

Relations Among Certain Generalized Hyper-Geometric Functions Suggested by N-fractional Calculus

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Abstract: The subject of fractional calculus has gained importance and popularity during the past three decades. Based upon the N-fractional calculus we introduce a new N-fractional operators involving hyper-geometric function. By means of these N-fractional operators a number of operational relations among the hyper-geometric functions of two, three, four and several variables are then found. Other closely-related results are also considered.

Keywords: N-fractional Calculus Operators, Horn's Functions, Appell Functions, Saran Functions, Quadruple Functions, Hyper-Geometric of Several Variables

1. Introduction

The subject of fractional calculus is one of the most intensively developing areas of mathematical analysis, mainly due to its fields of application range from biology through physics and electrochemistry to economics, probability theory, special functions and statistics (see [16] and [17]). Indeed, on behalf of the nature of their definitions the fractional derivatives and integrals provide an excellent instrument for the modeling of memory and hereditary properties of various materials and processes. Half-order derivatives and integrals prove to be more useful for the formulation of certain electrochemical problems than the classical methods [19]. Appell (see [1]; also [21, p. 22-23]) defined the four hyper-geometric functions of two variables which he denoted by F_1, F_2, F_3 and F_4 . Other hyper-geometric functions of two variables has been defined by Horn [6]. Seven of them he denoted by $H_1, H_2, H_3, H_4, H_5, H_6$ and H_7 (see e.g. [21, p. 24] and [22, p. 56-57]).

Appell's and Horn's functions are all generalization of Gaussian hypergeometric function

$${}_2F_1[a, b; c; x] = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m m!} x^m, \quad (1)$$

where $(a)_m = \frac{\Gamma(a+m)}{\Gamma a}$, Γ : Gamma function.

In 1893, Lauricella [15] further generalized the four Appell's functions F_1, F_2, F_3 and F_4 to functions of n-variables $F_A^{(n)}, F_B^{(n)}, F_C^{(n)}$ and $F_D^{(n)}$ respectively. After a gap of long time, Saran [18] initiated a systematic study of ten series of three variables with the notations $F_E, F_F, F_G,$

$F_K, F_M, F_N, F_P, F_R, F_S$ and F_T . In 1982, Extton [8] has studied 20 triple hyper-geometric series. He denoted four of them by X_6, X_8, X_{14} and X_{17} . Srivastava and Karlsson [21] provide an impressive tabulation and a wealth of information on the construction of the set of all 205 distinct triple Gaussian hypergeometric series. In their work symbols and references are given for those series that have been introduced previously and they denoted the new series by the symbol (cf. [21, p. 273]):

$$F \left[\begin{matrix} , , , , , \\ , , , , , \end{matrix} \middle| x, y, z \right].$$

For instance, the series F_{4d} has the definition

$$\sum_{m,n,p=0}^{\infty} \frac{(a)_{m+n} (b)_m (c)_n (d)_p (e)_{p-m}}{(f)_n (g)_p m! n! p!} x^m y^n z^p,$$

and it would be presented in the form

$$F \left[\begin{matrix} a, b; a, c; d, e \\ e; f; g \end{matrix} \middle| x, y, z \right].$$

Sharma and Parihar [20] introduced 83 complete hyper-

$$\begin{aligned} F_9^{(4)} = K_1, F_1^{(4)} = K_2, F_{38}^{(4)} = K_3, F_{10}^{(4)} = K_4, F_2^{(4)} = K_5, F_{56}^{(4)} = K_6, F_{34}^{(4)} = K_7, \\ F_{11}^{(4)} = K_8, F_{12}^{(4)} = K_9, F_3^{(4)} = K_{10}, F_{60}^{(4)} = K_{11}, F_{40}^{(4)} = K_{12}, F_{13}^{(4)} = K_{13}, F_{71}^{(4)} = K_{14}, \\ F_{78}^{(4)} = K_{15}, F_{79}^{(4)} = K_{16}, F_{82}^{(4)} = K_{19}, F_8^{(4)} = K_{20}, F_{83}^{(4)} = K_{21}. \end{aligned}$$

For the purpose of this work, we recall here some definitions.

