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Strong reflection principles and large cardinal axioms

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Abstract: In this article an possible generalization of the Löb's theorem is considered. We proved so-called uniform strong reflection principle corresponding to formal theories which has ω-models.Main result is: let κ be an inaccessible cardinal and H_κ is a set of all sets having hereditary size less than κ,then:$\neg\mathrm{Con}(\mathrm{ZFC}+(\mathrm{V}=\mathrm{H}_\kappa))$.

Keywords: Löb'stheorem, Second Gödelincompleteness Theorem, Consistency, Formal System, Uniform Reflection Principles, Ω-Model Of *ZFC*, Standard Model Of *ZFC*, Inaccessible Cardinal, Weakly Compact Cardinal

1. Introduction

Let Th be some fixed, but unspecified, consistent formal theory.

Theorem1.[1].(Löb's Theorem).

If $Th \vdash \exists x \mathrm{Prov}_{Th}(x, \bar{n}) \rightarrow \Phi_n$ where x is the Gödel number of the proof of the formula with Gödel number n, and $\bar{n}$ is the numeral of the Gödel number of the formulaΦ_n then$Th \vdash \Phi_n$. Taking into account the second Gödelincompleteness theorem it is easyto see that Φ_n not be able to prove$\exists x \mathrm{Prov}_{Th}(x, \bar{n}) \rightarrow \Phi_n$, for disprovable (refutable) and undecidable formulasΦ_n Thus Löb's theorem says that for refutable or undecidable formulasφ_n, the intuition "if there is exists proof ofΦ_n [i.e.$Th \vdash \exists x \mathrm{Prov}_{Th_\omega}(x, \bar{n})$]then$\Phi_n$ [i.e. Th$\vdash \Phi_n$]" is fails.The reason of this phenomenon, consist in thatthe concept of natural numbers is not absolute and therefore in general case statement $Th \vdash \exists x \mathrm{Prov}_{Th_\omega}(x, \bar{n})$ does not asserts that:$Th \vdash \Phi_n$.

Definition 1.LetM_ω^{Th} . be an ω-model of the Th. We said that, Th#is a nice theory over Thor a nice extension of the Thiff:

(i) $Th^{\#}$contains *Th*;
(ii) LetΦ be any closed formula, then

$$[Th \vdash Pr_{Th}([\Phi]^c)] \wedge [M_\omega^{Th} \vDash \Phi]$$

implies. , $Th^{\#} \vdash \Phi$.Here $[\Phi]^c$ is a code of Φ [2].

Definition2.We said that, Th#is a maximally nice theory over Th or a maximally nice extension of the ThiffTh#is consistent and for any consistent nice extension $Th'$of the Th: $Ded(Th^{\#}) \subseteq Ded(Th')$ implies:. $Ded(Th^{\#}) = Ded(Th')$.

Theorem2.(Generalized Löb's Theorem).Assume that (i) Con(Th) and (ii) Thhas an ω-modelM_ω^{Th} . Then theoryThcan be extended to a maximally consistent nice theory Th#.

Theorem3.(Strong Reflection Principle corresponding to ω-model) Assume that (i) $Con(Th)$, (ii) Th has an ω-model M_ω^{Th}.LetΦ be a Th-sentence and let $\Phi_\omega$be a Th-sentence Φ relativized to a model M_ω^{Th}.Then

$$Th_\omega \vdash \Phi_\omega \leftrightarrow Th_\omega \vdash \mathrm{Pr}_{Th_\omega}([\Phi_\omega]^c),$$

$$Th_\omega \vDash \Phi_\omega \leftrightarrow Th_\omega \vDash \mathrm{Pr}_{Th_\omega}([\Phi_\omega]^c).$$

Theorem4.Let κ be an inaccessible cardinal.Then ¬Con(ZFC+∃κ).
Theorem5. ¬Con(NFUA).
Theorem6.¬Con(NFUB).

2. Preliminaries

Let *Th*be some fixed, but unspecified, consistent formal theory. For later convenience, we assume that the encoding is done in some fixed formal theory *S* and that *Th*contains *S*. We do not specify *S* --- it is usually taken to be a formal system of arithmetic, although a weak set theory is often more convenient. The sense in which *S*is contained in *Th* is better exemplified than explained: If *S*is a formal system of arithmetic and *Th*is, say, *ZFC*, then *Th* contains *S* in the sense that there is a well-known embedding, or interpretation, of *S* in*Th*. Since encoding is to take place in *S*, it will have to have a large supply of constants and closed terms to be used as codes (e. g. in formal arithmetic, one has$\bar{0}, \bar{1}, \ldots$).Swill also have certain function symbols to be described shortly. To each formulaΦ of the language of *Th* is assigned a closed term $[\Phi]^c$called the code ofΦ . IfΦ is a formula with a free variable *x*, then$[\Phi(x)]^c$ is a closed term encoding the formula$\Phi(x)$ with x viewed as a syntactic object and not as a parameter. Corresponding to the logical connectives and quantifiers are function symbols: $neg(\cdot), imp(\cdot)$, etc., such that, for all formulae Φ,Ψ:$S \vdash neg([\Phi]^c) = [\neg\Phi]^c, S \vdash imp(\cdot) = [\Phi \to \Psi]^c$, etc. Of particular importance is the substitution operator, represented by the function symbol$ub(\cdot,\cdot)$. For formulae $\Phi(x)$, terms *t* with codes$[t]^c$:

$$S \vdash sub([\Phi(x)]^c, [t]^c) = [\Phi(t)]^c. \quad (2.1)$$

Iteration of the substitution operator *sub* allows one to define function symbols $sub_1, sub_2, \ldots, sub_n$,such that

$$S \vdash sub_n([\Phi(x_1, x_2, \ldots, x_n,)]^c, [t_1]^c, [t_2]^c, \underset{c}{\ldots}, [t_n]^c,) = \\ = [\Phi(t_1, t_2, \ldots, t_n)]. \quad (2.2)$$

It well known [2],[3] that one can also encode derivations and have a binary relation $\mathrm{Prov}_{Th}(x, y)$ (read "x proves y" or "x is a proof of y") such that for closed$t_1, t_2 : S \vdash \mathrm{Prov}_{Th}(t_1, t_2)ifft_1$ is the code of a derivation in Th of the formula with code t_2. It follows that

$$Th \vdash \Phi \leftrightarrow S \vdash \ \mathrm{Prov}_{Th}(t, [\Phi]^c) \quad (2.3)$$

for some closed term *t*. Thus one can define predicate $\mathrm{Pr}_{Th}(y)$:

$$\mathrm{Pr}_{Th}(y) \leftrightarrow \exists x \mathrm{Prov}_{Th}(x, y) \quad (2.4)$$

and therefore one obtain a predicate asserting provability.

Remark2.1.

