopy perturbation method. He's homotopy perturbation method [13 - 17] has been used by many researcher in physics and engineering to solve various problem [18 - 22]. In this paper we extend the modification of homotopy perturbation method (NHPM) [23 - 26] to obtain the exact solution of variables coefficients fourth order parabolic equations.

2. Basic Ideas of the (HPM)

To solve equation (1) with initial condition (2) by NHPM, we construct the following homotopy:

$$(1-p)\left(\frac{\partial^2 v}{\partial t^2}-u_0\right)+p\left(\frac{\partial^2 v}{\partial t^2}+\mu(x,y,z)\frac{\partial^2 v}{\partial x^2}+\lambda(x,y,z)\frac{\partial^2 v}{\partial y^2}+\eta(x,y,z)\frac{\partial^2 v}{\partial z^2}-g(x,y,z)\right)=0 \tag{3}$$

Or:

$$\frac{\partial^2 v}{\partial t^2}=u_0-p\left(u_0+\mu(x,y,z)\frac{\partial^2 v}{\partial x^2}+\lambda(x,y,z)\frac{\partial^2 v}{\partial y^2}+\eta(x,y,z)\frac{\partial^2 v}{\partial z^2}-g(x,y,z)\right) \tag{4}$$

Applying the inverse operator, $L^{-1}=\int_0^t\int_0^t(\circ)\,dtdt$, to both sides of Eq. (4) we obtain:

$$v(x,y,z,t)=v(x,y,z,0)+tv_t(x,y,z,0)+\int_0^t\int_0^t u_0\,dtdt-p\int_0^t\int_0^t\left(u_0+\mu(x,y,z)\frac{\partial^2 v}{\partial x^2}+\lambda(x,y,z)\frac{\partial^2 v}{\partial y^2}+\eta(x,y,z)\frac{\partial^2 v}{\partial z^2}-g(x,y,z)\right)dtdt \tag{5}$$

Where, $v(x,y,z,0)=f_0(x,y,z)$ and $v_t(x,y,z,0)=f_1(x,y,z)$.

Assume the solution of Eq. (4) has the form:

$$v=v_0+pv_1+p^2v_2+\ldots \tag{6}$$

Suppose the initial approximation to the solution $u_0(x,y,z,t)$ has the form:

$$u_0(x,y,z,t)=\sum_{n=0}^{\infty}a_n(x,y,z)\,\rho_n(t) \tag{7}$$

Where, $a_0,a_1,a_2,\ldots$ are unknown coefficients and are specific functions depending on problem. Substituting Eq. (6) into Eq (5) and equating terms of like power p, and equating each coefficients of p to zero we get:

$$\begin{aligned}
&p^0: v_0(x,y,z,t)=v_0(x,y,z,0)+tv_{0t}(x,y,z,0)+\int_0^t\int_0^t u_0\,dtdt.\\
&p^1: v_1(x,y,z,t)=-\int_0^t\int_0^t\left(u_0+\mu(x,y,z)\frac{\partial^4 v_0}{\partial x^4}+\lambda(x,y,z)\frac{\partial^4 v_0}{\partial y^4}+\eta(x,y,z)\frac{\partial^4 v_0}{\partial z^4}-g(x,y,z)\right)dtdt\\
&p^2: v_2(x,y,z,t)=-\int_0^t\int_0^t\left(\mu(x,y,z)\frac{\partial^4 v_1}{\partial x^4}+\lambda(x,y,z)\frac{\partial^4 v_1}{\partial y^4}+\eta(x,y,z)\frac{\partial^4 v_1}{\partial z^4}\right)dtdt
\end{aligned} \tag{8}$$

By solving these equation in such a way that $v_1(x,y,z)=0$ then Eq. (7) result in $v_j(x,y,z,t)=0$, $j=1,2,3,\ldots$, therefore the exact solution may be obtained as follows:

$$u(x,y,z,t)=v_0(x,y,z,t)=f_0(x,y,z)+tf_1(x,y,z)+\sum_{n=0}^{\infty}a_n(x,y,z)\int_0^t\int_0^t\rho_n(t)dtdt$$

It is worth mentioning that if $g(x,y,z,t)$ and $u_0(x,y,z,t)$ are analytic at $t=t_0$ then their Taylor series written as;

$$u_0(x,y,z,t)=\sum_{n=0}^{\infty}a_n(x,y,z)(t-t_0)^n$$

$$g(x,y,z,t)=\sum_{n=0}^{\infty}a_n^*(x,y,z)(t-t_0)^n$$

Can be used in Eq. (8), where $a_0,a_1,a_2,\ldots$ are unknown coefficients and $a_0^*,a_1^*,a_2^*,\ldots$ are known ones, which must be computed.

