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Developments on Beal Conjecture from Pythagoras´ and Fermat´s Equations

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Abstract: The Beal Conjecture was formulated in 1997 and presented as a generalization of Fermat's Last Theorem, within the number theory´s field. It states that, for X, Y, Z, n_1, n_2 and n_3 positive integers with n_1, n_2, $n_3 > 2$, if $X^{n1}+Y^{n2}=Z^{n3}$ then X, Y, Z must have a common prime factor. This article aims to present developments on Beal Conjecture, obtained from the correspondences between the real solutions of the equations in the forms $A^2 + B^2 = C^2$ (here simply refereed as Pythagoras´ equation), $\delta^n + \gamma^n=\alpha^n$ (here simply refereed as Fermat´s equation) and $X^{n1}+Y^{n2}=Z^{n3}$ (here simply referred as Beal´s equation). Starting from a bibliographical research on the Beal Conjecture, prime numbers and Fermat's Last Theorem, these equations were freely explored, searching for different aspects of their meanings. The developments on Beal Conjecture are divided into four parts: geometric illustrations; correspondence between the real solutions of Pythagoras´ equation and Fermat's equation; deduction of the transforms between the real solutions of Fermat's equation and the Beal´s equation; and analysis and discussion about the topic, including some examples. From the correspondent Pythagoras´ equation, if one of the terms A, B or C is taken as an integer reference basis, demonstrations enabled to show that the Beal Conjecture is correct if the remaining terms, when squared, are integers.

Keywords: Beal Conjecture, Fermat´s Last Theorem, Diophantine Equations, Number Theory, Prime Numbers

1. Introduction

The Beal Conjecture is a proposition within the number theory´s field that was formulated by Andrew Beal, according to which, for X, Y, Z, n_1, n_2 and n_3 positive integers with $n_1, n_2, n_3 > 2$, if $X^{n_1} + Y^{n_2} = Z^{n_3}$ then X, Y, Z must have a common prime factor. Stated another way, there is no solution in integers for $X^{n_1} + Y^{n_2} = Z^{n_3}$ in the case of X, Y, Z, n_1, n_2, n_3 positive integers and $n_1, n_2, n_3 > 2$ if X, Y and Z are coprime [1].

Darmon and Granville have also worked in this kind of problem while investigating integer solutions of the superelliptic equation $z^m = F(x,y)$, where F is a homogeneous polynomial with integer coefficients of the generalized Fermat equation $Ax^p + By^q = Cz^r$ [2].

The Beal Conjecture is presented by Mauldin [1] as a generalization of Fermat's Last Theorem. Concerning the latter, Rubin and Silverberg [3] mention that Pierre de Fermat (1601-1665) wrote that *"it is impossible to separate a cube into two cubes, or a fourth power into two fourth powers, or in general, any power higher than the second into two like powers. I have discovered a truly marvelous proof of this, which this margin is too narrow to contain"*.

Written in mathematical notation, Fermat's Last Theorem states that if $n > 2$, $\delta^n + \gamma^n = \alpha^n$ has no solutions in nonzero integers. As Fermat used not to annotate the proofs of his theorems, this and other statements inspired many generations of mathematicians, who went on to develop important math advances while seeking solutions. All statements of Fermat were eventually proved - except one that was refuted, but in this case, Fermat did not actually say that he knew a proof [4]. The only statement that remained unproved was the above one, which became known as "Fermat´s Last Theorem" (not because it was the last to be written, but the last to be proved).

According to Stewart [4], Fermat's Last Theorem became notorious and Euler proved that it is valid for cubes, Fermat himself proved that it applies to the fourth power, Peter Lejeune Dirichlet dealt with the fifth and fourteenth powers, Gabriel Lamé and Kummer for all powers to the 100th except 37, 59 and 67, and Friedrich Gauss tried to correct Lamé´s attempt to the 7th power, but gave up after not getting success. The author also points out that in 1980 other mathematicians proved the theorem for all powers to the 125,000th until the mathematician Andrew Wiles (with the collaboration of Richard Taylor) proved it definitively in 1995, using, however, ideas and methods that did not exist at the time of Fermat. These materials have become a permanent and important increase to the arsenal of mathematics.