Definition 1. (By K. Nishimoto (see [16] and [17]))

Let $D = \{D_-, D_+\}$, $C = \{C_-, C_+\}$,

C_- be a curve along the cut joining two points z and $-\infty + ilm(z)$,

C_+ be a curves along the cut joining two points z and $\infty + ilm(z)$,

D_- be a domain surrounded by C_- , D_+ be a domain surrounded by C_+

Moreover, let $f = f(z)$ be analytic (regular) function in D and it has no branch point inside C and on C , and

$$f_v = (f)_v(z) = \frac{\Gamma(v+1)}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta-z)^{v+1}} d\zeta, \quad v \in \mathbb{Z}^- \quad (2)$$

$$(f)_{-n} = \lim_{v \rightarrow -n} (f)_v, \quad (n \in \mathbb{Z}^+).$$

where $-\pi \leq \arg(\zeta - z) \leq \pi$, for C_- , $0 \leq \arg(\zeta - z) \leq 2\pi$, for $v \in C_+$, $\zeta \neq z$, then $(f)_v$ is the fractional differ-integration of arbitrary order v (derivatives of order v for $v > 0$, and integrals of order $-v$ for $v < 0$), with respect to z , of the function f , if $|(f)_v| < \infty$.

Furthermore, let \mathcal{N}^v be Nishimoto's operator defined by [Refer to (2)]

$$\mathcal{N}^v = \left(\frac{\Gamma(v+1)}{2\pi i} \int_C \frac{d\zeta}{(\zeta-z)^{v+1}} \right), \quad v \in \mathbb{Z}^- \quad (3)$$

with $\mathcal{N}^{-m} = \lim_{v \rightarrow -m} \mathcal{N}^v$, $m \in \mathbb{Z}^+$.

Lemma 1. We have

$$(z^\beta)_{\alpha(z)} = e^{-i\pi\alpha} \frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)} z^{\beta-\alpha}, \quad \frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)} < \infty. \quad (4)$$

Operational representations and relations involving one and more variables hypergeometric series have been given considerable in the literature, see for example, Chen and Srivastava [5], Goyal, Jain and Gaur ([7], [8]) Kalla [11], Kalla and Saxena ([12] and [13]), Kant and Koul [14], Chyan and Srivastava [5]. The present sequel to these

geometric series of four variables $F_1^{(4)}, F_2^{(4)}, \dots, F_{83}^{(4)}$. It is remarkable that out of these 83 series, the following 19 series had already been appeared in the literature due to Exton (see [9]) in the different notations:

earlier papers is motivated largely by the aforementioned work of Bin-Saad and Maisoon [2] in which a number of operational relations among the hypergeometric functions of two and three variables are found. The aim of this paper is to introduce some N-fractional operators involving certain hypergeometric functions. Based upon these operators, we aim here to derive operational relations among the above said hypergeometric functions of two, three, four and multiple variables. The structure of this paper is the following: In section 2, we establish N-fractional operators. Section 3 deals with same applications of these operators and the presentation of operational relations between functions of two, three and four variables. Section 4 aims at introducing multivariable generalization of the N-fractional operators in section 2 and establishing some operational relations among hypergeometric functions of several variables.

2. N-fractional Calculus Operators

By using definitions (2) and (3), we introduce three kinds of N-fractional operators in the following definitions:

Definition 2. Let $\mathcal{M}_z^{\alpha, \beta, \gamma}$ be a N-fractional operator defined by

$$\begin{aligned} \mathcal{M}_z^{\alpha, \beta, \gamma} &= \mathcal{N}^{\alpha-\beta-1} \{ z^{\alpha-1} (1-z)^{-\gamma} \} \\ &= (z^{\alpha-1} (1-z)^{-\gamma})_{\alpha-\beta} \\ &= \frac{\Gamma(\alpha-\beta)}{2\pi i} \int_C \frac{\zeta^{\alpha-1} (1-\zeta)^{-\gamma}}{(\zeta-z)^{\alpha-\beta}} d\zeta, \end{aligned} \quad (5)$$

where $\gamma, \alpha, \beta \in \mathbb{C}$ and $(\alpha-\beta) \notin \mathbb{Z}^+$.

