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Generalized order–*k* Pell–Padovan–like numbers by matrix methods

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Abstract: Weconsider the third – order recurrence relation $Q_n = 2Q_{n-2} + Q_{n-3}$ with initial conditions$Q_0 = 1$, $Q_1 = 0$ and$Q_2 = 2$ and define these numbers as Pell – Padovan – like numbers.We extend this definition generalized order – k Pell – Padovan – like numbers and give some relations between thesenumbers and the Fibonacci numbers. Wealso obtain some relations of thesenumbers and matrices by using matrix methods.

Keywords: Fibonacci Sequence, Pell – Padovan's Sequence, Generating Function, Binet Formula, Matrix Methods

1. Introduction

For $n \geq 2$, Fibonacci sequence is defined by

$$F_n = F_{n-1} + F_{n-2}$$

with initial conditions $F_0 = 0$ and$F_1 = 1$. Kalman [3] defined a generalized Fibonacci sequence as follows

$$F_{n+k} = c_0F_n + c_1F_{n+1} + \cdots + c_kF_{n+k-1} \quad (1)$$

where$c_0, c_1, \cdots, c_k$ are real constants. Er [2] defined k sequences of the generalized order$-k$ Fibonacci numbers as; for n> 0 and $1 \leq i \leq k$

$$g_n^i = \sum_{j=1}^{k} c_j g_{n-j}^i$$

with initial conditions for $1 - k \leq n \leq 0$

$$g_n^i = \begin{cases} 1 & \text{if } i + n = 1, \\ 0 & \text{otherwise}, \end{cases}$$

Where$c_0, c_1, \cdots, c_k$ are constant coefficients.

In [6], Pell–Padovan's sequence was defined, which is a third order recurrence relation

$$P_n = 2P_{n-2} + P_{n-3}, \text{ for every integer } n \geq 1 \quad (2)$$

with initial conditions $P_0 = P_1 = P_2 = 1$. In [5], an extended form of this sequence was given as

$$P_{n+3} = rP_{n+1} + sP_n, \text{ for every integer } n \geq 1 \quad (3)$$

with initial conditions $P_1 = a, P_2 = b, P_3 = c$ where *a, b, c, r* and *s* are fixed non-negative integers.

If we take $a = 1, b = 0, c = 2, r = 2$ and $s = 1$ in Eq.(3), we obtain a sequence which satisfies the recurrence relation (2). Let's show this sequence with $\{Q_n\}_{n \geq 0}$ and define it as Pell–Padovan–like sequence. For every integer $n \geq 3$, this sequence satisfies

$$Q_n = 2Q_{n-2} + Q_{n-3}$$

with initial conditions $Q_0 = 1$, $Q_1 = 0$ and $Q_2 = 1$.First few terms of this sequence are

1, 0, 2, 1, 4, 4, 9, 12, 22, 33, 56, 88, 145, 232, …

There is a connection between this sequence and the Fibonacci sequence given in the following lemma.

Lemma 1.1

For every non-negative integer n, we have

$$Q_n = F_n + (-1)^n \quad (4)$$

whereF_n is the n^{th} Fibonacci number.

Proof: We prove the Lemma by induction on n. We have $F_0 + (-1)^0 = 1 = Q_0$ and $F_1 + (-1)^1 = 0 = Q_1$ so the statement (4) holds when$n = 0$ and 1.

We assume that the statement (4) holds for every positive integer smaller than $k$where$k$>1. Thus

$$\begin{aligned} Q_k &= 2Q_{k-2} + Q_{k-3} \\ &= 2[F_{k-2} + (-1)^{k-2}] + F_{k-3} + (-1)^{k-3} \\ &= 2F_{k-2} + F_{k-3} + (-1)^{k-2} \end{aligned}$$

$$= F_{k-1} + F_{k-2} + (-1)^{k-2}$$
$$= F_k + (-1)^k.$$

So, the statement (4) holds for every non-negative integer n, and this proves the Lemma. ■

This sequenceappears in "The Online Encyclopedia of Integer Sequences" [8] with the code "A008346".