The aim of this work is to provide developments on Beal Conjecture with the use of basic mathematics.

Starting from a bibliographical research on the Beal Conjecture, prime numbers and Fermat's Last Theorem, the equations in the general forms $A^2 + B^2 = C^2$ (here simply refereed as Pythagoras´ equation), $\delta^n + \gamma^n = \alpha^n$ (here simply refereed as Fermat´s equation) and $X^{n_1} + Y^{n_2} = Z^{n_3}$ (here simply referred as Beal´s equation) were freely explored, searching for different aspects of their meanings.

The developments on Beal Conjecture are divided into four parts. The first part consists on the presentation of geometric illustrations that allow a view of the argument to be adopted. The second part presents the deduction of the correspondence between the real solutions of Pythagoras´ equation and Fermat's equation, obtaining the transforms that enable this operation. The third part contains the deduction of the transforms between the real solutions of Fermat's equation and the Beal´s equation, and the fourth part includes analysis and discussion about the topic, including some examples.

The presented language is simple, sometimes making use of geometrical illustrations to explore analytical aspects of the problem, which is believed to be helpful for the solution´s understanding. It is thus intended that the statements are sufficiently clear and straightforward, allowing analysis by the scientific community and by non-experts enthusiasts on the subject.

2. Developments on Beal Conjecture

2.1. Geometric Illustrations

Using a geometric approach to illustrate this proposition, one can take the figures below, in which are represented two squares of sides A and B, with areas A^2 and B^2, respectively. The forms are out of scale and possibly out of proportion what, however, do not imply in loss of the problem´s understanding.

By adding the areas of the squares of sides A and B one can obtain an area numerically equal to $S = A^2 + B^2$. There are infinite geometric surfaces that may represent the area S, among which some are exemplified in figure 2.

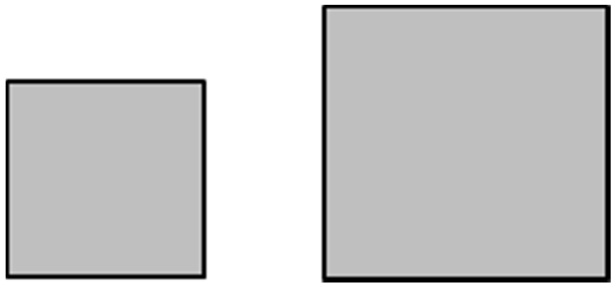

Figure 1. *Squares of sides A and B, and areas A^2 and B^2, respectively. Source: author.*

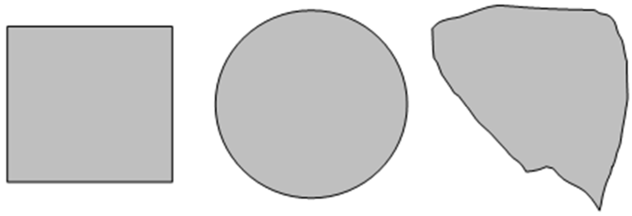

Figure 2. *Examples of figures with the same area S. Source: author.*

Choosing a square of side C to represent the area S, there comes that $S = A^2 + B^2 = C^2$, resulting in the Pythagorean equation for real numbers. That is, from the sum of two squares it is possible to obtain a third square, which can be graphically represented in Figure 3:

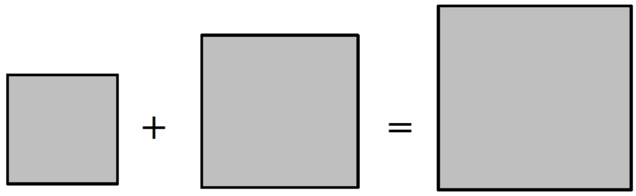

Figure 3. *Graphic representation of Pythagoras´ equation for squares with sides A, B and C, respectively. Source: author.*