Definition 3. Let $\mathcal{M}_{\alpha_2, \beta_2, \gamma_2; z, w}^{\alpha_1, \beta_1, \gamma_1}$ be a N-fractional operator defined by

$$\begin{aligned} &\mathcal{M}_{\alpha_2, \beta_2, \gamma_2; z, w}^{\alpha_1, \beta_1, \gamma_1} \\ &= \mathcal{M}_w^{\alpha_2, \beta_2, \gamma_2} \left\{ w^{\alpha_2-1} (1-w)^{-\gamma_2} \mathcal{M}_z^{\alpha_1, \beta_1, \gamma_1} \{ z^{\alpha_1-1} (1-z)^{-\gamma_1} \} \right\} \\ &= (w^{\alpha_2-1} (1-w)^{-\gamma_2} (z^{\alpha_1-1} (1-z)^{-\gamma_1})_{\alpha_1-\beta_1(z)})_{\alpha_2-\beta_2(w)} \end{aligned}$$

$$= \frac{\Gamma(\alpha_1 - \beta_1) \Gamma(\alpha_2 - \beta_2)}{(2\pi i)^2} \int_c \int_c \frac{\zeta^{\alpha_1 - 1} (1 - \zeta)^{-\gamma_1}}{(\zeta - z)^{\alpha_1 - \beta_1}} \frac{\xi^{\alpha_2 - 1} (1 - \xi)^{-\gamma_2}}{(\xi - w)^{\alpha_2 - \beta_2}} d\zeta d\xi \quad (6)$$

where $\{\gamma_1, \gamma_2, \alpha_1, \alpha_2, \beta_1, \beta_2\} \in \mathbb{C}$ and $\{(\alpha_1 - \beta_1), (\alpha_2 - \beta_2)\} \notin \mathbb{Z}^-$.

$$\begin{aligned} \mathcal{M}_{\alpha_2, \beta_2, \gamma_2; z, w}^* \alpha_1, \beta_1, \gamma_1 &= \mathcal{M}_w^{\alpha_2, \beta_2, \gamma_2} \left\{ w^{\alpha_2 - 1} (1 - w)^{-\gamma_2} \mathcal{M}_z^{\alpha_1, \beta_1, \gamma_1} \left\{ z^{\alpha_1 - 1} \left(1 - \frac{z}{1 - w}\right)^{-\gamma_1} \{f\} \right\} \right\} \\ &= \left(w^{\alpha_2 - 1} (1 - w)^{-\gamma_2} \left(z^{\alpha_1 - 1} \left(1 - \frac{z}{1 - w}\right)^{-\gamma_1} \{f\} \right)_{\alpha_1 - \beta_1(z)} \right)_{\alpha_2 - \beta_2(w)} \\ &= \frac{\Gamma(\alpha_1 - \beta_1) \Gamma(\alpha_2 - \beta_2)}{(2\pi i)^2} \int_c \int_c \frac{\zeta^{\alpha_1 - 1} (1 - \frac{\zeta}{1 - w})^{-\gamma_1}}{(\zeta - z)^{\alpha_1 - \beta_1}} \frac{\xi^{\alpha_2 - 1} (1 - \xi)^{-\gamma_2}}{(\xi - w)^{\alpha_2 - \beta_2}} \{f\} d\zeta d\xi, \end{aligned} \quad (7)$$

where $\{\gamma_1, \gamma_2, \alpha_1, \alpha_2, \beta_1, \beta_2\} \in \mathbb{C}$ and $\{(\alpha_1 - \beta_1), (\alpha_2 - \beta_2)\} \notin \mathbb{Z}^-$.

In view of lemma 1 and the binomial theorem

$$(1 - z)^{-\gamma} = \sum_{m=0}^{\infty} \frac{(\gamma)_m}{m!} z^m,$$

the operators $\mathcal{M}_z^{\alpha, \beta, \gamma}$, $\mathcal{M}_{\alpha_2, \beta_2, \gamma_2; z, w}^* \alpha_1, \beta_1, \gamma_1$ and $\mathcal{M}_{\alpha_2, \beta_2, \gamma_2; z, w}^* \alpha_1, \beta_1, \gamma_1$, can be written in the forms:

$$\mathcal{M}_z^{\alpha, \beta, \gamma} = A {}_2F_1 [\gamma, \alpha; \beta; z] \quad (8)$$

$$\mathcal{M}_{\alpha_2, \beta_2, \gamma_2; z, w}^* \alpha_1, \beta_1, \gamma_1 = B {}_2F_1 [\alpha_1, \gamma_1; \beta_1; x] \times {}_2F_1 [\alpha_2, \gamma_2; \beta_2; w], \quad (9)$$