3. Application

In this section, we present three examples to illustrate our shames for variable coefficient parabolic equation.

Example 1.

Consider the one dimensional, variable coefficients fourth order parabolic equation [1];

$$\frac{\partial^2 u}{\partial t^2}+(1+x)\frac{\partial^4 u}{\partial x^4}=\left(x^4+x^3-\frac{6}{7!}\right)\cos t,\ 0<x<1\ \ ,t>0 \quad (9)$$

With the initial conditions;

$$u(x,0)=\frac{6}{7!}x^7\ ,\ \frac{\partial u}{\partial t}(x,0)=0 \quad (10)$$

And the boundary conditions;

$$u(0,t)=0\quad u(1,t)=\frac{6}{7!}x^7\cos t$$

$$\frac{\partial^2 u}{\partial x^2}(0,t)=0\quad \frac{\partial^2 u}{\partial x^2}(1,t)=\frac{1}{20}\cos t \quad (11)$$

To solve Eq. (9) by (NHPM), we construct the following homotopy:

$$\frac{\partial^2 v}{\partial t^2}=u_0-p\left[u_0+(1+x)\frac{\partial^4 v}{\partial x^4}-\left(x^4+x^3-\frac{6}{7!}\right)\cos t\right] \quad (12)$$

By integration of Eq. (12) we have:

$$v(x,t)=v(x,0)+tv_0(x,0)+\int_0^t\int_0^t u_0\,dtdt$$

$$-p\int_0^t\int_0^t\left(u_0+(1+x)\frac{\partial^4 v}{\partial x^4}-\left(x^4+x^3-\frac{6}{7!}\right)\cos t\right)dtdt \quad (13)$$

Assume the solution of Eq. (13) has the following form:

$$v=v_0+pv_1+p^2v_2+\ldots \quad (14)$$

Substituting Eq. (14) in to Eq. (13) and equating the coefficients of like powers of p, results are:

$$p^0: v_0(x,t)=v_0(x,0)+tv_{0t}(x,0)+\int_0^t\int_0^t u_0\,dtdt.$$

$$p^1: v_1(x,t)=-\int_0^t\int_0^t\left(u_0+(1+x)\frac{\partial^4 v_0}{\partial x^4}-\left(x^4+x^3-\frac{6}{7!}\right)\cos t\right)dtdt$$

$$p^2: v_2(x,t)=-\int_0^t\int_0^t(1+x)\frac{\partial^4 v_1}{\partial x^4}\,dtdt\ .$$

Assuming $u_0(x,t)=\sum_{n=0}^{\infty}a_n(x)t^n$, $v(x,0)=u(x,0)$, $v_t(x,0)=u_t(x,0)$ and $v_1(x,t)=0$

Then we have:

$$v_1(x,t)=\left(-\frac{a_0}{2}-\frac{1}{2}\frac{6x^7}{7!}\right)t^2+\left(-\frac{a_1}{6}\right)t^3$$
$$+\left(-\frac{a_{0_{xxxx}}}{24}-\frac{a_{0_{xxxx}}x}{24}-\frac{x^4}{24}-\frac{x^3}{24}-\frac{1}{24}\frac{6x^7}{7!}-\frac{a_2}{12}\right)t^4$$
$$+\left(-\frac{a_{3_{xxxx}}}{20}\right)t^5+\ldots$$

It can be easily to shown that:

$$a_0=-\frac{6}{7!}x^7\ ,\ a_1=0\ ,\ a_2=\frac{1}{2}\frac{6}{7!}x^7\ ,\ a_3=0\ ,..$$

Therefore we obtain the solution of Eq. (9);

$$u(x,t)=\frac{6x^7}{7!}+a_0(x)\frac{t^2}{2}+a_1(x)\frac{t^3}{6}$$
$$+a_2(x)\frac{t^4}{12}+a_3(x)\frac{t^5}{20}+\ldots=\frac{6}{7!}x^7\cos t$$

This is an exact solution.