Assuming that each square has an infinitesimal thickness du, the infinitesimal volumes of the elements correspond to: $dV_A = A^2du$, $dV_B = B^2du$ and $dV_C = C^2du$. In fact, the sum of the areas can be converted into volumes by simply multiplying the scalar du on the Pythagoras´ equation, resulting in:

$$A^2du + B^2du = C^2du \tag{1}$$

Or, equivalently:

$$dV_A + dV_B = dV_C \tag{2}$$

As illustrated in Al Shenk [5], one can integrate the infinitesimal volumes along an axis u transverse to plane of the figures (therefore linearly independent to it, [6]) until some arbitrary thickness U, and then have:

$$\int_0^U dV_A + \int_0^U dV_B = \int_0^U dV_C \tag{3}$$

$$\int_0^U A^2du + \int_0^U B^2du = \int_0^U C^2du \tag{4}$$

$$A^2U + B^2U = C^2U \tag{5}$$

By making U numerically equal to A, that is, making the thickness set to all elements be equal to the side of the first element, there comes that:

$$A^2A + B^2A = C^2A \tag{6}$$

$$A^3 + B^2A = C^2A \tag{7}$$

As one can see, by setting U = A, the cube of side A on the first element was obtained in a direct way. Being C always greater than A and B, and assuming B different from A (just for the purpose of visualization, but this is not a condition at first), the corresponding volumes to the second and third elements will necessarily be different from their respective cubes, corresponding to tridimensional solids in the shapes of parallelepipeds.

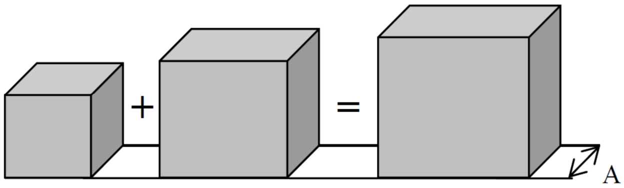

Figure 4. Graphic representation of the sum of volumes corresponding to figures with main faces with sides A, B and C, respectively, all of them with thickness (depth) A. Source: author.

In an analogous manner to the areas (figure 2), a volume V can be represented by various tridimensional solids. It is well known that the volume contained in a parallelepiped can also be represented by a cube of an equivalent volume, with side equal to $L = \sqrt[3]{V}$. Relying on this artifice, it can be obtained a cubic equivalency to the volumes of the second and third solids, resulting in:

$$V_B = B^2A = L_B^3 \tag{8}$$

$$V_C = C^2A = L_C^3 \tag{9}$$

$$L_B = \sqrt[3]{B^2A} \tag{10}$$

$$L_C = \sqrt[3]{C^2A} \tag{11}$$

The equation of volumes can then be rewritten in the form:

$$A^3 + L_B^3 = L_C^3 \tag{12}$$

One can note that the equation now assumes the form of a sum of cubes, revealing that from the solutions for $A^2 + B^2 = C^2$ in reals one can obtain equivalent solutions for $A^3 + L_B^3 = L_C^3$ in reals and vice versa, using the transforms $L_B = \sqrt[3]{B^2A}$ and $L_C = \sqrt[3]{C^2A}$.

2.2. Correspondence between Real Solutions of Pythagoras´ Equation and Fermat´s Equation

In case the equation of volumes is integrated again and by making U=A, one can obtain:

$$A^4 + B^2A^2 = C^2A^2 \tag{13}$$

From this moment, the tridimensional visualization feature cannot be applied, since it requires a 4-dimensional space. If the integrations continue in a n-dimensional Euclidean space [7] until it is obtained the n-power of A [8], there will be:

$$A^n + B^2A^{n-2} = C^2A^{n-2} \tag{14}$$

Similarly to the approach performed in tridimensional space illustrated above, one can obtain in n-space the equivalent elements that allow rewriting the equation to the n-power. By making $\gamma^n = B^2A^{n-2}$ and $\alpha^n = C^2A^{n-2}$, the equation becomes:

$$A^n + \gamma^n = \alpha^n \tag{15}$$

This is the general form of Fermat´s equation. It is highlighted that the deductions are being made within the framework of real numbers, not addressing in particular the existence of entire solutions.

