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Stability of a Jensen's functional equation in Banach spaces

Seyed Habib Shakoory

Electrical and Computer Engineering Department, Seraj Higher Education Institute, Tabriz, Iran

Email address:
Habibshakoory@yahoo.com

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Abstract: Our aim in this note is to study the Hyers -Ulam stability and we investigate the Hyers-Ulam stability of generalized Jensen's Functional Equation in Banach Spaces: $f((x + y)/2) = (f(x) + f(y))/2$

Keywords: Hyers-Ulam Stability, Jensen's Functional Equation, Approximation Theory

1. Introduction

The Stability of Functional Equations seem to have been first studied by Ulam [5] in 1940 and solved in the next year for the Cuachy Functional Equations by Hyers [2]. In the following year, Hyers theorem gave a partial affirmative answer to the question of Ulam-Hyers theorem and it was generalized by Aoki [1]. For additive mapping by Rassias [4] and for linear mapping by considering an unbounded Cauchy difference during the past decades, several stability problems for various Functional equations have been investigated by a number of mathematicians, and we refer the reader to [cf. Gajda 1991, Gavruta 1994, Vaezi and park 2008, vaezi and zamani 2009, vaezi and dehgan 2010, Vaezi and Shakoory 2012, shakoory 2013]and also to the references cited therein. Another famous equation that is closely related to the Cauchy functional equation is Jensen's functional equation:

$$f((x + y)/2) = (f(x) + f(y))/2$$

The first result on the stability of the classical Jensen's functional equation was given by Kominek [3]. In this paper, we investigate the Hyers-Ulam stability of the Jensen's functional equation for linear bounded mapping from Banach spaces in to Banach spaces.

2. Main Results

2.1. Definition

Let X be a set. A function $d: X \times X \to [0, \infty)$ is called a generalized metric on X if d satisfies:

(1) $d(x, y) = 0$if and only if $x = y$.
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$.
(3) $d(x; z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

2.2. Definition

A (real) complex normed space is a (real) complex vector space X together with a map: $X \longrightarrow R$, called the norm and denoted $\|.\|$, such that

(1) $\|x\| \geq 0$ for all $x \in X$, and $\|x\| = 0$ if and only if $x = 0$.
(2) $\|ax\| = |\alpha| \|x\|$, for all $x \in X$ and all $\alpha \in R$ or (C).
(3) $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in X$.

2.3. Definition

A complete normed space is called a Banach space. Thus, a normed space X is Banach space if every Cuchy sequence in X converges (where X is given the metric space structure as outlined above).

2.4. Definition

Let W be a Banach space. f a differentiable map on V into W so that

$$\|f((x + y)/2) - f(x/2) - f(y/2)\|_B \leq \varepsilon$$

for all $x, y \in V$,we say that the Hyers-Ulam stability holds for f ,if there exist a $K > 0$ and a differentiable map T on V into W such that

$$\|T((x+y)/2) - T(x/2) - T(y/2)\|_B \le \varepsilon$$

and

$$\|f(x) - T(x)\|_B \le K\varepsilon$$

holds for every x, y ∈ V ,we call such K a Hyers-Ulam stability constant for Jensen's functional equation:

$$f((x+y)/2) = (f(x) + f(y))/2$$

Now, the main result of this work is given in the following theorem.

2.4.1. Theorem

Let V and W be Banach spaces. A mapping f: V →W with f (0) = 0 satisfies the functional equation:

$$2n\, f((x1 + x2 + \ldots + x2n)/(2^n)) = \sum_{i=1}^{2^n} f(xi)$$

for all x1, x2, ..., x2n ∈V if and only if the mapping f: V → W satisfies the additive Cauchy equation f (x + y) = f (x) + f (y)for all x, y ∈ V.

Proof:

Assume that a mapping f: V → W satisfies

$$2^n f((x1 + x2 + \ldots + x2n)/(2^n)) = \sum_{i=1}^{2^n} f(xi)$$

Put

$$x_1 + x_2 + \ldots + x_{2^{n-1}} = x$$

and

$$x_{2^{n-1}} + 1 = x_{2^{n-1}} + 2 = \ldots = x_{2^n} = y$$

in

$$2^n f((x1 + x2 + \ldots + x2n)/(2n)) = \sum_{i=1}^{2^n} f(xi)$$

then

$$2^n f((x+y)/(2)) = 2n{-}1 f(x) + 2n{-}1 f(y)$$

for all x, y ∈ V . So

$$2f((x+y)/(2)) = f(x) + f(y)$$

for all x, y ∈ V. Hence f is additive. The convergence is obvious.