In [4, 7], the authors generalize the Perrin sequence and the Padovan sequence respectively which are third-order integer sequences to k-th order and using matrix methods,give some properties of these sequences. We use similar methods for finding properties of the generalized order – k Pell–Padovan–like numbers.

2. A Generalization of the Pell-Padovan-Like Numbers

Throughout this paper k denotes an integer greater than 2. We definek–sequences of the generalized order–k Pell–Padovan–like numbers;for n >0 and $1 \le i \le k$

$$Q_n^i = 2Q_{n-2}^i + Q_{n-3}^i + \cdots + Q_{n-k}^i \tag{5}$$

Withinitial conditions

$$Q_n^i = \begin{cases} 1, & \text{if } i+n=1, \\ 0, & \text{otherwise,} \end{cases} \text{for} 1-k \le n \le 0.$$

Q_n^i is the n^{th} term of the i^{th} sequence. For $k=3$ and $i=1$, the generalized sequence reduces to the Pell–Padovan–like series$\{Q_n\}_{n\ge 0}$.

We define kxk square matrix A such that

$$A = \begin{bmatrix} 0 & 2 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

From Eq.(5), we have

$$\begin{bmatrix} Q_{n+1}^i \\ Q_n^i \\ \vdots \\ Q_{n-k+2}^i \end{bmatrix} = A \begin{bmatrix} Q_n^i \\ Q_{n-1}^i \\ \vdots \\ Q_{n-k+1}^i \end{bmatrix}. \tag{6}$$

Now we define

$$P_n = \begin{bmatrix} Q_n^1 & Q_n^2 & \cdots & Q_n^k \\ Q_{n-1}^1 & Q_{n-1}^2 & \cdots & Q_{n-1}^k \\ \vdots & \vdots & \ddots & \vdots \\ Q_{n-k+1}^1 & Q_{n-k+1}^2 & \cdots & Q_{n-k+1}^k \end{bmatrix}. \tag{7}$$

By expanding the vectors on the both sides of Eq.(6) to $k-$ columns and multiplying the obtained matrix on the right-hand side by A, we obtain

$$P_n = AP_{n-1}.$$

From the last equation, by an inductive argument, we have

$$P_n = A^{n-1}P_1$$

and since $P_1 = A$, we are in a position to prove the following lemma.

Lemma 2.1

For all integer $n \ge 1$, we have
$$P_n = A^n.$$

Corollary 2.2

For $n \ge 1$, we have

$$\det P_n = \begin{cases} 1, & \text{if} k \text{ is odd,} \\ (-1)^n, & \text{if} k \text{ is even.} \end{cases}$$

Proof:Since$P_n = A^n$, we have

$$\det P_n = \det A^n = (\det A)^n$$

with $\det A = (-1)^{k-1}$. This equation completes the proof. ■

For later use, we need the following lemma.

Lemma 2.3

For $n > 0$, we have

$$Q_{n+1}^k = Q_n^1,$$

$$Q_{n+1}^1 = Q_n^2,$$

$$Q_{n+1}^2 = 2Q_n^1 + Q_n^3,$$

and

$$Q_{n+1}^i = Q_n^1 + Q_n^{i+1}.$$

Proof:From Eq.(7), we can see that

$$P_{r+1} = P_rP_1 = P_1P_r. \tag{8}$$

From the first equation in Eq.(8), by using matrix multiplication, the lemma can be obtained easily. ■

Now we give some relations of the generalized order$-k$ Pell–Padovan–like numbers.