It was demonstrated, therefore, that it is possible to obtain real solutions for $A^n + \gamma^n = \alpha^n$ from the solutions for $A^2 + B^2 = C^2$ in reals and vice versa, using the transforms $\gamma = \sqrt[n]{B^2A^{n-2}}$ and $\alpha = \sqrt[n]{C^2A^{n-2}}$. The term A remains the same, that is, not transformed.

2.3. Correspondence between Real Solutions of Fermat's Equation and Beal´s Equation

Taking the Beal´s equation in the form $X^{n_1} + Y^{n_2} = Z^{n_3}$ one can realize that it is possible to transform it into the Fermat´s equation by making $Y = \gamma^{n_1/n_2}$ and $Z = \alpha^{n_1/n_3}$, resulting in:

$$X^{n_1} + (\gamma^{n_1/n_2})^{n_2} = (\alpha^{n_1/n_3})^{n_3} \tag{16}$$

$$X^{n_1} + \gamma^{n_1} = \alpha^{n_1} \tag{17}$$

This is the general form of the Fermat´s equation for $\delta = A = X$ and $n = n_1$. Having established that $Y = \gamma^{n_1/n_2}$, $\gamma = \sqrt[n]{B^2A^{n-2}}$, $Z = \alpha^{n_1/n_3}$, $\alpha = \sqrt[n]{C^2A^{n-2}}$ and making A=X e $n = n_1$, one can have:

$$Y = \left(\sqrt[n_1]{B^2X^{n_1-2}}\right)^{n_1/n_2} = \sqrt[n_2]{B^2X^{n1-2}} \tag{18}$$

$$Z = \left(\sqrt[n_1]{C^2X^{n_1-2}}\right)^{n_1/n_3} = \sqrt[n_3]{C^2X^{n_1-2}} \tag{19}$$

As demonstrated, one can obtain real solutions X, Y, Z for the Beal´s equation from real solutions A, B, C for the Pythagoras´ equation, using the transforms $X = A$, $Y = \sqrt[n_2]{B^2X^{n_1-2}}$ and $Z = \sqrt[n_3]{C^2X^{n_1-2}}$.

2.4. Analysis

From the transforms $Y = \sqrt[n_2]{B^2X^{n_1-2}}$ and $Z = \sqrt[n_3]{C^2X^{n_1-2}}$, one can note that, in principle, unless that $n_1 = 2$ (situation in which the power of X is zero, resulting in the unit) or X=0 (trivial solution), the variable X is always present in the transforms.

Assuming X, Y, Z integers and $n_1 > 2$, B^2 and C^2 must be rational numbers, because they are necessarily written as quotients of integers:

$$B^2 = \frac{Y^{n_2}}{X^{n_1-2}} \tag{20}$$

$$C^2 = \frac{Z^{n_3}}{X^{n_1-2}} \tag{21}$$

Once the primes are infinite [9] and W an integer number by hypothesis, one can necessarily write W in the form of the product of the infinite primes raised to their respective powers [10]. As the powers represent the number of times a given prime appears in the factorization of W, the absent primes in the factorization will have exponents 0, corresponding to the unit and thus not altering the product. Adopting the notation of infinite product, the number W can be written as:

$$W = \prod_{i=1}^{\infty} P_i^{k_{W,i}} = P_1^{k_{W,1}} P_2^{k_{W,2}} \ldots P_{\infty}^{k_{W,\infty}} \tag{22}$$