We note that is not always the case that [2]-[3]:

$$Th \vdash \Phi \leftrightarrow S \vdash \mathrm{Pr}_{Th}([\Phi]^c). \quad (2.5)$$

It well known [3] that the above encoding can be carried out in such a way that the following important conditions D1,D2 and D3 are met for allsentences [2],[3]:

$$D1.\ Th \vdash \Phi \to S \vdash \mathrm{Pr}_{Th}([\Phi]^c),$$

$$D2.\ \mathrm{Pr}_{Th}([\Phi]^c) \to \mathrm{Pr}_{Th}([\mathrm{Pr}_{Th}([\Phi]^c)]^c), (2.6)$$

$$D3.\ \mathrm{Pr}_{Th}([\Phi]^c) \wedge \mathrm{Pr}_{Th}([\Phi \to \Psi]^c) \to \mathrm{Pr}_{Th}([\Psi]^c).$$

Conditions D1,D2 and D3 are called the Derivability Conditions.

Assumption2.1.

We assume throughout that:

(i) the language of *Th* consists of: numerals $\bar{0}, \bar{1}, \ldots$)
countableset of the numerical variables:$\{v_0, v_1, \ldots\}$
countable set*F* of the set variables:

$$F = \{x, y, z, X, Y, Z, \mathfrak{I}, \mathfrak{K}, \mathfrak{M}, \mathfrak{N}, \ldots\};$$

countable set of the *n*–ary function symbols: $f_0^n, f_1^n, \ldots, f_m^n, \ \ldots$;
countable set of the *n*–ary relation bols:$R_0^n, R_1^n, \ldots, R_m^n, \ \ldots$;
connectives:$\neg, \to$;
quantifier:$\forall$.

(ii) *Th*contains a theory *Th**:

$$Th^* = ZFC + \exists(\omega - model\ \ of\ ZFC).$$

(iii) *Th*has anω-model M_ω^{Th} .

Theorem2.1.

(*Löb's Theorem*). Let be (1) *Con*(*Th*) and (2) Φbe closed. Then

$$Th \vdash \mathrm{Pr}_{Th}([\Phi]^c) \to \Phi. \quad (2.7)$$

It well known that replacing the induction scheme in Peano arithmetic *PA* by the ω-rule with the meaning "if the formula$A(n)$ is provable for all *n*, then the formula$A(x)$is provable" :

$$\frac{A(0), A(1), \ldots, A(n)}{\forall x A(x)} \quad (2.8)$$

leads to complete and sound systemPA_∞ where each true arithmetical statement is provable. S.Feferman showed that an equivalent formal system $Th^\#$ can be obtained by erecting on *Th=PA* a transfinite progression of formal temsPA_α according to the following scheme

$$PA_0 = PA,$$

$$PA_{\alpha+1} = PA_\alpha + \{\forall x Pr_{PA_\alpha}([A(\dot{x})]^c) \to \forall x A(x)\}, (2.9)$$

$$PA_\lambda = \cup_{\alpha<\lambda} PA_0,$$

where$A(x)$ is a formula with one free variable and where λis a limit ordinal. Then$Th^\# = \cup_{\alpha \in O} PA_0$,Obeing Kleene's system of ordinal notations, is equivalentto a theory:.PA_∞. It is easy to see that a theory$Th^\# = PA^\# = PA_\infty$, i.e.$Th^\#$ $Th^\#$ is a maximally nice extension of thePA.

Generalized Löb's Theorem. Strong Reflection Principle Corresponding to ω-model.

Definition3.1.

AnTh-wffΦ)(well-formed formula Φ)is closed i.e., Φ) is a Th- sentence iff it has no free variables; a wff Ψ is open if it has free variables. We'll use the slang 'k – place open wff' to mean a wff with k distinct free variables. Given a modelM^{Th} of the Thand a Th–sentenceΦ, we assume known the meaning of$M^{Th} \vDash \Phi$- i. e. Φ is true inM^{Th}, (see for example [4],[5],[6]).

Definition3.2.

Let M_ω^{Th} be an ω-model of the Th. We said that,$Th^\#$ is a

nice theory over Th or a nice extension of the Thiff:
(i) $Th^{\#}$ contains a theoryTh;
(ii) Let Φbe any closed formula, then

$$[\ Th \vdash \mathrm{Pr}_{Th}([\Phi]^c)] \wedge \left[M\ \ _{\omega}^{Th} \vDash \Phi\right]$$

impliesthat:$Th^{\#} \vdash \Phi$.

Definition3.3.

We said that $Th^{\#}$ is a maximally nice theory over *Th*or a maximally nice extension of the *Th*iff$Th^{\#}$ is consistent and for any consistent nice extension$Th'$of the *Th*: $Ded(Th^{\#}) \subseteq Ded(Th')$implies$Ded(Th^{\#}) = Ded(Th')$.
Lemma3.1. Assume that: (*i*)*Con*(*Th*)and (*ii*) $Th \vdash Pr_{Th}([\Phi]^c)$,where Φ is a closed formula. Then: $Th \nvdash \mathrm{Pr}_{Th}([\neg\Phi]^c)$.
Proof.Let $Con_{Th}(\Phi)$ be the formula

$$Con_{Th}(\Phi) \triangleq$$

$$\forall t_1 \forall t_2 \neg[\mathrm{Prov}_{Th}(t_1, [\Phi]^c) \wedge \mathrm{Prov}_{Th}(t_2, neg([\Phi]^c))]$$

$$\leftrightarrow$$

$$\neg\exists t_1 \neg\exists t_2[\mathrm{Prov}_{Th}(t_1, [\Phi]^c) \wedge \mathrm{Prov}_{Th}(t_2, neg([\Phi]^c))] \tag{3.1}$$

wheret_1, t_2 is a closed term. We note that under canonicalobservation, one obtain
$Th + Con(Th) \vdash Con_{Th}(\Phi)$ for any closedwffΦ.
Suppose that: $Th \vdash Pr_{Th}([\neg\Phi]^c)$, then assumption (ii) gives

$$Th \vdash\ \mathrm{Pr}_{Th}([\Phi]^c) \wedge \mathrm{Pr}_{Th}([\neg\Phi]^c) \tag{3.2}$$

From (3.1) and (3.2) one obtain(3.3)

$$\exists t_1 \neg\exists t_2[\mathrm{Prov}_{Th}(t_1, [\Phi]^c) \wedge\ \mathrm{Prov}_{Th}(t_2, neg([\Phi]^c))]$$

But the formula (3.3) contradicts the formula (3.1). Therefore:$Th \nvdash \mathrm{Pr}_{Th}([\neg\Phi]^c)$.

Lemma3.2.

Assume that: (i)Con(Th)and (ii) $Th \vdash \mathrm{Pr}_{Th}([\neg\Phi]^c)$, where Φ is a closed formula. Then $Th \nvdash \mathrm{Pr}_{Th}([\Phi]^c)$.

Theorem3.1.

[7],[8]. (Generalized Löb's Theorem). Assume that:Con(Th). Then theory Th can be extended to a maximally consistent nice theory $Th^{\#}$ over Th.

