Pure and Applied Mathematics Journal
2013; 2(2) : 98-100
Published online April 2, 2013 (http://www.sciencepublishinggroup.com/j/pamj)
doi: 10.11648/j.pamj.20130202.17

Sub Hilbert spaces in a Bi-Disk

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To cite this article:
Niteesh Sahni. Sub Hilbert Spaces in a Bi-Disk, *Pure and Applied Mathematics Journals.* Vol. 2, No. 2, 2013, pp.98-100.
doi: 10.11648/j.pamj.20130202.17

Abstract: Recently, Sahni and Singh [7] have solved an open problem posed by Yousefi and Hesameddini [12] regarding Hilbert spaces contained algebraically in the Hardy space $H^2(T)$. In fact the result obtained by Sahni and Singh is much more general than the open problem. In the present note we examine the validity of the main results of [7] and [12] in two variables.

Keywords: Hardy Space, Beurling Type Result, Isometry, Wold Type Decomposition

1. Introduction

In [12], Yousefi and Hesameddini have characterized the invariant subspaces of all Hilbert spaces H that are vector subspaces of the Hardy space $H^2(T)$, with $\langle .,. \rangle_H$(the inner product on H) satisfying the following axioms:

A1.If there are four functions $f_1, f_2, g_1, g_2 \in H$ such that $\langle f_1, g_1\rangle_H = \langle f_2, g_2\rangle_H$, then we have $\langle f_1, g_1\rangle_{H^2} = \langle f_2, g_2\rangle_{H^2}$.

A2. If ϕ is any inner function, then $\varphi f \in H$ and $\langle \varphi f, \varphi g\rangle_H = \langle f, g\rangle_H$ for all f,g $\in$ H.

The characterization of H, however is left as the following open question in [12]: Is every Hilbert space H satisfying axioms A1 and A2 of the form φH^2, for some $\varphi \in H^\infty$, and does there exist a constant k such that $\langle f, g\rangle_H = k\langle f, g\rangle_{H^2}$ for every f,g $\in$ H?

Sahni and Singh [7] have settled this open problem in the affirmative. They in fact prove a far more general version of the above problem. Below we record a special case of their main result which clearly generalized the open problem mentioned above:

Theorem 1.1 ([7], Corollary 3.2). Let H be a Hilbert space which is algebraically contained in H2 and invariant under multiplication by z. Further if the inner product on H satisfies the following conditions

i. $\langle zf, zg\rangle_H = \langle f, g\rangle_H$ for all f,g $\in$ H;
ii. for any $f_1, f_2, g_1, g_2 \in H$ such that $\langle f_1, g_1\rangle_H = \langle f_2, g_2\rangle_H$, then we have

$$\langle f_1, g_1\rangle_{H^2} = \langle f_2, g_2\rangle_{H^2}.$$

then there exists a unique inner function $b \in H^\infty$ and a constant k such that $H = bH^2$ and $\langle f, g\rangle_H = k\langle f, g\rangle_{H^2}$ for every f,g $\in$ H.

In [7], the authors use the above characterization of H to derive the following invariant subspace characterization:

Theorem 1.2 (**[7]**, **Theorem 3.3**). Let H be a Hilbert space satisfying all the conditions of Theorem 1.1. If M is a closed subspace of H and invariant under multiplication by z, then there exists a unique inner function $\varphi \in H^\infty$ such that M = ϕH.

It is further noted in [7] that the characterization obtained in [12] comes as a special case of Theorem 1.2. In the present paper, we obtain a two variable version of Theorem 1.1 for the Hardy space $H_2(T_2)$ on the torus T_2, and also examine the validity of Theorem 1.2 in two variables.