In which k_{Wi} represents the number of times the i-th prime appears in the factorization of W. Writing X^{n_1-2}, Y^{n_2} and Z^{n_3} in infinite product notation, one can have:

$$X^{n_1-2} = \prod_{i=1}^{\infty} P_i^{(n_1-2)k_{X,i}} = P_1^{(n_1-2)k_{X,1}} P_2^{(n_1-2)k_{X,2}} \ldots P_{\infty}^{(n_1-2)k_{X,\infty}} \tag{23}$$

$$Y^{n_2} = \prod_{i=1}^{\infty} P_i^{n_2 k_{Y,i}} = P_1^{n_2 k_{Y,1}} P_2^{n_2 k_{Y,2}} \ldots P_{\infty}^{n_2 k_{Y,\infty}} \tag{24}$$

$$Z^{n_3} = \prod_{i=1}^{\infty} P_i^{n_3 k_{Z,i}} = P_1^{n_3 k_{Z,1}} P_2^{n_3 k_{Z,2}} \ldots P_{\infty}^{n_3 k_{Z,\infty}} \tag{25}$$

Therefore, B^2 and C^2can be written as:

$$B^2 = \frac{P_1^{n_2 k_{Y,1}} P_2^{n_2 k_{Y,2}} \ldots P_{\infty}^{n_2 k_{Y,\infty}}}{P_1^{(n_1-2)k_{X,1}} P_2^{(n_1-2)k_{X,2}} \ldots P_{\infty}^{(n_1-2)k_{X,\infty}}} \tag{26}$$

$$C^2 = \frac{P_1^{n_3 k_{Z,1}} P_2^{n_3 k_{Z,2}} \ldots P_{\infty}^{n_3 k_{Z,\infty}}}{P_1^{(n_1-2)k_{X,1}} P_2^{(n_1-2)k_{X,2}} \ldots P_{\infty}^{(n_1-2)k_{X,\infty}}} \tag{27}$$

Or equivalently:

$$B^2 = P_1^{n_2 k_{Y,1}-(n_1-2)k_{X,1}} P_2^{n_2 k_{Y,2}-(n_1-2)k_{X,2}} \ldots P_{\infty}^{n_2 k_{Y,\infty}-(n_1-2)k_{X,\infty}} \tag{28}$$

$$C^2 = P_1^{n_3 k_{Z,1}-(n_1-2)k_{X,1}} P_2^{n_3 k_{Z,2}-(n_1-2)k_{X,2}} \ldots P_{\infty}^{n_3 k_{Z,\infty}-(n_1-2)k_{X,\infty}} \tag{29}$$

Since B^2 and C^2 are rational numbers, three situations may occur:

1) Situation 1: B^2 and C^2 are integers;
2) Situation 2: B^2 or C^2 is integer and the other is non-integer;
3) Situation 3: B^2 and C^2 are non-integers.

2.4.1. Situation 1 Analysis (B^2 and C^2 Integers)

Assuming that B^2 and C^2 are integers, all the powers of the primes factors must be positive or zero. If not, then the prime factor would be raised to a negative power, going to the denominator and leading to a non-integer result. This condition can be expressed in the general form as

$$n_2 k_{Y,i} - (n_1-2)k_{X,i} = k_{B^2,i} \therefore n_2 k_{Y,i} = k_{B^2,i} + (n_1-2)k_{X,i} \tag{30}$$

$$n_3 k_{Z,i} - (n_1-2)k_{X,i} = k_{C^2,i} \therefore n_3 k_{Z,i} = k_{C^2,i} + (n_1-2)k_{X,i} \tag{31}$$

As one can note, if $k_{B^2,i} \geq 0$ for all primes and a certain prime factor P_m exists in X ($k_{X,m} \geq 0$), then necessarily $k_{Y,m} \geq 0$, that is, P_m is also present in Y. The same is valid for X and Z once $k_{C^2,i} \geq 0$, that is, if a prime factor P_v is present in X ($k_{X,v} \geq 0$), then necessarily it will be present in Z ($k_{Z,v} \geq 0$).