Example 2.

We next consider the fourth - order parabolic equation in two variables;

$$\frac{\partial^2 u}{\partial t^2}+2\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\frac{\partial^4 u}{\partial x^4}+2\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\frac{\partial^4 u}{\partial y^4}=0$$
$$\frac{1}{2}<x,y<1\ \ ,t>0 \quad (15)$$

With the initial conditions:

$$u(x,y,0)=0\ ,\ \frac{\partial u}{\partial t}(x,y,0)=2+\frac{x^6}{6!}+\frac{y^6}{6!}, \quad (16)$$

And the boundary conditions:

$$u\left(\frac{1}{2},y,t\right)=\left(2+\frac{(0.5)^6}{6!}+\frac{y^6}{6!}\right)\sin t\ ,$$

$$u(1,y,t)=\left(2+\frac{1}{6!}+\frac{y^6}{6!}\right)\sin t$$

$$\frac{\partial^2 u}{\partial x^2}\left(\frac{1}{2},y,t\right)=\frac{(0.5)^6}{24}\sin t\quad ,\ \frac{\partial^2 u}{\partial x^2}(1,y,t)=\frac{1}{24}\sin t \quad (17)$$

$$\frac{\partial^2 u}{\partial x^2}\left(x,\frac{1}{2},t\right)=\frac{(0.5)^6}{24}\sin t\ ,\ \frac{\partial^2 u}{\partial x^2}(x,1,t)=\frac{1}{24}\sin t$$

To solve Eq. (15) by (NHPM), we construct the following homotopy:

$$\frac{\partial^2 v}{\partial t^2}=u_0-p\left(u_0+2\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\frac{\partial^4 v}{\partial x^4}+2\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\frac{\partial^4 v}{\partial y^4}\right) \quad (18)$$

Applying the inverse operator, $L^{-1}=\int_0^t\int_0^t(\circ)\,dtdt$, to both sides of Eq. (18) we obtain:

$$v(x,y,t)=v(x,y,0)+tv_t(x,y,0)+\int_0^t\int_0^t u_0\,dtdt$$

$$-p\int_0^t\int_0^t\left(u_0+2\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\frac{\partial^4 v}{\partial x^4}+2\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\frac{\partial^4 v}{\partial y^4}\right)dtdt \quad (19)$$

Suppose the solution of Eq. (18) have the form Eq. (14) substituting Eq. (14) into Eq. (19) equating the coefficients of like powers of p we get:

$$p^0: v_0(x,y,t)=v_0(x,y,0)+tv_{0t}(x,y,0)+\int_0^t\int_0^t u_0(x,y,t)\,dtdt.$$

$$p^1: v_1(x,y,t)=-\int_0^t\int_0^t\left(u_0+2\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\frac{\partial^4 v_0}{\partial x^4}+2\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\frac{\partial^4 v_0}{\partial y^4}\right)dtdt\,.$$

$$p^2: v_2(x,y,t)=-\int_0^t\int_0^t\left(2\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\frac{\partial^4 v_1}{\partial x^4}+2\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\frac{\partial^4 v_1}{\partial y^4}\right)dtdt\,.$$

Assume $u_0(x,y,t)=\sum_{n=0}^{\infty}a_n(x,y)t^n$, $v(x,y,0)=u(x,y,0)$, $v_t(x,y,0)=u_t(x,y,0)$ and solving the above equation for $v_1(x,y,t)$ leads to;

$$v_1(x,t)=(-\frac{a_0}{2})t^2+\left(-\frac{a_1}{6}-\frac{1}{6}\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\right)t^3$$
$$+\left(\frac{a_2}{12}-\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\frac{a_{0_{xxxx}}}{12}-\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\frac{a_{0_{yyyy}}}{12}\right)t^4$$
$$+\left(-\frac{a_5}{20}-\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\frac{a_{1_{xxxx}}}{120}-\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\frac{a_{0_{yyy}}}{120}\right)t^5+\ldots$$