2. Notations and Preliminary Results

Let T denote the unit circle in the complex plane, and T^2 stands for the cartesian product of T, that is, $T^2 = \{(e^{i\theta_1}, e^{i\theta_2}): 0 \le \theta_1 \le 2\pi, 0 \le \theta_2 \le 2\pi\}$. We shall denote the coordinate functions $e^{i\theta_1}$and $e^{i\theta_2}$by z and w respectively. The Lebesgue space $L^2(T^2)$ on the torus is a collection of complex valued functions f on the torus such that $\int |f|^2\, dm$ is finite, where *dm* is the normalized Lebesgue measure on T^2. It is well known that $L^2(T^2)$ is a Hilbert space with respect to the inner product

$$\langle f, g\rangle_2 = \int_{T^2} f g\, dm,$$

and the collection $\{z^n w^m : m, n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(T^2)$. The Hardy space $H^2(T^2)$ comprises of those $L^2(T^2)$ functions f(z,w) that have the Fourier expansion of the form $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn} z^m w^n$, and hence the norm of f is given by $\|f\|_2 = (\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} |a_{mn}|^2)^{1/2}$.

The space $H^2(T^2)$ turns out to be a closed subspace of $L^2(T^2)$ and hence a Hilbert space under the above norm. The class of essentially bounded functions in $H^2(T^2)$ is denoted by H^∞. Let S_1 and S_2 be operators on $H^2(T^2)$ defined by $S_1f(z,w) = zf(z,w)$, and $S_2f(z,w) = wf(z,w)$ respectively.

A function $\phi \in H^2(T^2)$ is called an inner function if $|\phi| = 1$ a.e.. Equivalently, this means that $\langle \varphi f, \varphi g\rangle_2 = \langle f, g\rangle_2$ for every $f,g \in H^2(T^2)$.

Theorem 2.1 (Singh [11]). Let M be a Hilbert space which is a vector subspace of $H^2(T^2)$ such that $S_1(M) \subset M$, $S_2(M) \subset M$; and S_1, S_2 are doubly commuting isometries on M. Then there exists $b \in H^\infty$ such that $M = bH^2(T^2)$, and $\|bf\|_M = \|f\|_2$ for all $f \in H^2(T^2)$.

The following lemma provides a necessary and sufficient condition for an H^∞ function to be an inner function:

Lemma 2.2. A function $\varphi \in H^\infty$ is inner if and only if $\{z^m w^n \varphi : m.n = 0,1,2,\ldots\}$ is an orthonormal set in $H^2(T^2)$.

Proof. Suppose ϕ is inner. So $|\phi| = 1$ a.e., and therefore

$$\langle z^{m_1}w^{n_1}\varphi, z^{m_2}w^{n_2}\varphi\rangle_2 = \langle z^{m_1}w^{n_1}, z^{m_2}w^{n_2}\rangle_2 = \begin{cases} 1 & if\ (m_1,n_1) = (m_2,n_2) \\ 0 & if\ (m_1,n_1) \neq (m_2,n_2) \end{cases}.$$

Conversely, the orthonormality of the set $\{z^m w^n \varphi : m.n = 0,1,2,\ldots\}$ yields the following equations:

$\langle \varphi, \varphi z^m w^n\rangle_2 = 0 = \overline{\langle \varphi, \varphi z^m w^n\rangle_2}$ for all $m > 0$ and $n > 0$,

$\langle \varphi z^m, \varphi w^n\rangle_2 = 0 = \overline{\langle \varphi z^m, \varphi w^n\rangle_2}$ for all $m > 0$ and $n > 0$,

$\langle \varphi, \varphi z^m\rangle_2 = 0 = \overline{\langle \varphi, \varphi z^m\rangle_2}$ for all $m > 0$,

and $\langle \varphi, \varphi w^n\rangle_2 = 0 = \overline{\langle \varphi, \varphi w^n\rangle_2}$ for all $n > 0$.

From the above four equations we conclude that

$\int |\varphi|^2 z^m w^n dm = 0$ for all $(m,n) \neq (0,0)$.

Therefore $|\phi|^2$ is a constant, say c.