Proof.Let .,$\Phi_1, \dots, \Phi_i$, ...be an enumeration of all wff's of the theory Th (this can be achieved if the set of propositional variables can be enumerated). Define a chain

$$\wp = \{Th_i | i \in \mathbb{N}\}, Th_1 = Th,$$

of the consistent theories inductively as follows: assume that theory Th_i is defined.
(i) Suppose that a statement (3.4) is satisfied

$$Th \vdash \mathrm{Pr}_{Th}([\Phi_i]^c)\text{and}$$

$$[Th \nvdash \Phi_i] \wedge \left[M\ \ _{\omega}^{Th} \vDash \Phi_i\right] \tag{3.4}$$

Then we define theory Th_{i+1} as follows:

$$Th_{i+1} \triangleq Th_i \cup \{\Phi_i\}$$

(ii) Suppose that a statement (3.5) is satisfied

$$Th \vdash \mathrm{Pr}_{Th}([\neg\Phi_i]^c)\text{and}$$

$$[Th \nvdash \neg\Phi_i] \wedge \left[M\ \ _{\omega}^{Th} \vDash \neg\Phi_i\right]. \tag{3.5}$$

Then we define theoryTh_{i+1}as follows

$$Th_{i+1} \triangleq Th_i \cup \{\neg\Phi_i\}.$$

(iii) Suppose that a statement (3.6) is satisfied

$$Th \vdash \mathrm{Pr}_{Th}([\Phi_i]^c)\text{and}$$

$$Th_i \vdash \Phi_i. \tag{3.6}$$

Then we define theory Th_{i+1} as follows:

$$Th_{i+1} \triangleq Th_i \cup \{\Phi_i\}$$

(iv) Suppose that a statement (3.7) is satisfied

$$Th \vdash \mathrm{Pr}_{Th}([\neg\Phi_i]^c)\text{and}$$

$$Th_i \vdash \neg\Phi_i. \tag{3.7}$$

Then we define theory Th_{i+1} as follows: $Th_{i+1} \triangleq Th_i$.
We define now theory $Th^{\#}$ as follows

$$Th^{\#} = \cup_{i \in \mathbb{N}} Th_i. \tag{3.8}$$

First, notice that eachTh_i is consistent. This is done by induction on i and by Lemmas 3.1-3.2. By assumption, the case is true when $i = 1$. Now, suppose Th_i is consistent. Then its deductive closure$Ded(Th_i)$ is also consistent. If a statement (3.6) is satisfied, i.e. $Th \vdash \mathrm{Pr}_{Th}([\Phi_i]^c)andTh_i \vdash \Phi_i$then clearly$Th_{i+1} \triangleq Th_i \cup \{\Phi_i\}$ is consistent since it is a subset of closure $Ded(Th_i)$.If a statement (3.7) is satisfied, i.e., $Th \vdash \mathrm{Pr}_{Th}([\neg\Phi_i]^c)$ and $Th_i \vdash \neg\Phi_i$ then clearly $Th_{i+1} \triangleq Th_i \cup \{\neg\Phi_i\}$ is consistent since it is a subset of closure $Ded(Th_i)$.Otherwise:
(i) if a statement (3.4) is satisfied, i.e. $Th \vdash \mathrm{Pr}_{Th}([\Phi_i]^c)$ and $Th \nvdash \Phi_i$ then clearly $Th_{i+1} \triangleq Th_i \cup \{\Phi_i\}$ is consistent by Lemma 3.1 and by one of the standard properties of consistency: $\Delta \cup \{A\}$ is consistent iff$\Delta \nvdash \neg A$;
(ii) if a statement (3.5) is satisfied, i.e. $Th \vdash \mathrm{Pr}_{Th}([\neg\Phi_i]^c)andTh \nvdash \neg\Phi_i$, then clearly$Th_{i+1} \triangleq Th_i \cup \{\neg\Phi_i\}$ is consistent by Lemma 3.2 and by one of the standard properties of consistency: $\Delta \cup \{\neg A\}$is consistentiff$\Delta \nvdash \neg A$.

Next, notice $Ded(Th^{\#})$ is a maximally consistent nice extension of the set $Ded(Th)$. A set$Ded(Th^{\#})$ is consistent because, by the standard Lemma 3.3 below, it is the union of a chain of consistent sets. To see that $Ded(Th^{\#})$ is maximal, pick any wffΦ. Then Φ is some Φ_i in the enumerated list of all wff's. Therefore for any Φ such that $Th \vdash \Pr_{Th}([\Phi_i]^c)$ or $Th \vdash \Pr_{Th}([\neg\Phi_i]^c)$ either $\Phi \in Th^{\#}$or$\neg\Phi \in Th^{\#}$.

Since $Ded(Th_{i+1}) \subseteq Ded(Th^{\#})$ we have $\Phi \in Ded(Th^{\#})$ or $\neg\Phi \in Ded(Th^{\#})$, which implies that $Ded(Th^{\#})$, is maximally consistent nice extension of the $Ded(Th)$.

Lemma3.3.

The union of a chain$\wp = \{\Gamma_i | i \in \mathbb{N}\}$ of theconsistentsetsΓ, ordered by$\subseteq$, is consistent.

Definition3.4.

(a) Assume that a theoryThhas ω-model M_{ω}^{Th} and Φ is aTh-sentence. Let $\Phi_\omega$be a Th-sentence $\Phi$with all quantifiers relativized to ω-model M_{ω}^{Th} [9];

(b) Assume that a theory Thhas standard model SM^{Th}

And Φ is a Th-sentence. Let Φ_{SM} be a Th-sentence Φ with all quantifiers relativized to a model SM^{Th} [9].

Remark3.1.

In some special cases we denote a sentence Φ_ω by a symbol: $\Phi[M_\omega^{Th}]$.

Definition3.5.

(a)Assume that Thhas an ω-model M_{ω}^{Th}. Let Th_ωbe a theory Threlativized toa model M_{ω}^{Th} - i.e., any Th_ω-sentenceΨ has a form Φ_ω for some Th-sentence Φ[9];

(b) Assume that Th has an standard model SM^{Th}. Let Th_{SM}be a theory Th relativized to a model SM^{Th}- i.e., any Th_{SM}-sentence Ψ has a form Φ_{SM} for some Th-sentence Φ [9].

Remark3.2.

In some special cases we denote a theory Th_ω by a symbol:$Th[M_\omega^{Th}]$.

Definition3.6.

(a) For a givenω-model M_{ω}^{Th} of the *Th*and for any Th_ω-sentenceΦ_ω, we define relation$M_{\omega}^{Th} \vDash_* \Phi_\omega$ such that the nextequivalence:

$$M_{\omega}^{Th} \vDash_* \Phi_\omega \text{ iff}[Th^\dagger \vdash \Phi_\omega] \wedge$$

$$\wedge \left[\left(Th_\omega \vdash \Pr_{Th_\omega}([\Phi_\omega]^c) \leftrightarrow Th^\dagger \vdash \Phi_\omega\right)\right], \quad (3.9.a)$$

where$Th^\dagger \triangleq Th + \exists M_\omega^{Th}$ is satisfied.