By taking $v_1(x,y,t)=0$, coefficients $a_n, n=1,2,3,\ldots$ can be determined as the following:

$$a_0=0\,,\ a_1=-\left(2+\frac{x^6}{6!}+\frac{y^4}{6!}\right),\ a_2=0\,,$$

$$a_3=\frac{1}{6}\left(2+\frac{x^6}{6!}+\frac{y^4}{6!}\right),\ldots$$

Therefore, an exact solution of the Eq. (15) can be expressed as:

$$u(x,y,t)=v_0(x,y,t)$$
$$=\left(2+\frac{x^6}{6!}+\frac{y^4}{6!}\right)t+a_0(x,y)\frac{t^2}{2}+$$
$$a_1(x,y)\frac{t^3}{6}+a_2(x,y)\frac{t^4}{12}+a_3(x,y)\frac{t^5}{20}+\ldots$$
$$=\left(2+\frac{x^6}{6!}+\frac{y^4}{6!}\right)\sin t$$

Example 3.

We finally consider the following three - dimensional fourth - order parabolic Equation;

$$\frac{\partial^2 u}{\partial t^2}+\left(\frac{1}{4!z}\right)\frac{\partial^4 u}{\partial x^4}+\left(\frac{1}{4!x}\right)\frac{\partial^4 u}{\partial y^4}+\left(\frac{1}{4!y}\right)\frac{\partial^4 u}{\partial z^4}=$$
$$-\left[\frac{x}{y}+\frac{y}{z}+\frac{z}{x}-\frac{1}{x^5}-\frac{1}{y^5}-\frac{1}{z^5}\right]\cos t\,,$$

$$\frac{1}{2}<x,y,z<1\quad ,t>0 \quad (20)$$

With the initial conditions:

$$u(x,y,z,0)=\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\quad ,\ \frac{\partial u}{\partial t}(x,y,z,0)=0 \quad (21)$$

And the boundary conditions:

$$u\left(\frac{1}{2},y,z,t\right)=\left(\frac{1}{2y}+\frac{y}{z}+2z\right)\cos t\,,$$
$$u(1,y,z,t)=\left(\frac{1}{y}+\frac{y}{z}+z\right)\cos t$$

$$u\left(x,\frac{1}{2},z,t\right)=\left(2x+\frac{1}{z}+\frac{z}{x}\right)\cos t\,,$$
$$u(x,1,z,t)=\left(x+\frac{1}{z}+\frac{z}{x}\right)\cos t$$

$$u\left(x,y,\frac{1}{2},t\right)=\left(\frac{x}{y}+2y+\frac{1}{2x}\right)\cos t$$
$$u(x,y,1,t)=\left(\frac{x}{y}+y+\frac{1}{x}\right)\cos t \quad , \quad (22)$$

$$\frac{\partial u}{\partial x}\left(\frac{1}{2},y,z,t\right)=\left(\frac{1}{y}-4z\right)\cos t,\ \frac{\partial u}{\partial x}(1,y,z,t)=\left(\frac{1}{y}-z\right)\cos t$$

$$\frac{\partial u}{\partial y}\left(x,\frac{1}{2},z,t\right)=\left(-4x+\frac{1}{z}\right)\cos t$$

$$\frac{\partial u}{\partial y}(x,1,z,t)=\left(-x+\frac{1}{z}\right)\cos t$$

$$\frac{\partial u}{\partial z}\left(x,y,\frac{1}{2},t\right)=\left(-4y+\frac{1}{x}\right)\cos t,$$

$$\frac{\partial u}{\partial z}(x,y,1,t)=\left(-y+\frac{1}{x}\right)\cos t$$

To solve Eq. (20) by (NHPM), Eq. (5), in this equation will be in conical form as the following:

$$v(x,y,z,t)=v(x,y,z,0)+tv_t(x,y,z,0)+\int_0^t\int_0^t u_0\,dtdt$$
$$-p\int_0^t\int_0^t\Bigg(u_0+\left(\frac{1}{4!z}\right)\frac{\partial^4 v}{\partial x^4}+\left(\frac{1}{4!x}\right)\frac{\partial^4 v}{\partial y^4}+\left(\frac{1}{4!y}\right)\frac{\partial^4 v}{\partial z^4}$$
$$+\left[\frac{x}{y}+\frac{y}{z}+\frac{z}{x}-\frac{1}{x^5}-\frac{1}{y^5}-\frac{1}{z^5}\right]\cos t\Bigg)dtdt \tag{23}$$