Theorem 2.4

For $m \ge 0$and $n \ge 1$, we have

$$Q_{n+m}^i = \sum_{j=1}^{k} Q_m^j Q_{n-j+1}^i.$$

Proof:From Lemma 2.1, we know that $P_n = A^n$.So, we can write

$$P_n = P_{n-1}P_1 = P_1P_{n-1}.$$

By generalizing this equation, we have

$$P_{n+m} = P_nP_m = P_mP_n.$$

From the last equation, we say that an element of P_{n+m} is the product of a row P_n and a column P_m. Therefore, the proof is complete. ■

If we take $k = 3$and $i = 1$, we have

$$Q_{n+m}^1 = Q_m^1 Q_n^1 + Q_m^2 Q_{n-1}^1 + Q_m^3 Q_{n-2}^1$$

$$= Q_m^1 Q_n^1 + (2Q_{m-1}^1 + Q_{m-1}^3)Q_{n-1}^1 + Q_{m-1}^1 Q_{n-2}^1$$

$$= Q_m^1 Q_n^1 + (2Q_{m-1}^1 + Q_{m-2}^1)Q_{n-1}^1 + Q_{m-1}^1 Q_{n-2}^1$$

and

$$Q_{n+m}^1 = Q_n^1 Q_m^1 + 2Q_{n-1}^1 Q_{m-1}^1 + Q_{n-1}^1 Q_{m-2}^1 + Q_{n-2}^1 Q_{m-1}^1. \quad (9)$$

Corollary 2.5

For usual Pell–Padovan–like numbers, we have

$$Q_{2n} = (Q_n)^2 + 2Q_{n-1}F_n \quad (10)$$

Where$F_n$is the n^{th} Fibonacci numbers, and

$$Q_{2n+1} = (Q_{n-1})^2 + 2Q_n Q_{n+1}.$$

Proof:If we take $m = n$ in Eq.(9), we have

$$Q_{2n} = (Q_n)^2 + 2(Q_{n-1})^2 + 2Q_{n-1}Q_{n-2}$$

$$= (Q_n)^2 + 2Q_{n-1}(Q_{n-1} + Q_{n-2})$$

$$= (Q_n)^2 + 2Q_{n-1}F_n.$$

If we take $m = n + 1$ in Eq.(9), we obtain

$$Q_{2n+1} = Q_n Q_{n+1} + 2Q_{n-1}Q_n + (Q_{n-1})^2 + Q_{n-2}Q_n$$

$$= (Q_{n-1})^2 + Q_n(Q_{n+1} + 2Q_{n-1} + Q_{n-2})$$

$$= (Q_{n-1})^2 + 2Q_n Q_{n+1}.$$ ■

If we write $Q_n = F_n + (-1)^n$, Eq.(10) becomes

$$F_{2n} + 1 = [F_n + (-1)^n]^2 + 2[F_{n-1} - (-1)^n]F_n.$$

After some simplifications, we have the following well–known identity [1]

$$F_n^2 = F_{2n} - 2F_n F_{n-1}.$$

3. Generating Function

In this section, we give the generating function for the order $-k$ generalized Pell–Padovan–like sequence. We begin the generating function for the sequence $\{Q_n\}_{n\geq 0}$. Let

$$H_k(x) = Q_0^k + Q_1^k x + Q_2^k x^2 + \cdots + Q_k^k x^k + \cdots.$$

Then

$$H_k(x) - 2x^2 H_k(x) - x^3 H_k(x) - \cdots - x^k H_k(x)$$

$$= (1 - 2x^2 - x^3 - \cdots - x^k)H_k(x)$$

$$= Q_0^k + Q_1^k x + (Q_2^k - 2Q_0^k)x^2 + (Q_3^k - 2Q_1^k - Q_0^k)x^3$$

$$+ \cdots + \left(Q_{k-1}^k - 2Q_{k-3}^k - Q_{k-4}^k - \cdots - Q_0^k\right)x^{k-1} + \cdots.$$

From the definition of the generalized order $-k$ Pell–Padovan–like numbers, we obtain

$$(1 - 2x^2 - x^3 - \cdots - x^k)H_k(x) = Q_0^k$$

and since $Q_0^k = 1$, we have

$$H_k(x) = (1 - 2x^2 - x^3 - \cdots - x^k)^{-1}$$

for $0 \leq 2x^2 + x^3 + \cdots + x^k < 1$.