(b) For a given standard model SM^{Th} of the theoryThand for anyTh_{SM}-sentenceΦ_{SM} wedefine relation$M_{SM}^{Th} \vDash_* \Phi_{SM}$ such that the next equivalence:

$$M_{SM}^{Th} \vDash_* \Phi_{SM} \text{iff } (Th^\dagger \vdash \Phi_{SM}) \wedge \quad (3.9.b)$$

$$\wedge \left(Th_{\mathrm{SM}} \vdash \Pr_{Th_{SM}}([\Phi_{SM}]^c) \leftrightarrow Th^\dagger \vdash \Phi_{SM}\right),$$

where $Th^\dagger \triangleq Th + \exists M_{SM}^{Th}$, issatisfied.

Theorem3.2.(Strong Reflection Principle correspondingto ω-model).Assume that: (i) Con(Th),(ii) Th has ω-model M_{ω}^{Th},i.e.$M_{\omega}^{Th} \vDash Th_\omega$.Let Φ be a Th-sentence. Then

$$(a)\, Th_\omega \vdash \Pr_{Th_\omega}([\Phi_\omega]^c) \leftrightarrow Th_\omega \vdash \Phi_\omega,$$

$$(b)\, Th_\omega \vDash \Pr_{Th_\omega}([\Phi_\omega]^c) \leftrightarrow Th_\omega \vDash \Phi_\omega (3.10)$$

Proof.(a) Let Φ is any axiom of thetheory *Th.* Then statement (3.10) immediately follows from Definition 3.6 (a).The one direction is obvious. For the other, assume that

$Th_\omega \vdash \Pr_{Th_\omega}([\Phi_\omega]^c)$,(3.11)

$Th_\omega \nvdash \Phi_\omega$and $Th_\omega \vdash \neg\Phi_\omega$. Then

$Th_\omega \vdash \Pr_{Th_\omega}([\neg\Phi_\omega]^c)$. (3.12)

Notethat (i)+(ii) implies $Con(Th_\omega)$.Let Con_{Th_ω}be the formula:

$Con_{Th_\omega} \triangleq \forall t_1 \forall t_2 \forall t_3 (t_3 = [\Phi_\omega]^c) \neg\left[\mathrm{Prov}_{Th_\omega}(t_1, [\Phi_\omega]^c) \wedge \mathrm{Prov}_{Th_\omega}(t_2, neg([\Phi_\omega]^c))\right] \leftrightarrow$(3.13)

$\neg\exists t_1 \neg\exists t_2 \neg\exists t_3 (t_3 = [\Phi_\omega]^c)\left[\mathrm{Prov}_{Th_\omega}(t_1, [\Phi_\omega]^c) \wedge \mathrm{Prov}_{Th_\omega}(t_2, neg([\Phi_\omega]^c))\right]$,

Heret_1, t_2, t_3 is a closed term. Note that in any ω-model M_{ω}^{Th} by the canonical observation one obtain the equivalence: $Con(Th_\omega) \leftrightarrow\ Con_{Th_\omega}$, But the formulae: (3.11) – (3.12)contradicts the formula (3.13). Therefore $Th_\omega \nvdash \Phi_\omega$ and $Th_\omega \nvdash \Pr_{Th_\omega}([\neg\Phi_\omega]^c)$.

Then a theory $Th'_\omega = Th_\omega + \neg\Phi_\omega$ is consistent and from the above observation one have obtain that:

$$Con(Th'_\omega) \leftrightarrow\ Con_{Th'_\omega}, \text{ where}$$

$$Con_{Th'_\omega} \leftrightarrow \quad (3.14)$$

$$\leftrightarrow \neg\exists t_1 \neg\exists t_2 \neg\exists t_3 (t_3 = [\Phi_\omega]^c)\left[\mathrm{Prov}_{Th'_\omega}(t_1, [\Phi_\omega]^c) \wedge \mathrm{Prov}_{Th'_\omega}(t_2, neg([\Phi_\omega]^c))\right],$$

On the other hand one obtain

$$Th'_\omega \vdash \Pr_{Th'_\omega}([\Phi_\omega]^c),\ Th'_\omega \vdash \Pr_{Th'_\omega}([\neg\Phi_\omega]^c)\ (3.15)$$

But the formulae (3.15), contradicts the formula (3.14). This contradiction completed the proof.

(b) The one direction is obvious. For the other, assume that:(i)$Th_\omega \vDash \Pr_{Th_\omega}([\Phi_\omega]^c)$ *and* (ii)$Th_\omega \vDash\ \neg\Phi_\omega$. From ($ii$) using derivability condition D1 (see Remark 2.1) one obtain $Th_\omega \vDash \Pr_{Th_\omega}([\neg\Phi_\omega]^c)$. Therefore one obtainthe contradiction

$$Th_\omega \vDash \Pr_{Th_\omega}([\Phi_\omega]^c) \wedge \Pr_{Th_\omega}([\neg\Phi_\omega]^c)\,.$$

This contradiction completed the proof.

Definition3.7.(a)Assume that: (i) *Th*has an ω-model M_{ω}^{Th},(ii) $M_{\omega,\vDash}^{Th}$ is a set and (iii) $M_{\omega}^{Th} \vDash_* Th_\omega$. Then we said that M_{ω}^{Th} is a strong ω-model of the *Th* and denote such ω-model of the Thas$M_{\omega,\models*}^{Th}$.

(b) Assume that: (*i*) *Th*has an standard modelSM^{Th},(*ii*) $SM_{\models*}^{Th}$ is a set and (*iii*)$SM^{Th} \models* Th_{SM}$. Then we said that $SM^{Th}$is a strong standard model of the *Th* and denote such standard model of the *Th* as $SM_{\models*}^{Th}$.

Remark3.3. Note that there exists formal theories*Th*-which has not a strong standard models.For example a theory *ZFC*+(*V=L*) has not any strong standard model.

Definition3.8.(a) Assume that *Th*has a strong ω-model $M_{\omega,\models*}^{Th}$.Then we said that *Th*is a *strongly consistent*.

(b) Assume that *Th* has a strong standard model $SM_{\models*}^{Th}$

Then we said that *Th* is a *strongly SM-consistent*

Definition3.9.(a) Assume that a theory*Th*has a strong ω-model $M_{\omega,\models*}^{Th}$ .and Φ is a*Th*–sentence. Let $\Phi_{\omega,\models*}$ be a *Th*–sentence Φ with all quantifiers relativized to a strong ω-model $M_{\omega,\models*}^{Th}$.

(b) Assume that *Th* has a strong standard model $SM_{\models*}^{Th}$and Φ is a *Th*-sentence Let $\Phi_{SM,\models*}$ be a *Th*-sentence Φ with all quantifiers relativized toa model$SM_{\models*}^{Th}$.

Remark3.4.

In some special cases we denote a sentence $\Phi_{\omega,\models*}$by a symbol:$\Phi\left[M_{\omega,\models*}^{Th}\right]$.