WritingX^{n_1}, Y^{n_2}, Z^{n_3} with highlights to the P_m and P_v prime factors, one can have:

$$X^{n_1} = P_1^{n_1 k_{X,1}} \ldots P_m^{n_1 k_{X,m}} \ldots P_v^{n_1 k_{X,v}} \ldots P_{\infty}^{n_1 k_{X,\infty}} \tag{32}$$

$$Y^{n_2} = P_1^{n_2 k_{Y,1}} \ldots P_m^{n_2 k_{Y,m}} \ldots P_v^{n_2 k_{Y,v}} \ldots P_{\infty}^{n_2 k_{Y,\infty}} \tag{33}$$

$$Z^{n_3} = P_1^{n_3 k_{Z,1}} \ldots P_m^{n_3 k_{Z,m}} \ldots P_v^{n_3 k_{Z,v}} \ldots P_{\infty}^{n_3 k_{Z,\infty}} \tag{34}$$

Therefore the equation $X^{n1} + Y^{n2} = Z^{n3}$ becomes:

$$\left(P_1^{n_1 k_{X,1}} \dots P_m^{n_1 k_{X,m}} \dots P_v^{n_1 k_{X,v}} \dots P_\infty^{n_1 k_{X,\infty}}\right) + \left(P_1^{n_2 k_{Y,1}} \dots P_m^{n_2 k_{Y,m}} \dots P_v^{n_2 k_{Y,v}} \dots P_\infty^{n_2 k_{Y,\infty}}\right)$$
$$= P_1^{n_3 k_{Z,1}} \dots P_m^{n_3 k_{Z,m}} \dots P_v^{n_3 k_{Z,v}} \dots P_\infty^{n_3 k_{Z,\infty}} \quad (35)$$

Once P_m is present in X ($k_{X,m} \geq 0$) and in Y ($k_{Y,m} \geq 0$), it can be put at evidence, resulting

$$P_m[\left(P_1^{n_1 k_{X,1}} \dots P_m^{n_1 k_{X,m-1}} \dots P_v^{n_1 k_{X,v}} \dots P_\infty^{n_1 k_{X,\infty}}\right) + \left(P_1^{n_2 k_{Y,1}} \dots P_m^{n_2 k_{Y,m-1}} \dots P_v^{n_2 k_{Y,v}} \dots P_\infty^{n_2 k_{Y,\infty}}\right)] = Z^{n_3} \quad (36)$$

As the content of the brackets is integer (here named M), it comes that $Z^{n_3} = P_m M$, that is, Z^{n_3} also has the prime factor P_m in its factorization. As Z is assumed integer by initial hypothesis, then Z and Z^{n_3} have the same prime factors, resulting that P_m is also present in Z.

Doing the same for P_v, once P_v is present in X ($k_{X,v} \geq 0$) and in Z ($k_{Z,v} \geq 0$), it can be put at evidence, resulting

$$P_v\left[\left(P_1^{n_3 k_{Z,1}} \dots P_m^{n_3 k_{Z,m}} \dots P_v^{n_3 k_{Z,v-1}} \dots P_\infty^{n_3 k_{Z,\infty}}\right) - \left(P_1^{n_1 k_{X,1}} \dots P_m^{n_1 k_{X,m}} \dots P_v^{n_1 k_{X,v-1}} \dots P_\infty^{n_1 k_{X,\infty}}\right)\right] = Y^{n_2} \quad (37)$$

As the content of the brackets is integer (here named N), it comes that $Y^{n_2} = P_v N$, that is, Y^{n_2} also has the prime factor P_v in its factorization. As Y is assumed integer by initial hypothesis, then Y and Y^{n_2} have the same prime factors, resulting that P_v is also present in Y.