Let $h_k(x) = 2x^2 + x^3 + \cdots x^k$. Then $0 \leq h_k(x) < 1$ and we can give exponential representation for generalized order$-k$ Pell–Padovan–like numbers

$$\ln H_k(x) = \ln[1 - h_k(x)]^{-1}$$

$$= -\ln[1 - h_k(x)]$$

$$= h_k(x) + \frac{1}{2}[h_k(x)]^2 + \cdots + \frac{1}{n}[h_k(x)]^n + \cdots$$

$$= x^2 \sum_{n=0}^{\infty} \frac{1}{n}(2 + x + x^2 + \cdots + x^{k-2})^n.$$

Thus, we obtain

$$H_k(x) = \exp\left[x^2 \sum_{n=0}^{\infty} \frac{1}{n}(2 + x + x^2 + \cdots + x^{k-2})^n\right].$$

4. Generalized Binet Formula

The characteristic equation of the matrix A is $x^k - 2x^{k-2} - x^{k-3} - x^{k-4} - \cdots - 1$ and this equation is also the characteristic equation of the generalized order$-k$ Pell–Padovan–like numbers. We should show that the eigenvalues of the companion matrix A are distinct to give the generalized Binet formula.

Lemma 4.1

For $k \geq 3$, none of the roots of the equation $x^{k+1} - x^k - 2x^{k-1} + x^{k-2} + 1$ is multiple root.

Proof:We define

$$f(x) = x^k - 2x^{k-2} - x^{k-3} - \cdots - x - 1,$$

$$h(x) = (x - 1)f(x) = x^{k+1} - x^k - 2x^{k-1} + x^{k-2} + 1.$$

Tough 1 is a root of $h(x)$, it is not a multiple root because $f(1) \neq 0$. Suppose that α is a multiple root of $h(x)$ such that $\alpha \neq 0$and $\alpha \neq 1$. We have

$$h(\alpha) = \alpha^{k+1} - \alpha^k - 2\alpha^{k-1} + \alpha^{k-2} + 1$$

$$= \alpha^{k-2}[\alpha^3 - \alpha^2 - 2\alpha + 1] + 1$$

and

$$h'(\alpha) = (k+1)\alpha^k - k\alpha^{k-1} - 2(k-1)\alpha^{k-2} + (k-2)\alpha^{k-3}$$

$$= \alpha^{k-3}[(k+1)\alpha^3 - k\alpha^2 - 2(k-1)\alpha + k - 2]$$

$$= 0.$$

The roots of the equation $(k+1)\alpha^3 - k\alpha^2 - 2(k-1)\alpha + k - 2$ are

$$\alpha_1 = \frac{1}{6(k+1)}[\gamma + 4(7k^2-6)\gamma^{-1} + 2k]$$

$$\alpha_{2,3} = -\frac{1}{12(k+1)}[\gamma + 4(7k^2-6)\gamma^{-1} - 4k] \pm i\frac{\sqrt{3}}{12(k+1)}[\gamma - 4(7k^2-6)\gamma^{-1}]$$

where

$$\gamma = [u + (k+1)v]^{1/3},$$

$$u = -28k^3 + 252k + 216$$

and

$v = 12(-147k^4 + 294k^3 - 147k^2 - 84k + 420)^{1/2}$.

It can be seen that α_1, α_2 and α_3 are distinct from each other. We write

$$0 = h(\alpha_i) = \alpha^{k-2}[\alpha^3 - \alpha^2 - 2\alpha + 1] - 1.$$

If we choose $k = 3$, we have

$$h(\alpha_1) \approx -0.433903264 - i(0.585166688)10^{-10} \neq 0,$$

$$h(\alpha_2) \approx -0.161491954 - i(0.1659503261)10^{-9} \neq 0,$$

$$h(\alpha_3) \approx 1.114926467 + i(0.4718000278)10^{-10} \neq 0,$$

and this contradiction proves the lemma. ■

As a result of this lemma, The characteristic equation $x^k - 2x^{k-2} - x^{k-3} - x^{k-4} - \cdots - 1$ of the matrix A does not have multiple roots for $k \geq 3$. Thus, the eigenvalues of the matrix A are distinct. Let $\lambda_1, \lambda_2, \cdots, \lambda_k$ be these eigenvalues. Let V be the following Vandermonde matrix

$$V = \begin{bmatrix} \lambda_1^{k-1} & \lambda_1^{k-2} & \cdots & \lambda_1^1 & 1 \\ \lambda_2^{k-1} & \lambda_2^{k-2} & \cdots & \lambda_2^1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_k^{k-1} & \lambda_k^{k-2} & \cdots & \lambda_k^1 & 1 \end{bmatrix}.$$