Definition3.10. Assume that a theory*Th*has a strong ω-model $M_{\omega,\models*}^{Th}$.Let $Th_{\omega,\models*}$ be a theory *Th*relativised to a model $M_{\omega,\models*}^{Th}$ i.e., any $Th_{\omega,\models*}$-sentence Ψ has a form $\Phi_{\omega,\models*}$ for some*Th*-sentence Φ.

Remark3.5.

In some special cases we denote a theory $Th_{\omega,\models*}$by a symbol:$Th\left[M_{\omega,\models*}^{Th}\right]$.

Definition3.11.

(a) Let *Th*be a theory such that Assumption 2.1 is satisfied. Let $\widehat{Con}\left(Th; M_{\omega,\models*}^{Th}\right)$be a predicate in *Th*asserting that$M_{\omega,\models*}^{Th}$ is a strong ω-model of the *Th*.Then a tence$Con\left(Th; M_{\omega,\models*}^{Th}\right)$ such that

$$Con\left(Th; M_{\omega,\models*}^{Th}\right) \leftrightarrow \exists M_{\omega,\models*}^{Th}\widehat{Con}\left(Th; M_{\omega,\models*}^{Th}\right)$$

isa sentence in *Th* asserting that *Th* has a strong ω-model $M_{\omega,\models*}^{Th}$.(b) Let *Th**be a theory:

$$Th^{*} = Th + Con\left(Th; M_{\omega,\models*}^{Th}\right)$$

Let$Con\left(Th^{*}; M_{\omega,\models*}^{Th^{*}}\right)$ be a sentence in $Th*$ asserting that $Th*$ has a strong ω-model$M_{\omega,\models*}^{Th^{*}}$.

Lemma3.4.Assumethat a theory *Th* has a strong ω-model $SM_{\omega,\models*}^{Th}$ and a theory $Th*$ has a strong ω-model $M_{\omega,\models*}^{Th^{*}}$.Then: (*i*) a sentence$Con\left(Th; M_{\omega,\models*}^{Th}\right)$ is a*Th*-sentence, (*ii*);a sentence $Con\left(Th^{*}; M_{\omega,\models*}^{Th^{*}}\right)$ is a Th*-sentence.

Proof.Immediately follows from Definition 3.6 and Definition 3.11.

Assumption3.1.We now assume,throughout this subsection that *Th* is a strongly consistent, i.e. a tence$Con\left(Th; M_{\omega,\models*}^{Th}\right)$ is true in any ω-model M_{ω}^{Th} of the *Th*.

Remark3.6.

Note that:

$$Con\left(Th; M_{\omega,\models*}^{Th}\right) \leftrightarrow Con_{Th_{\omega,\models*}} \tag{3.16}$$

where

$$Con_{Th_{\omega,\models*}} \leftrightarrow \neg\mathrm{Pr}_{Th_{\omega,\models*}}\left(\left[\Phi_{\omega,\models*}\right]^{c}\right) \tag{3.17}$$

Herea sentence$\Phi_{\omega,\models*}$is refutablein$Th_{\omega,\models*}$.

Remark3.6.Note that:

$$Con\left(Th^{*}; M_{\omega,\models*}^{Th^{*}}\right) \leftrightarrow Con_{Th_{\omega}^{*}} \tag{3.18}$$

where

$$Con_{Th_{\omega,\models*}^{*}} \leftrightarrow \neg\mathrm{Pr}_{Th_{\omega,\models*}^{*}}\left(\left[\Phi_{\omega,\models*}^{*}\right]^{c}\right). \tag{3.19}$$

Herea sentence$\Phi_{\omega,\models*}^{*}$is refutablein$Th_{\omega,\models*}^{*}$.

Lemma3.5.

Ander Assumption3.1 a theory Th* is a strongly consistent.

Proof. Assume that a theoryTh*is no strongly consistent, that is, has not any strong ω-model $M_{\omega,\models*}^{Th^{*}}$ of the Th*. This means that there is no any model M^{Th} of the theory Th in which a sentence$Con\left(Th; M_{\omega,\models*}^{Th}\right)$is true and therefore a sentence $\neg Con\left(Th; M_{\omega,\models*}^{Th}\right)$ is true in any model $M^{Th}$of the theoryTh.In particular a sentenceΘ:

$\Theta \triangleq \neg Con\left(Th; M_{\omega,\models*}^{Th}\right)$ (3.20)is true in any strong ω-model $M_{\omega,\models*}^{Th}$ of theTh.Therefore from formula (3.16) one obtain, that a formula $\neg Con_{Th_{\omega,\models*}}$is true in any strong ω-model $M_{\omega,\models*}^{Th}$ oftheTh,i.e.

$$\widetilde{M}_{\omega,\models*}^{Th} \models \neg Con_{\widetilde{Th}_{\omega,\models*}} \tag{3.21}$$

Here $\widetilde{Th}_{\omega,\models*} \triangleq Th_{\omega,\models*}\left[\widetilde{M}_{\omega,\models*}^{Th}\right]$, i.e. $\widetilde{Th}_{\omega,\models*}$. is a theory $Th_{\omega,\models*}$ relativized to a strongω-model $\widetilde{M}_{\omega,\models*}^{Th}$, seeRemark3.2.From formulae (3.17) and (3.21) one obtain

$$\widetilde{M}_{\models*}^{Th}\models\mathrm{Pr}_{\widetilde{Th}_{\omega,\models*}}\left(\left[\widetilde{\Phi}_{\omega,\models*}\right]^{c}\right). \tag{3.22}$$

Here $\widetilde{\Phi}_{\omega,\models*} \triangleq \Phi_{\omega,\models*}\left[\widetilde{M}_{\omega,\models*}^{Th}\right]$, i.e. $\widetilde{\Phi}_{\omega,\models*}$ is a sentence$\Phi_{\omega,\models*}$relativized to a strong ω-model $\widetilde{M}_{\omega,\models*}^{Th}$,see Remark 3.1.So from formula (3.22) using a Strong Reflection Principle (Theorem3.2.b) one obtain

$\widetilde{M}_{\omega,\models*}^{Th} \models \widetilde{\Phi}_{\omega,\models*}$, where a sentence $\widetilde{\Phi}_{\omega,\models*}$ is refutable in a theory $\widetilde{Th}_{\omega,\models*}$, i.e. $\widetilde{Th}_{\omega,\models*} \vdash \neg\widetilde{\Phi}_{\omega,\models*}$ andtherefore $\widetilde{M}_{\omega,\models*}^{Th} \models \neg\widetilde{\Phi}_{\omega,\models*}$.

Thus a sentence$\widetilde{\Theta}_{1} \triangleq \widetilde{\Phi}_{\omega,\models*} \wedge \neg\widetilde{\Phi}_{\omega,\models*}$is satisfiedina model$\widetilde{M}_{\omega,\models*}^{Th}$, i.e.$\widetilde{M}_{\omega,\models*}^{Th} \models \widetilde{\Theta}_{1}$.But a sentence $\widetilde{\Theta}_{1}$ contrary to the assumption that *Th* is a strongly consistent. This contradiction completed the proof.