As already mentioned, the exception occurs when $n_1 = 2$, when Y and Z do not depend on the variable X anymore. One can note that, in principle, it seems not to be necessary that $n_2, n_3 > 2$ for the rule to be considered valid, but only $n_1 > 2$. However, there are cases of known integer solutions of X, Y, Z coprime in which $n_1 > 2$ and $n_2 = 2$ or $n_3 = 2$, i.e. $7^3 + 13^2 = 2^9$ e $2^7 + 17^3 = 71^2$ [2]. This aspect will be clarified in section 2.5.

2.4.2. Situation 2 Analysis (B^2 or C^2 Non-Integers)

Writing B^2(non-integer) in the notation form $I_B + \varepsilon$, in which I_B represents the whole part of B^2 and ε is the decimal part, and as $X^2 + B^2 = C^2$, it follows that:

$$X^2 + I_B + \varepsilon = C^2 \quad (38)$$

$$(X^2 + I_B) + \varepsilon = C^2 \quad (39)$$

Since X^2 and I_B are integers, then $(X^2 + I_B) = I_C$, in which I_C is the whole part of C^2. This results that $C^2 = I_C + \varepsilon$, that is, if B^2 is a non-integer real, then C^2 also is, and the decimal part ε is common to both. Therefore, situation 2 is not possible, leaving only situation 3.

2.4.3. Situation 3 Analysis (B^2 and C^2 Non-integers)

Assuming that B^2 and C^2 are non-integers, there are one or more prime factors with negative powers in their ratio form. If not, then the prime factors would be raised only to positive powers, leading to B^2 and C^2 integers.

Naming the prime factors with negative powers in B^2 as P_{m^*} and the prime factors with negative powers in C^2 as P_{v^*}, there comes that:

$$n_2 k_{Y,m^*} - (n_1 - 2) k_{X,m^*} = k_{B^2,m^*} \quad (40)$$

$$n_3 k_{Z,v^*} - (n_1 - 2) k_{X,v^*} = k_{C^2,v^*} \quad (41)$$

Developments on this section are further explored in Gregorio [11] and may lead to exceptions to Beal Conjecture, in case of certain conditions are attended.

2.5. Extending the approach for Situation 1

In fact, the initial approach adopted the first shape as the reference for the development (figure 4), what led to $A^n + B^2 A^{n-2} = C^2 A^{n-2}$, and consequently $X = A$, $Y = \sqrt[n_2]{B^2 X^{n_1-2}}$, $Z = \sqrt[n_3]{C^2 X^{n_1-2}}$.

One can also adopt the second element as the basis, resulting in $A^2 B^{n-2} + B^n = C^2 B^{n-2}$, and $X = \sqrt[n_1]{A^2 B^{n_2-2}}$, $Y = B$, $Z = \sqrt[n_3]{C^2 B^{n_2-2}}$. The same applies if one chooses the third element to be the reference, coming to $A^2 C^{n-2} + B^2 C^{n-2} = C^n$ and $X = \sqrt[n_1]{A^2 C^{n_3-2}}$, $Y = \sqrt[n_2]{B^2 C^{n_3-2}}$, $Z = C$.

As one can see, a rational solution of $A^2 + B^2 = C^2$(in which at least A or B or C is integer) may lead to three different types of solutions for $X^{n_1} + Y^{n_2} = Z^{n_3}$:

Table 1. *Types of solutions for $X^{n_1} + Y^{n_2} = Z^{n_3}$ that can be obtained using the first, second and third elements as unaltered basis, respectively. Source: author.*

SOLUTIONS FOR $X^{n1} + Y^{n_2} = Z^{n_3}$	X	Y	Z
I	A	$\sqrt[n_2]{B^2 X^{n_1-2}}$	$\sqrt[n_3]{C^2 X^{n_1-2}}$
II	$\sqrt[n_1]{A^2 B^{n_2-2}}$	B	$\sqrt[n_3]{C^2 B^{n_2-2}}$
III	$\sqrt[n_1]{A^2 C^{n_3-2}}$	$\sqrt[n_2]{B^2 C^{n_3-2}}$	C