and

$$\mathcal{M}_{\alpha_2, \beta_2, \gamma_2; z, w}^* \alpha_1, \beta_1, \gamma_1 =$$

$$B \sum_{m=0}^{\infty} \frac{(\alpha_2)_m (\gamma_2)_m}{(\beta_2)_m m!} w^m {}_3F_2 [\gamma_2 + m, \gamma_1, \alpha_1; \beta_1, \gamma_2; x], \quad (10)$$

respectively, where throughout this work

$$A = e^{-i\pi(\alpha - \beta)} \frac{\Gamma(1 - \beta)}{\Gamma(1 - \alpha)} z^{\beta - 1} \quad (11)$$

and

$$B = e^{-i\pi(\alpha_1 + \alpha_2 - \beta_1 - \beta_2)} z^{\beta_1 - 1} w^{\beta_2 - 1} \frac{\Gamma(1 - \beta_1)}{\Gamma(1 - \alpha_1)} \frac{\Gamma(1 - \beta_2)}{\Gamma(1 - \alpha_2)} \quad (12)$$

Definition 4. Let $\mathcal{M}_{\alpha_2, \beta_2, \gamma_2; z, w}^* \alpha_1, \beta_1, \gamma_1$ be a N-fractional operator defined by

On replacing α , β and γ by $\alpha - \gamma + 1$, $2 - \gamma$ and $\beta - \gamma + 1$ respectively, relation (8) reduces to a known result due to Nishimoto [9, p. 78 (3)]. Further, if $\gamma_1 = \gamma_2 = \gamma$, relation (10) yields

$$\mathcal{M}_{\alpha_2, \beta_2, \gamma; z, w}^* \alpha_1, \beta_1, \gamma = B {}_2F_2 [\gamma, \alpha_1, \alpha_2; \beta_1, \beta_2; x, w]. \quad (13)$$

Whereas, if $\alpha_1 = \gamma_2 = \gamma$ relation (10) yields

$$\mathcal{M}_{\alpha_2, \beta_2, \gamma; z, w}^* \gamma, \beta_1, \gamma_1 = B {}_2F_2 [\gamma, \gamma, \alpha_2; \beta_1, \beta_2; x, w]. \quad (14)$$

where F_2 is Appell's function of two variables [12, p. 23 (3)].

3. Applications

By choosing a suitable hyper-geometric function the operators (5), (6) and (7) can be applied to deduce relationships involving a fairly variety of hyper-geometric functions of one and more variables.

i. Relations among two and three variables functions

By making use of the N-fractional operator (5), we establish the following relationships between hyper-geometric series of two and three variables.

$$\mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ F_1 \left[\lambda, \sigma, \gamma; \mu; x, \frac{y}{1 - z} \right] \right\} = A {}_F_M [\alpha, \lambda, \lambda, \gamma, \sigma, \gamma; \beta, \mu, \mu; z, x, y] \quad (15)$$

$$\mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ F_1 \left[\gamma, \delta, \sigma; \mu; \frac{x}{1 - z}, \frac{y}{1 - z} \right] \right\} = A {}_F_G [\gamma, \gamma, \gamma, \alpha, \delta, \sigma; \beta, \mu, \mu; z, x, y] \quad (16)$$

$$\mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ F_2 \left[\lambda, \sigma, \gamma; \mu, \eta; x, \frac{y}{1 - z} \right] \right\} = A {}_F_K [\alpha, \lambda, \lambda, \gamma, \sigma, \gamma; \beta, \mu, \eta; z, x, y] \quad (17)$$

$$\mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ F_2 \left[\gamma, \sigma, \delta; \mu, \eta; \frac{x}{1 - z}, \frac{y}{1 - z} \right] \right\} = A {}_F_A^{(3)} [\gamma, \alpha, \sigma, \delta; \beta, \mu, \eta; z, x, y] \quad (18)$$

$$\mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ F_3 \left[\lambda, \gamma, \sigma, \delta; \mu; x, \frac{y}{1 - z} \right] \right\} = A {}_F_N [\alpha, \lambda, \delta, \gamma, \sigma, \gamma; \beta, \mu, \mu; z, x, y] \quad (19)$$

$$\mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ F_3 \left[\frac{\gamma}{2}, \sigma, \frac{\gamma+1}{2}, \delta; \mu; \frac{4x}{(1-z)^2}, y \right] \right\} = A X_{20} [\gamma, \alpha, \sigma, \delta; \mu, \beta; x, z, y] \quad (20)$$