Theorem3.3.

Th has notany strong ω-model $M_{\omega,\vDash*}^{Th}$.Proof. By Lemma 3.5 and formula (3.17) one obtain that $Th_{\omega,\vDash*}^{*} \vdash Con_{Th_{\omega,\vDash*}^{*}}$. But Gödel's Second Incompleteness Theorem applied to $Th_{\omega,\vDash*}^{*}$ asserts that a sentence $Con_{Th_{\omega,\vDash*}^{*}}$ is in $Th_{\omega,\vDash*}^{*}$. This contradiction completed the proof.

Theorem3.4.

ZFC has not any strong ω-model $M_{\omega,\vDash*}^{ZFC}$.

Proof. Immediately follows from Theorem 3.3 and definitions.

Theorem3.5.

ZFC has not any strong standard model.$SM_{\vDash*}^{ZFC}$

Proof. Immediately follows from Theorem 3.4 and definitions.

Theorem3.6.

ZFC+*Con*(*ZFC*) is incompatible with all the usual large cardinal axioms [10],[11] which imply the existence of a strong standard model of *ZFC*.

Proof.Theorem3.6 immediately follows from Theorem 3.5.

Lemma3.6.

Let κ be an inaccessible cardinal and let H_κ be a set of all sets having hereditary size less then, κ.Suppose that $Con(ZFC + \exists\kappa)$.Then$H_\kappa$forms a strong standard model of *ZFC*.

Proof. From definitions one obtain that H_κ forms some standard model*SM* of *ZFC*. Let φ^{ZFC} be any axiom of ZFC and $Th^{\dagger} \triangleq Th + \exists H_\kappa$.Thenbydefinitions one obtain

$$(Th^{\dagger} \vdash \varphi^{ZFC}[H_\kappa])\wedge \tag{3.23}$$
$$\wedge(Th_{SM} \vdash \varphi^{ZFC}[H_\kappa] \leftrightarrow Th^{\dagger} \vdash \varphi^{ZFC}[H_\kappa]).$$

Using Strong Reflection Principle (see Theorem3.2) from statement (3.23)one obtain that RHS of the formula (3.9.b) is satisfied. Thus $H_\kappa \vDash* ZFC$.

Theorem3.7.

Letκ be an inaccessible cardinal.Then $\neg Con(ZFC + \exists\kappa)$.

Proof. Let H_κ be a set of all sets having hereditary size less then κ.From Lemma3.6 we know that $H_\kappa$forms a strong standard model of *ZFC.*Therefore Theorem 3.7 immediately follows from Theorem 3.6.

New Foundation [*NF* for short] was introduced by Quine [13].It well known that his approach for blocking paradoxes of naïve set theory was to introduced a special stratification condition in the comprehension schema. Jensen [14] introduced*NFU*, the [slight?]version of *NF* in which axiom of the extensionality was weakened to allow ur-elements which are not sets. The theory *NFU* has a universal class,*V*,which contains all of its subsets.

Definition3.12.

We say that a set *S* is a *Cantorian*iffthere is a bijection of *S* with the set *USC*(*S*)consistingof all the singletons whose members lie in *S*. A set is *strongly Cantorian*iff the map $x \mapsto \{x\}$provides a bijection of *S* withaset *USC*(*S*).

Holmes [15],[16],[17]introduced the system *NFUA* which is obtained from *NFU* by adjoining the axiom that:"EveryCantorian set is a strongly Cantorian."

Theorem3.8.(Solovay,1995)[18].The following theories are equiconsistent:

(i) $Th_1 \triangleq$ ZFC+{"there is n-Mahlo cardinal": $n \in \omega$},

(ii) $Th_2 \triangleq NFUA$.

Holmes also introduced stronger theory*NFUB* [19].Note that in *NFU* one can introduce ordinal such that any ordinal$\mathcal{R}$ consists of the class of all well-orderings which are order-isomorphic to a given well-ordering.

Definition3.13.

We say that an ordinal$\mathfrak{H}$is Cantorian if the underlying sets of the well-orderings which are its members are all Cantorian.

Definition3.14.

We say thata subcollectionΣ of the Cantorian ordinals is *coded* if there is there is some set σ whose members among the Cantorian ordinals are precisely the members of Σ.

The system *NFUB* is obtained from the system *NFUA* by adding the axiom schema which asserts that any subcollection Σ of the Cantorian ordinals which is definable by a formula of the language of *NFUB*[possible unstratified and possible with parameters] is coded by some set σ.

Theorem3.9.

(Solovay,1997)[19].Let ZFC^{-}be a theory consisting all the axioms of ZFC except the power set axiom.The following theories are equiconsistent:

(i) $Th_1 \triangleq$
$ZFC^{-} + \{$there is a weakly compact cardinal$\}$,

(ii) $Th_2 \triangleq$ *NFUB.*

Remark3.7.

The formulation of "weak compactness"we shall use is: κ is weakly compact if κ is strongly inaccessible and every κ-tree has a branch [19].

Theorem3.10.¬Con(NFUA).

Proof. Theorem3.10 immediately follows from Theorem3.7 and Theorem3.8.

Theorem3.11.¬Con(NFUB)

Proof.Theorem3.11 immediately follows from Theorem3.7, Theorem3.9 and definitions.

Strong Reflection Principle Corresponding to Nonstandard Models. Let*Th* be consistent formal theory. When interpretedas a

proof within first order theory *Th*,Dedekind's categoricity proof for *PA*shows that the each modelM^{Th} of the *Th* has the unique sub-model $M^{PA} \subset M^{ZFC} \subseteq M^{Th}$of the*PA*arithmetic, up to isomorphism, that imbedsas an initial segment of all models of *PA* contained within model M^{ZFC} of set theory *ZFC*.In the standard model of the *Th* this smallest model of the *PA* is the standard model$SM^{PA} \approx \mathbb{N}$of *PA*.

Remark4.1.

Note that.in any nonstandard model$NsM^{Th}$of the *Th*it may by a nonstandard model $NsM^{PA}$of the *PA*.

Remark4.2.

Note thatin any nonstandard model of the *PA*, the terms$\bar{0}$, $S\bar{0} = \bar{1}, SS\bar{0} = \bar{2}, \ldots$ comprise the initial segment isomorphic to SM^{PA}. This initial segment is called the standard cutThe order type of any nonstandard model of *PA*is equal to ℕ+A×ℤ for some linear order A[12].

Definition4.1.

Let NsM^{Th}bea nonstandard model of the *Th* and Φ is a *Th*-sentence. Let Φ_{NsM} be a Th-sentence Φ with all quantifiers relativized to nonstandardmodel NsM^{Th}.In somespecial cases we denote thissentence by symbol $\Phi[NsM^{Th}]$.

Definition4.2.Let Th_{NsM} be a theory *Th* relativized to a model NsM^{Th}.In some special cases we denote this theory as:$Th[NsM^{Th}]$

Definition4.3.