Suppose the solution of Eq. (23) has the following form:

$$v=v_0+pv_1+p^2v_2+\ldots \tag{24}$$

Substituting Eq. (24) into Eq. (23) and equating the coefficients of like powers of p, to obtain:

$$p^0: v_0(x,y,z,t)=v_0(x,y,z,0)+tv_{0t}(x,y,z,0)+\int_0^t\int_0^t u_0(x,y,z,t)\,dtdt.$$

$$p^1: v_1(x,y,z,t)=\int_0^t\int_0^t\Bigg(u_0+\left(\frac{1}{4!z}\right)\frac{\partial^4 v_0}{\partial x^4}$$
$$+\left(\frac{1}{4!x}\right)\frac{\partial^4 v_0}{\partial y^4}+\left(\frac{1}{4!y}\right)\frac{\partial^4 v_0}{\partial z^4}$$
$$+\left[\frac{x}{y}+\frac{y}{z}+\frac{z}{x}-\frac{1}{x^5}-\frac{1}{y^5}-\frac{1}{z^5}\right]\cos t\Bigg)dtdt$$

$$p^2: v_2(x,y,t)=-\int_0^t\int_0^t\left(\left(\frac{1}{4!z}\right)\frac{\partial^4 v_1}{\partial x^4}+\left(\frac{1}{4!x}\right)\frac{\partial^4 v_1}{\partial y^4}+\left(\frac{1}{4!y}\right)\frac{\partial^4 v_1}{\partial z^4}\right)dtdt$$

By assuming $u_0(x,y,z,t)=\sum_{n=0}^{\infty}a_n(x,y,z)t^n$,

$v(x,y,z,0)=u(x,y,z,0)$,

$v_t(x,y,z,0)=u_t(x,y,z,0)$, And solving $v_1(x,y,t)$, leads to the following result:

$$v_1(x,t)=\left(-\frac{a_0}{2}-\frac{1}{2}\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\right)t^2$$
$$+\left(-\frac{a_1}{6}\right)t^3+\Bigg(\frac{a_2}{12}-\frac{1}{24}\left(\frac{a_{0_{xxxx}}}{4!z}+\frac{a_{0_{yyyy}}}{4!x}+\frac{a_{0_{zzzz}}}{4!y}\right)$$
$$+\frac{1}{24}\left(\frac{1}{x^5}+\frac{1}{y^5}+\frac{1}{z^5}\right)-\frac{1}{24}\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\Bigg)t^4$$
$$+\Bigg(\Bigg(\frac{a_3}{20}-\frac{1}{120}\left(\frac{a_{0_{xxxx}}}{4!z}+\frac{a_{0_{yyyy}}}{4!x}+\frac{a_{0_{zzzz}}}{4!y}\right)$$
$$-\frac{1}{120}\left(\frac{1}{x^5}+\frac{1}{y^5}+\frac{1}{z^5}\right)+\frac{1}{120}\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\Bigg)\Bigg)t^5+\ldots$$

By taking $v_1(x,y,z,t)=0$, coefficients $a_n, n=1,2,3,\ldots$ can be determined as the following:

$$a_0=-\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right),\ a_1=0,\ a_2=\frac{1}{2}\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right),\ a_3=0,..$$

Therefore, the exact solution of the Eq. (20) can be expressed as:

$$u(x,y,z,t)=v_0(x,y,z,t)$$
$$=\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+a_0(x,y,z)\frac{t^2}{2}+a_1(x,y,z)\frac{t^3}{6}$$
$$+a_2(x,y,z)\frac{t^4}{12}+a_3(x,y,z)\frac{t^5}{20}+\ldots$$
$$=\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\cos t.$$

4. Conclusions

In this article, NHPM, has been introduced for solving variable coefficients fourth - order parabolic partial differential equations, this method give the exact solution of the problems. This result reveals the method explicit, effective and easy to use.

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