$$\mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ F_4 \left[\gamma, \lambda; \mu, \eta; \frac{x}{1-z}, \frac{y}{1-z} \right] \right\} = A F_E [\gamma, \gamma, \gamma, \alpha, \lambda, \lambda; \beta, \mu, \eta; z, x, y] \quad (21)$$

$$\mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ F_4 [\lambda, \sigma; \mu, 1-\gamma; x, y(z-1)] \right\} = A F_{8c} \left[\begin{matrix} \lambda, \sigma; \lambda, \sigma; \gamma, \alpha \\ \mu; \gamma; \beta \end{matrix} \middle| x, y, z \right] \quad (22)$$

$$\mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ H_1 \left[\lambda, \sigma, \gamma; \mu; x, \frac{y}{1-z} \right] \right\} = A F_{10d} \left[\begin{matrix} \sigma, \lambda; \sigma, \gamma; \gamma, \alpha \\ \mu; \lambda; \beta \end{matrix} \middle| x, y, z \right] \quad (23)$$

$$\mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ H_1 \left[\lambda, \gamma, \sigma; \mu; \frac{x}{1-z}, \frac{y}{1-z} \right] \right\} = A F_{17b} \left[\begin{matrix} \gamma, \lambda; \gamma, \sigma; \gamma, \alpha \\ \mu; \lambda; \beta \end{matrix} \middle| x, y, z \right] \quad (24)$$

$$\mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ H_1 [\lambda, \sigma, \delta; 1-\gamma; x(z-1), y] \right\} = A F_{4i} \left[\begin{matrix} \sigma, \lambda; \sigma, \delta; \alpha, \gamma \\ \gamma; \lambda; \beta \end{matrix} \middle| x, y, z \right] \quad (25)$$

$$\mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ H_2 [\lambda, \sigma, \delta, \varepsilon; 1-\gamma; x(z-1), y] \right\} = A F_{1c} \left[\begin{matrix} \alpha, \gamma; \sigma, \lambda; \delta, \varepsilon \\ \beta; \gamma; \lambda \end{matrix} \middle| z, x, y \right] \quad (26)$$

$$\mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ H_2 [\gamma, \sigma, \delta, \varepsilon; \mu; \frac{x}{1-z}, y(1-z)] \right\} = A F_{4b} \left[\begin{matrix} \sigma, \gamma; \alpha, \gamma; \delta, \varepsilon \\ \mu; \beta; \gamma \end{matrix} \middle| x, z, y \right] \quad (27)$$

$$\mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ H_2 \left[\lambda, \sigma, \gamma, \delta; \mu; x, \frac{y}{1-z} \right] \right\} = A F_{4d} \left[\begin{matrix} \gamma, \delta; \gamma, \alpha; \sigma, \lambda \\ \lambda; \beta; \mu \end{matrix} \middle| y, z, x \right] \quad (28)$$

$$\mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ H_3 \left[\gamma, \sigma; \mu; \frac{x}{(1-z)^2}, \frac{y}{1-z} \right] \right\} = A X_6 [\gamma, \sigma, \alpha; \mu, \beta; x, y, z] \quad (29)$$

$$\mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ H_3 \left[\lambda, \gamma; \mu; x, \frac{y}{1-z} \right] \right\} = A X_{14} [\lambda, \gamma, \alpha; \mu, \beta; x, y, z] \quad (30)$$

$$\mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ H_3 [\lambda, \sigma; 1-\gamma; x(z-1), y(z-1)] \right\} = A F_{30d} \left[\begin{matrix} \lambda, \lambda; \lambda, \sigma; \alpha, \gamma \\ \gamma; \gamma; \beta \end{matrix} \middle| x, y, z \right] \quad (31)$$

$$\mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ H_4 \left[\gamma, \sigma; \mu, \eta; \frac{x}{(1-z)^2}, \frac{y}{1-z} \right] \right\} = A X_8 [\gamma, \sigma, \alpha; \mu, \eta, \beta; x, y, z] \quad (32)$$

$$\mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ H_4 \left[\lambda, \gamma; \mu, \eta; x, \frac{y}{1-z} \right] \right\} = A X_{17} [\lambda, \gamma, \alpha; \mu, \eta, \beta; x, y, z] \quad (33)$$