One can define a predicate $\mathrm{Pr}_{Th_{NsM}}(y)$such that for all y∈ NsM^{Th}theequivalence::

$$\mathrm{Pr}_{Th_{NsM}}(y) \leftrightarrow \exists x(x \in M^{PA})\mathrm{Prov}_{Th_{NsM}}(x, y). \qquad (4.1)$$

is satisfied.Therefore one obtain a predicate asserting provability in a theoryTh_{NsM}.

Definition4.4.

For a given nonstandard model $NsM^{Th}$of the *Th*and any Th_{NsM} -sentence Φ_{NsM} we define relation $NsM^{Th} \vDash_* \Phi_{NsM}$such the next equivalence:

$$NsM^{Th} \vDash_* \Phi_{NsM} \text{iff } (Th^{\dagger} \vdash \Phi_{NsM}) \wedge \qquad (4.2)$$

$$\wedge \left[\left(Th_{SM} \vdash \mathrm{Pr}_{Th_{NsM}}([\Phi_{NsM}]^c)\right) \leftrightarrow (Th^{\dagger} \vdash \Phi_{NsM})\right],$$

where $Th^{\dagger} \triangleq Th + \exists NsM^{Th}$, issatisfied.

Theorem4.1.(StrongReflectionPrinciple corresponding to nonstandard model)Assume that:(*i*) $Con(Th)$,(*ii*)*Th* has annonstandard modelNsM^{Th}, i.e. $NsM^{Th} \vDash Th_{NsM}$ .LetΦis a *Th*-sentence. Then

$$\mathrm{Pr}_{Th_{NsM}}([\Phi_{NsM}]^c) \leftrightarrow Th_{NsM} \vdash \Phi_{NsM}. \qquad (4.3)$$

Proof. The proof completely to similarly a proof of the Theorem 3.2.

Definition4.5.Assume that: (*i*) *Th* has an nonstandard model NsM^{Th} ,(*ii*) $NsM^{Th}_{\vDash_*}$ is a set and (*ii*)$NsM^{Th} \vDash_* \Phi_{NsM}$. Then we said that $NsM^{Th}$is astrong nonstandard model of the *Th* anddenotesuchnonstandard model as $NsM^{Th}_{\vDash_*}$.

Definition4.6.

Assume that a theory*Th* has astrononstandard el $NsM^{Th}_{\vDash_*}$. Then we said that a theory Th is a strongly *NsM*-consistent.

Definition4.7.

Assume that a theory *Th* has a strongnonstandard el $NsM^{Th}_{\vDash_*}$ and Φ is a *Th*-sentence. Let $\Phi_{NsM,\vDash_*}$ be a *Th*-sentence Φ with all quantifiers and all constants relativized to a model $NsM^{Th}_{\vDash_*}$.

Remark4.3.

In some special cases we denote a sentence$\Phi_{NsM,\vDash_*}$by a symbol: $\Phi[NsM^{Th}_{\vDash_*}]$.

Definition4.8.

Assume that a theory *Th* has a strong nonstandard el $NsM^{Th}_{\vDash_*}$. Let $Th_{NsM,\vDash_*}$be a theory *Th*relativized to a model $NsM^{Th}_{\vDash_*}$ i.e., any $Th_{NsM,\vDash_*}$-sentence Ψ has a form$\Phi_{NsM,\vDash_*}$for some Th-sentence Φ.

Remark4.4.In some special cases we denote atheory$Th_{NsM,\vDash_*}$by a symbol:$Th[NsM^{Th}_{\vDash_*}]$.

Assumption4.1.

We now assume throughout this subsectionthat *Th* is a strongly*NsM*-consistent, i.e. a sentence$Con(Th; NsM^{Th}_{\vDash_*})$ is true in any nonstandard model NsM^{Th} of the *Th*.

Definition4.9.(a) Let *Th* be a theory such that Assumption 3.1 is satisfied. Let $\widehat{Con}(Th; NsM^{Th}_{\vDash_*})$ be a predicate in *Th*asserting that $NsM^{Th}_{\vDash_*}$ is a strong nonstandard model of the *Th*. Then a sentence $Con(Th; NsM^{Th}_{\vDash_*})$such that:

$$Con(Th; NsM^{Th}_{\vDash_*}) \leftrightarrow \exists NsM^{Th}_{\vDash_*}\widehat{Con}(Th; NsM^{Th}_{\vDash_*})(4.4)$$

is a sentence in *Th*asserting that *Th* has a strong non-standard model $NsM^{Th}_{\vDash_*}$. (b) Let *Th** be a theory*Th**=*Th*+ $Con(Th; NsM^{Th}_{\vDash_*})$ .Let $Con\left(Th^*; NsM^{Th^*}_{\vDash_*}\right)$ bea sentence in*Th** asserting that *Th** has a strongnonstandard model$NsM^{Th}_{\vDash_*}$.

Lemma4.1.

Assume that a theory *Th* has a strong nonstandard model $NsM^{Th}_{\vDash_*}$ anda theory *Th** has a strong nonstandard el $NsM^{Th^*}_{\vDash_*}$. Then: (*i*) a sentence $Con(Th; NsM^{Th}_{\vDash_*})$ is a*Th*-sentence, (*ii*)a sentence $Con\left(Th^*; NsM^{Th^*}_{\vDash_*}\right)$ is a *Th**-sentence.

Proof.Immediately follows from Definition 4.4 and Definition 4.9.

Remark4.5.Note that:

$$Con(Th; NsM_{\models *}^{Th}) \leftrightarrow Con_{Th_{NsM,\models *}}, \quad (4.5)$$

where

$$Con_{Th_{NsM,\models *}} \leftrightarrow \neg \mathrm{Pr}_{Th_{NsM,\models *}}([\Phi_{NsM,\models *}]^{c}). \quad (4.6)$$

Here a sentence $\Phi_{NsM,\models *}$is refutable in $Th_{NsM,\models *}$.

Remark4.6.

Note that

$$Con(Th^{*}; NsM_{\models *}^{Th^{*}}) \leftrightarrow Con_{Th^{*}{}_{NsM,\models *}}, \quad (4.7)$$

where

$$Con_{Th^{*}{}_{NsM,\models *}} \leftrightarrow \neg \mathrm{Pr}_{Th^{*}{}_{NsM,\models *}}([\Phi^{*}{}_{NsM,\models *}]^{c}). \quad (4.8)$$

Here a sentence $\Phi_{NsM,\models *}^{*}$,isrefutable in$Th_{NsM,\models *}^{*}$.

Lemma4.2.Ander Assumption 4.1 a theory$Th*$ is a strongly *NsM*-consistent.

Proof. The proof uses formulae (4.5) and (4.6) and completely, to similarly a proof of the Lemma 3.5.

Theorem4.2.

Th has not any strong nonstandard model$NsM_{\models *}^{Th}$.

Proof. The proof uses formulae (4.7) and (4.8) and completely, to similarly a proof of the Theorem 3.3.

Theorem4.3.*ZFC*has not any strong nonstandard el$NsM_{\models *}^{ZFC}$.