Since $1 = \|\varphi\|_2 = \int |\varphi|^2 dm$, we have $c = 1$. This proves that ϕ is inner.

3. Main Results

Theorem 3.1. Let H be a Hilbert space which is a vector subspace of $H^2(T^2)$ such that $S_1(H) \subset H$, $S_2(H) \subset H$; and S_1, S_2 are doubly commuting isometries on H. Further, assume that $\langle .,. \rangle_H$ (the inner product on H) satisfies the following property:

(P) For any functions $f_1, f_2, g_1, g_2 \in H$ such that $\langle f_1, g_1\rangle_H = \langle f_2, g_2\rangle_H$, then we have $\langle f_1, g_1\rangle_{H^2} = \langle f_2, g_2\rangle_{H^2}$.

Then there exists a unique inner function b such that

$H = bH^2(T^2)$, and $\|bf\|_H = \|f\|_2$ for every $f \in H^2(T^2)$.

Proof. Since the operators S_1 and S_2 are doubly commuting isometries on the Hilbert space H, so all the the conditions of Theorem 2.1 are satisfied and hence there exists $b \in H^\infty$ such that $H = bH^2(T^2)$, and $\|bf\|_H = \|f\|_2$ for all $f \in H^2(T^2)$. It remains to be shown that b is inner. The proof of Theorem 2.1, as given in [11], reveals that H has the Slocinski-Wold type decomposition:

$$H = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \oplus\, S_1^m S_2^n (L_1 \cap L_2).$$

Here $L_k = H \ominus S_k(H)$, k=1,2 . It turns out that b is a unit vector in H that spans $L_1 \cap L_2$. From the above Wold type decomposition we see that $\langle b, bz^m w^n\rangle_H = 0$ for all $(m,n) \neq (0,0)$. The arguments used to establish this fact are similar to the ones adopted in the proof of Lemma 2.2. This, in view of property (P), implies that $\langle b, bz^m w^n\rangle_2 = 0$ for all $(m,n) \neq (0,0)$. Thus the set $\{bz^m w^n : m \geq 0 \text{ and } n \geq 0\}$ is orthogonal in $H^2(T^2)$. So the collection $\left\{\frac{b}{\|b\|_2} z^m w^n : m \geq 0 \text{ and } n \geq 0\right\}$ is orthonormal in $H^2(T^2)$. Hence by Lemma 2.2, the function $\frac{b}{\|b\|_2}$ is inner, and we have $H = bH^2(T^2) = \frac{b}{\|b\|_2} H^2(T^2)$.

The above theorem clearly generalizes the two variable Beurling type result proved by Mandrekar [5]:

Corollary 3.2 ([5], Theorem 2). Let M be a closed subspace of $H^2(T^2)$ which is invariant under S_1 and S_2 and that S_1, S_2 doubly commute on M. Then there exists a unique inner function b such that $M = bH^2(T^2)$.

Proof. Since the norm on M is the same as that on H^2, so the property (P) stated in Theorem 3.1 is automatically satisfied and hence the result follows.

Theorem 3.3. Let H be a Hilbert space satisfying the conditions of Theorem 3.1, and M be a closed subspace of H which is invariant under S_1 and S_2. Then there exists a unique inner function ϕ such that $M = \phi H$.

Proof. By Theorem 3.1, there exists an inner function b such that $H = bH^2(T^2)$. Since M is a closed subspace of H, so M is also a Hilbert space satisfying the assumptions of Theorem 3.1. Thus there exists an inner function c such that $M = cH^2(T^2)$. Note that $c = b\phi$ for some $\phi \in H^2(T^2)$. This implies that $|\phi| = 1$ a.e. Further, $M = \phi bH^2(T^2) = \phi H$.

Acknowledgements

The author thanks the Shiv Nadar University, Dadri, Uttar Pradesh (India), for support and facilities needed to complete the present work.

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