2.4.2. Theorem

Let V and W to be two Banach spaces with norms $\|.\|$ and $\|.\|$, respectively. Consider f:V →W to be a mapping such that f(tx) is continuous in t ∈ R for each fixed x ∈ V. Assume that there exist constants ε ≥ 0 and p ∈ [0, 1) such that:

$$\|f((x+y)/2) - f(x/2) - f(y/2)\| \le \varepsilon(\|x/2\|^p + \|y/2\|^p$$

for any x, y ∈ V. Then there exist a unique R-Linear mapping T: V → W such that :

$$\|f(x) - T(x)\| \le \varepsilon \|x\|^p (4/(2-2p))$$

for any x ∈ V, where Hyers-Ulam stability constant is for Jensen's functional equation.

Proof

Set

$$1/2T(x) = \lim_{n\to\infty} 1/2^n[f(2^n x)]$$

and multiplying the assumed equation

$$\|f((x+y)/2) - f(x/2) - f(y/2)\| \le \varepsilon(\|x/2\|p + \|y/2\|\ p$$

to 2^{n+1}:

$$\begin{aligned}\|f[2^n(x+y)] - f[2^n(x)] - f[2^n(y)]\| &\le \varepsilon(\|2^n x\|p + \|2^n x\|p)\\ &= 2^n p\ \varepsilon\ \|x\|p + \|y\|p)\end{aligned}$$

Therefore

$$1/2^n\|f[2^n(x+y)] - f[2^n(x)] - f[2^n(y)]\| \le 2^{n}(p\text{-}1)\ \varepsilon(\|x\|p + \|y\|p)$$

Or

$$\lim_{n\to\infty} 1/2^n \|f[2^n(x+y)] - f[2^n(x)] - f[2^n(y)]\| \le \lim_{n\to\infty} 2^n(p\text{-}^1)\ \varepsilon(\|x\|p + \|y\|p)$$

Or

$$\left\|\lim_{n\to\infty} 1/2^n f[2^n(x+y)] - \lim_{n\to\infty} 1/2^n f[2^n(x)] - \lim_{n\to\infty} 1/2^n f[2^n(y)]\right\| = 0$$

Or

$$\|T((x+y)/2) - T(x/2) - T(y/2)\| = 0$$

For any x, y∈V.

To prove stability our claim that:

$$\|f(2^n x)/2^n - f(x)\| \le \varepsilon\|2x\|p \sum_{m=0}^{n-1} 2^m(p\text{-}^1)$$

For any integer n, and some ε ≥ 0. The verification of above equation following by induction on n. Indeed the case n = 1 is clear because by the hypothesis we can find ε,that is greater or equal to zero, and p such that 0 ≤ p < 1 with:

$$\|f(2x)/2 - f(x)\| \le \varepsilon\|2x\|^p$$

Assume now that

$$\|f(2^n x)/2^n - f(x)\| \leq \varepsilon\|2x\|p\sum_{m=0}^{n-1} 2^m(p^{-1})$$

holds and we want to prove it for the case n + 1. However this is true, because by

$$\|f(2^n x)/2^n - f(x)\| \leq \varepsilon\|2x\|p\sum_{m=0}^{n-1} 2^m(p^{-1})$$

We obtain

$$\|f(2^n .2x)/2^n - f(x)\| \leq \varepsilon\|4x\|p\sum_{m=0}^{n-1} 2^m(p^{-1})$$

Therefore

$$\|f(2^n .2x)/2^n + ^1 - 1/2f(2x)\| \leq \varepsilon\|2x\|p\sum_{m=0}^{n-1} 2^m(p^{-1})$$

By the triangle inequality we obtain:

$$\|f(2^{n+1}x)/2^{n+1} - f(x)\| \leq \|1/2^{n+1} f(2^{n+1}x) - 1/2f(2x)$$

$$\| + \|1/2f(2x) - f(x)\| \leq \varepsilon\|\|2x\|p\sum_{m=0}^{n-1} 2^m(p^{-1})$$

and is valid for any integer n. It follows then that

$$\|f(2^n x)/2^n - f(x)\| \leq \varepsilon \|2x\|^p.(2/(2-2p))$$

because $\sum_{m=0}\infty 2^m(p^{-1})$ converges to (2/(2−2p)), as $0 \leq p < 1$ we obtain

$$\|T(x) - f(x)\| \leq \varepsilon \|2x\|^p.(2/(2-2p))$$

Hence

$$\|T(x) - f(x)\| \leq \varepsilon \|x\|^p.(4/(2-2))$$

which completes the proof.

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