$$\mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ H_4 [\lambda, \sigma; 1-\gamma, \mu; x(z-1), y] \right\} = A F_{29e} \left[\begin{matrix} \lambda, \lambda; \lambda, \sigma; \alpha, \gamma \\ \gamma; \mu; \beta \end{matrix} \middle| x, y, z \right] \quad (34)$$

$$\mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ H_4 [\lambda, \sigma; \mu, 1-\gamma; x, y(z-1)] \right\} = A F_{29d} \left[\begin{matrix} \lambda, \lambda; \lambda, \sigma; \alpha, \gamma \\ \mu; \gamma; \beta \end{matrix} \middle| x, y, z \right] \quad (35)$$

$$\mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ H_5 \left[\gamma, \sigma; \mu; \frac{x}{(1-z)^2}, \frac{y}{1-z} \right] \right\} = A F_{41b} \left[\begin{matrix} \gamma, \gamma; \gamma, \sigma; \gamma, \alpha \\ \sigma; \mu; \beta \end{matrix} \middle| x, y, z \right] \quad (36)$$

$$\mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ H_5 \left[\lambda, \gamma; \mu; x(z-1), \frac{y}{1-z} \right] \right\} = A F_{36c} \left[\begin{matrix} \lambda, \lambda; \lambda, \gamma; \alpha, \gamma \\ \sigma; \mu; \beta \end{matrix} \middle| x, y, z \right] \quad (37)$$

$$\mathcal{M}_z^{\alpha, \beta, \gamma} \{H_5[\lambda, \sigma; 1-\gamma; x, y(1-z)]\} = A F_{29k} \left[\begin{matrix} \lambda, \lambda; \lambda, \gamma; \gamma, \alpha \\ \gamma, \mu; \beta \end{matrix} \middle| x, y, z \right] \quad (38)$$

$$\mathcal{M}_z^{\alpha, \beta, \gamma} \{H_6[\gamma, \sigma, \delta; \frac{x}{(1-z)^2}, y(1-z)]\} = A F_{29g} \left[\begin{matrix} \gamma, \gamma; \alpha, \gamma; \delta, \sigma \\ \sigma; \beta; \gamma \end{matrix} \middle| x, z, y \right] \quad (39)$$

$$\mathcal{M}_z^{\alpha, \beta, \gamma} \{H_6[\lambda, \gamma, \sigma; x(1-z), \frac{y}{1-z}]\} = A F_{26c} \left[\begin{matrix} \sigma, \gamma; \alpha, \gamma; \lambda, \lambda \\ \lambda; \beta; \gamma \end{matrix} \middle| y, z, x \right] \quad (40)$$

$$\mathcal{M}_z^{\alpha, \beta, \gamma} \{H_6[\lambda, \sigma, \gamma; x, \frac{y}{1-z}]\} = A F_{26d} \left[\begin{matrix} \gamma, \sigma; \gamma, \alpha; \lambda, \lambda \\ \lambda; \beta; \sigma \end{matrix} \middle| y, z, x \right] \quad (41)$$

$$\mathcal{M}_z^{\alpha, \beta, \gamma} \{H_7[\gamma, \sigma, \delta; \mu; \frac{x}{(1-z)^2}, y(1-z)]\} = A F_{29b} \left[\begin{matrix} \gamma, \gamma; \alpha, \gamma; \sigma, \delta \\ \mu; \beta; \gamma \end{matrix} \middle| x, z, y \right] \quad (42)$$

$$\mathcal{M}_z^{\alpha, \beta, \gamma} \{H_7[\lambda, \sigma, \delta; 1-\gamma; x(z-1), y]\} = A F_{23c} \left[\begin{matrix} \sigma, \delta; \alpha, \gamma; \lambda, \lambda \\ \lambda; \beta; \gamma \end{matrix} \middle| y, z, x \right] \quad (43)$$

PROOF. To establish relation (15), we first write Appell's function F_1 in it's series form (see e.g. [12, p. 23 (3)]):

$$F_1[\lambda, \alpha, \beta; \mu; x, y] = \sum_{m, n=0}^{\infty} \frac{(\lambda)_{m+n} (\alpha)_m (\beta)_n}{(\mu)_{m+n}} \frac{x^m y^n}{m! n!}.$$