We define

$$C_k^i = \begin{bmatrix} \lambda_1^{n+k-i} \\ \lambda_2^{n+k-i} \\ \vdots \\ \lambda_k^{n+k-i} \end{bmatrix}$$

and define $V_j^{(i)}$ as a k −square matrix which is obtained from V by replacing the j^{th}column of V by C_k^i. Then the following theorem gives the generalized Binet formula.

Theorem 4.2

For $1 \leq i \leq k$, we have

$$Q_{n-i+1}^j = \frac{\det V_j^{(i)}}{\det V}.$$

Proof:A is a diagonalizable matrix since its eigenvalues are distinct. Let C be the transpose of V. We write

$$C^{-1}AC = diag(\lambda_1, \lambda_2, \cdots, \lambda_k) = D,$$

and obtain

$$A^nC = CD^n.$$

Since $P_n = A^n$, we have

$$\begin{aligned} p_{i1}\lambda_1^{k-1} + p_{i2}\lambda_1^{k-2} + \cdots + p_{ik} &= \lambda_1^{n+k-1} \\ p_{i1}\lambda_2^{k-1} + p_{i2}\lambda_2^{k-2} + \cdots + p_{ik} &= \lambda_2^{n+k-1} \\ &\vdots \\ p_{i1}\lambda_k^{k-1} + p_{i2}\lambda_k^{k-2} + \cdots + p_{ik} &= \lambda_k^{n+k-1} \end{aligned}$$

where $P_r = [p_{ij}]_{kxk}$. For $j = 1,2,\cdots,k$, we obtain

$$p_{ij} = \frac{\det V_j^{(i)}}{\det V},$$

So, we have the conclusion. ■

5. Sums of Generalized Pell –Padovan-Like Numbers

To obtain sums of generalized Pell–Padovan–like numbers, we define

$$S_n = \sum_{i=0}^{n-1} Q_i^1.$$

Since $Q_{r+1}^k = Q_r^1$, we have

$$S_n = \sum_{i=1}^{n} Q_i^k.$$

Let T and W_n be $(k+1)$ −square matrices such that

$$T = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & & & & \\ 0 & & & A & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix}$$

and

$$W_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ S_n & & & & \\ S_{n-1} & & & P_n & \\ \vdots & & & & \\ S_{n-k+1} & & & & \end{bmatrix}.$$

We easily obtain $W_n = W_{n-1}T$ and inductively, we have$W_n = W_1T^{n-1}$.

Since $S_{-i} = 0$ for $1 \leq i \leq k$ and $W_1 = T$, we infer $W_n = T^n$. So, we can write

$$W_{n+1} = W_n W_1 = W_1 W_n.$$

We consider the equation $W_n = W_1 W_{n-1}$. By multiplying the matrices on the right hand-side, we obtain the following recurrence relation

$$S_n = 1 + 2S_{n-2} + \sum_{j=3}^{k} S_{n-j}.$$

For usual Pell–Padovan–like numbers, if we take $k = 3$, we have

$$S_n = 1 + 2S_{n-2} + S_{n-3}.$$

Using the equation $S_n = Q_{n-1} + S_{n-1}$, after some elementary operation, we obtain

$$\begin{aligned} S_n &= \frac{1}{2}(3Q_{n-1} + 3Q_{n-2} + Q_{n-3} - 1) \\ &= \frac{1}{2}(Q_{n+1} + Q_n + Q_{n-1} - 1). \end{aligned}$$

Since $Q_n = F_n + (-1)^n$, the last equation gives

$$\sum_{i=0}^{n-1} Q_i = \frac{1}{2}[F_{n+1} + F_n + F_{n-1} - (-1)^n - 1]$$

where F_n is the n^{th} Fibonacci number.

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