Proof. Immediately follows from theorem 4.2 and definitions.

Definability in Second-Order Set Theory

Definition5.1.

Assume that: (*i*) *Con*(*Th*) and (*ii*)*Th* has an ω-model M_{ω}^{Th}. Let $\Psi(X)$ be one-place open *Th*-wiff and let $\Psi_{\omega}(X)$ be *Th*-wiff$\Psi(x)$ relativized to ω-model M_{ω}^{Th}. Assume that condition

$$Th_{\omega} \vdash \exists! X_{\Psi}(X_{\Psi} \in M_{\omega}^{Th})[\Psi_{\omega}(X_{\Psi})] \quad (5.1)$$

is satisfied. We say that an *Th*-wiff$\Psi(X)$ is a nice *Th*-wiff, iff condition (5.1) is satisfied.

Definition5.2.

Let us define a second-order predicate

$\Xi_{\omega}(\Psi(X_{\Psi}))$ suchthat equivalence

$$\Xi_{\omega}[\Psi(X_{\Psi})] \leftrightarrow Th_{\omega} \vdash \Delta_{\omega}^{\Psi(X_{\Psi})}, \quad (5.2)$$

where

$$\Delta_{\omega}^{\Psi(X_{\Psi})} \triangleq \exists! X_{\Psi}(X_{\Psi} \in M_{\omega}^{Th})[\Psi_{\omega}(X_{\Psi})] \quad (5.3)$$

is satisfied.We say that a set Y is a Th_{ω}-set iffthe second-order sentence:

$$\Sigma^{Y}[\Psi(X), X_{\Psi}] \triangleq$$

$$\exists\Psi(X_{\Psi})\{\Xi_{\omega}[\Psi(X_{\Psi})]\wedge(Y = X_{\Psi})\} \quad (5.4)$$

is satisfied inω-model M_{ω}^{Th}, i.e.

$$M_{\omega}^{Th} \models \exists\Psi(X_{\Psi})\{\Xi_{\omega}[\Psi(X_{\Psi})]\wedge(Y = X_{\Psi})\}. \quad (5.5)$$

Definition5.3.

Let$\Psi_1(x)$and $\Psi_2(x)$ is a nice *Th*-wiffs. Let us define equivalence relation$\Psi_1(x) \backsim \Psi_2(x)$ such that condition

$$\Psi_1(x) \backsim \Psi_2(x) \leftrightarrow X_{\Psi_1} = X_{\Psi_2} \quad (5.6)$$

is satisfied.

Assumption5.1.

Weassume now that: (*i*) a theory *Th* admitscanonical primitive recursive encoding of syntax and (*ii*) the set of codes of axiom of *Th*is primitive recursive.

Lemma5.1.Second-order predicate $\Xi_{\omega}[\Psi(X_{\Psi})]$ can be replaced by some equivalent first-order predicate:

$$\breve{\Xi}_{\omega}\{[\Psi(X_{\Psi})]^{c}, [X_{\Psi}]^{c}\}. \quad (5.7)$$

Proof. Let us rewrite a sentence (5.3) in equivalent form such that

$$\Delta_{\omega}^{\Psi(X_1)} \triangleq \exists! X_1(X_1 \in M_{\omega}^{Th})[\Psi_{\omega}(X_1)]. \quad (5.8)$$

Using a Strong Reflection Principle [formula (3.10.a)], one obtain equivalence

$$\mathrm{Pr}_{Th_{\omega}}\left(\left[\Delta_{\omega}^{\Psi(X_{\Psi})}\right]^{c}\right) \leftrightarrow Th_{\omega} \vdash \Delta_{\omega}^{\Psi(X_{\Psi})}.$$

Therefore

$$\breve{\Xi}_{\omega}\{[\Psi(X_{\Psi})]^{c}, [X_{\Psi}]^{c}\} \leftrightarrow \mathrm{Pr}_{Th_{\omega}}\left(\left[\Delta_{\omega}^{\Psi(X_{\Psi})}\right]^{c}\right) \quad (5.9)$$

Formula (5.9) andDefinition 5.2 completed the proof.Lemma5.2.Second-orderpredicate $\Sigma^{Y}[\Psi(X), X_{\Psi}]$ can be replaced by some equivalent first-order predicate:

$$\breve{\Sigma}^{Y}\left[[\Psi(X_{\Psi})]^{c}, [X_{\Psi}]^{c}, [Y]\right] \quad (5.10)$$

Proof. Let us rewrite formula(5.4) in the next equivalent form

$$\Sigma^{Y}[\Psi(X_1)] \triangleq$$

$$\exists\Psi(X_1)\{\Xi_{\omega}[\Psi(X_1)]\wedge(Y = X_1)\} \quad (5.11)$$

Using formula (5.7) one can rewrite RHS in the next equivalent first-order form

$$\exists \mathrm{t}(\mathrm{t} = [\Psi(X_1)]^{c})[\breve{\Xi}_{\omega}\{\mathrm{t}, [X_1]^{c}\}\wedge([Y]^{c} = [X_1]^{c})]. \quad (5.12)$$

Formula (5.12) completed the proof.

Remark5.2.

We now assume, throughout this subsection that encoding $[\circ]^{c}$means canonical Gödel encoding such that defined in

[20].Let (*i*) EVbl(x) be the predicate: x is a Gödel number of an expression consisting of a variable, (*ii*)Fr(y,x)be the predicate:y is the Gödel number of 1-place openwffof *Th* which contains free occurrences of the variable with Gödelnumber x [20].

Remark5.3.

Note that by using Remark 5.2, first-order predicate

$$\breve{\Xi}_\omega\{[\Psi(X_1)]^c,[X_1]^c\}, \tag{5.13}$$

one can by replace in equivalent form such that

$$\breve{\Xi}_\omega\{y_1,x_1\}, \tag{5.14}$$

$$\text{where} M_\omega^{Th} \vDash \mathrm{Fr}(y_{1,}, x_1).$$

Remark5.4.

Note that by using Remark 5.2 first-order predicate given by formula (5.12) one can replace by first-order predicate such that

6. Conclusion

In this paper we proved so-called strong reflection principles corresponding to formal theories *Th*which has ω-models M_ω^{Th} and in particular to formal theories *Th*,, which has a standard models SM^{Th}.Theassumption that there existsa standard model of *Th* is stronger than the assumption that there exists a model of *Th*. This paper examined some specified classes of the standard and nonstandard models of *ZFC* so-called *strongstandardmodels* of *ZFC* and *strong nonstandard models* of *ZFC* correspondingly. Such strong standard models of *ZFC* correspond tolarge cardinal axioms. In particular we proved that theory*ZFC*+*Con*(*ZFC*) is incompatible with existence of any inaccessible cardinal κ.Note that the statement:*Con*(*ZFC*+∃ some inaccessible cardinal κ) isΠ_1^0.Thus Theorem 3.6 asserts there is exist numerical counterexample which is turn would imply that a specific polynomial equation has at least one integer root.

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