The development performed for solution type I is analogous for the solutions II and III and will not be replicated to avoid redundancy. Considering the case of Situation 1, it is clear that:

1) In solution type I, for $n_1 > 2$ and if B^2 and C^2 are integers, then Y and Z depend on X;
2) In solution type II, for $n_2 > 2$ and if A^2 and C^2 are integers, then X and Z depend on Y;

3) In solution type III, for $n_3 > 2$ and if B^2 and A^2 are integers, then X and Y depend on Z.

One can note that, in the case of Situation 1, there are prime factors in common to X, Y, Z since in solution type I,$n_1 > 2$; in solution type II,$n_2 > 2$ and in solution type III,$n_3 > 2$. I. e. one can obtain infinite integer solutions for $X^{n_1} + Y^{n_2} = Z^{n_3}$ with common prime factors from the classic Pythagorean solution $3^2 + 4^2 = 5^2$, among which some of them are following presented:

Table 2. *Solutions for $X^{n_1} + Y^{n_2} = Z^{n_3}$ obtained from $3^2 + 4^2 = 5^2$. Source: author.*

SOLUTIONS FOR $X^{n1} + Y^{n_2} = Z^{n_3}$	X	Y	Z	n_1	n_2	n_3
I-a(3 is common)	3	6	45	6	4	2
I-b(3 is common)	3	36	45	6	2	2
II-a(2 is common)	6	4	10	2	3	2
II-b(2 is common)	6144	4	10240	2	13	2
III-a(5 is common)	15	20	5	2	2	4
III-b(5 is common)	75	10	5	2	4	6

The opposite is also true, that is, from one whole solution for $X^{n_1} + Y^{n_2} = Z^{n_3}$, one can obtain three different rational solutions for $A^2 + B^2 = C^2$. I. e. [1] presented the solution $3^3 + 6^3 = 3^5$, resulting in:

Table 3. *Types of rational solutions for $A^2 + B^2 = C^2$ obtained from $3^3 + 6^3 = 3^5$. Source: author.*

SOLUTIONS FOR $A^2 + B^2 = C^2$	A^2	B^2	C^2
I´	9 (A=X=3)	72	81
II´	4,50	36 (B=Y=6)	40,5
III´	1	8	9 (C=Z=3)

In the example presented, one can note that solution type I´ led to $B^2$and C^2 integers, solution type II´ led to $A^2$and C^2 non-integers and solution type III´ led to $A^2$and B^2 integers.

The Beal Conjecture states that, for $n_1, n_2, n_3 > 2$, the integer solutions for $X^{n1} + Y^{n2} = Z^{n3}$ have a common prime factor. Therefore, the Beal's Conjecture is correct for Situations 1 (possible to happen) and 2 (impossible to happen).

3. Conclusion

Demonstrations revealed the correspondence between real solutions of equations in the forms $A^2 + B^2 = C^2$ (Pythagoras), $\delta^n + \gamma^n = \alpha^n$ (Fermat) and $X^{n1} + Y^{n2} = Z^{n3}$ (Beal), what enabled to show that the Beal Conjecture is correct for possible situations in which:

1) $B^2$and C^2 are integers, if A is taken as an integer reference basis;
2) $A^2$and C^2 are integers, if B is taken as an integer reference basis;
3) $A^2$and B^2 are integers, if C is taken as an integer reference basis;

That is, from the correspondent Pythagoras´ equation, if one of the terms A, B or C is taken as an integer reference basis, demonstrations revealed that the Beal Conjecture is correct if the remaining terms, when squared, are integers.

However, cases different than the previous exposed (situation 3) may lead to exceptions to Beal Conjecture, in case of certain conditions are attended. Developments for Situation 3 are explored in Gregorio [11].

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