Interchange the order of summation and differ-integration, which is permissible due to the absolute convergence, we get

$$\begin{aligned} & \mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ F_1 \left[\lambda, \sigma, \gamma; \mu; x, \frac{y}{1-z} \right] \right\} \\ &= \sum_{m, n=0}^{\infty} \frac{(\lambda)_{m+n} (\alpha)_m (\beta)_n}{(\mu)_{m+n}} \frac{x^m y^n}{m! n!} \mathcal{M}_z^{\alpha, \beta, \gamma} \{ z^{\alpha-1} (1-z)^{-\gamma-n} \} \end{aligned}$$

Now, making use of (5), the identity $(a+m)_n = \frac{(a)_{m+n}}{(a)_n}$

and the definition of Saran's function F_M [18, p. 42 (5)], with a little simplification, the desired result is obtained. The proof of relations (16) to (43) would run as above.

ii. Relations among two and four variables functions

Now, we establish N-fractional relationships between hyper-geometric functions of two and four variables. These relations can be proved on the same lines as adopted in the proof of the relations in the previous section, considering lemma 1 and the definitions of the hyper-geometric functions during the proof. Thus, using (6) and (7), we obtain the following formulas:

$$\begin{aligned} & \mathcal{M}_{\alpha_2, \beta_2, \gamma_2; z, w}^{\alpha_1, \beta_1, \gamma_1} \left\{ F_1 \left[\lambda, \gamma_1, \gamma_2; \mu; \frac{x}{1-z}, \frac{y}{1-w} \right] \right\} \\ &= B F_{34}^{(4)} [\gamma_1, \gamma_1, \gamma_2, \gamma_2, \lambda, \alpha_1, \lambda, \alpha_2; \mu, \mu, \beta_1, \beta_2; x, y, z, w], \end{aligned} \quad (44)$$

$$\begin{aligned} & \mathcal{M}_{\alpha_2, \beta_2, \gamma_2; z, w}^{\alpha_1, \beta_1, \gamma_1} \left\{ F_1 \left[\gamma_1, \gamma_2, \lambda; \mu; \frac{x}{(1-z)(1-w)}, \frac{y}{(1-z)} \right] \right\} \\ &= B F_{22}^{(4)} [\gamma_1, \gamma_1, \gamma_1, \alpha_2, \gamma_2, \lambda, \alpha_1, \gamma_2; \mu, \mu, \beta_1, \beta_2; x, y, z, w] \end{aligned} \quad (45)$$

$$\begin{aligned} & \mathcal{M}_{\alpha_2, \beta_2, \gamma_2; z, w}^{\alpha_1, \beta_1, \gamma_1} \left\{ F_2 \left[\gamma_1, \gamma_2, \lambda; \mu, \eta; \frac{x}{(1-z)(1-w)}, \frac{y}{(1-z)} \right] \right\} \\ &= B F_6^{(4)} [\gamma_1, \gamma_1, \gamma_1, \alpha_2, \gamma_2, \lambda, \alpha_1, \gamma_2; \mu, \eta, \beta_1, \beta_2; x, y, z, w], \end{aligned} \quad (46)$$

$$\begin{aligned} & \mathcal{M}_{\alpha_2, \beta_2, \gamma_2; z, w}^{\alpha_1, \beta_1, \gamma_1} \left\{ F_2 \left[\lambda, \gamma_1, \gamma_2; \mu, \eta; \frac{x}{1-z}, \frac{y}{1-w} \right] \right\} \\ &= B F_8^{(4)} [\gamma_1, \gamma_1, \gamma_2, \gamma_2, \lambda, \alpha_1, \lambda, \alpha_2; \mu, \eta, \beta_1, \beta_2; x, y, z, w] \end{aligned} \quad (47)$$

$$\begin{aligned} & \mathcal{M}_{\alpha_2, \beta_2, \gamma_2; z, w}^{\alpha_1, \beta_1, \gamma_1} \left\{ F_3 \left[\gamma_1, \gamma_2, \lambda, \sigma; \mu; \frac{x}{(1-z)}, \frac{y}{(1-w)} \right] \right\} \\ &= B F_{35}^{(4)} [\gamma_1, \gamma_1, \gamma_2, \gamma_2, \lambda, \alpha_1, \sigma, \lambda, \alpha_2; \mu, \beta_1, \mu, \beta_2; x, z, y, w] \end{aligned} \quad (48)$$

$$\begin{aligned} & \mathcal{M}_{\alpha_2, \beta_2, \gamma_2; z, w}^{\alpha_1, \beta_1, \gamma_1} \left\{ F_3 \left[\gamma_1, \lambda, \gamma_2, \sigma; \mu; \frac{x}{(1-z)(1-w)}, y \right] \right\} \\ &= B F_{36}^{(4)} [\gamma_1, \gamma_1, \alpha_2, \lambda, \gamma_2, \alpha_1, \gamma_2, \sigma; \mu, \beta_1, \beta_2, \mu; x, z, w, y] \end{aligned} \quad (49)$$

$$\begin{aligned} & \mathcal{M}_{\alpha_2, \beta_2, \gamma_2; z, w}^{\alpha_1, \beta_1, \gamma_1} \left\{ F_4 \left[\gamma_1, \gamma_2; \mu, \eta; \frac{x}{(1-z)(1-w)}, \frac{y}{(1-z)(1-w)} \right] \right\} \\ &= B F_4^{(4)} [\gamma_1, \gamma_1, \gamma_1, \alpha_2, \gamma_2, \gamma_2, \alpha_1, \gamma_2; \mu, \eta, \beta_1, \beta_2; x, z, y, w] \end{aligned} \quad (50)$$

$$\begin{aligned} & \mathcal{M}_{\alpha_2, \beta_2, \gamma; z, w}^{\alpha_1, \beta_1, \gamma} \left\{ F_1 \left[\lambda, \sigma, \gamma; \mu; x, \frac{y}{(1-\frac{z}{1-w})(1-w)} \right] \right\} \\ &= B F_{25}^{(4)} [\gamma, \gamma, \gamma, \sigma, \lambda, \alpha_1, \alpha_2, \lambda; \mu, \beta_1, \beta_2, \mu; y, z, w, x] \end{aligned} \quad (51)$$

$$\begin{aligned} & \mathcal{M}_{\alpha_2, \beta_2, \gamma; z, w}^{\gamma, \beta_1, \gamma_1} \left\{ F_1 \left[\lambda, \gamma_1, \sigma; \mu; \frac{x}{1-\frac{z}{1-w}}, y \right] \right\} \\ &= B F_{31}^{(4)} [\lambda, \lambda, \gamma, \gamma, \gamma_1, \sigma, \gamma_1, \alpha_2; \mu, \mu, \beta_1, \beta_2; x, y, z, w] \end{aligned} \quad (52)$$

$$\begin{aligned} & \mathcal{M}_{\alpha_2, \beta_2, \gamma; z, w}^{\alpha_1, \beta_1, \gamma} \left\{ F_2 \left[\gamma, \lambda, \sigma; \mu, \eta; \frac{x}{(1-\frac{z}{1-w})(1-w)}, \frac{y}{(1-\frac{z}{1-w})(1-w)} \right] \right\} \\ &= B F_A^{(4)} [\gamma, \lambda, \sigma, \alpha_2, \alpha_1; \mu, \eta, \beta_1, \beta_2; x, y, z, w] \end{aligned} \quad (53)$$

$$\begin{aligned} & \mathcal{M}_{\alpha_2, \beta_2, \gamma; z, w}^{\gamma, \beta_1, \gamma_1} \left\{ F_3 \left[\gamma_1, \lambda, \sigma, \delta; \mu; \frac{x}{(1-\frac{z}{1-w})}, y \right] \right\} \\ &= B F_{37}^{(4)} [\gamma_1, \gamma_1, \alpha_2, \lambda, \gamma, \sigma, \gamma, \delta; \beta_1, \mu, \beta_2, \mu; z, x, w, y] \end{aligned} \quad (54)$$

$$\begin{aligned} & \mathcal{M}_{\alpha_2, \beta_2, \gamma; z, w}^{\gamma, \beta_1, \gamma_1} \left\{ F_4 \left[\gamma_1, \gamma; \mu, \eta; \frac{x}{(1-\frac{z}{1-w})}, \frac{y}{(1-\frac{z}{1-w})} \right] \right\} \\ &= B F_5^{(4)} [\gamma_1, \gamma_1, \gamma_1, \alpha_2, \lambda, \lambda, \gamma, \gamma; \mu, \eta, \beta_1, \beta_2; x, y, z, w]. \end{aligned} \quad (55)$$

iii. Relations among three and four variables functions

In this section, we derive a number of relationships

between hyper-geometric functions of three and four variables, which can be established on the same lines as in the proof of (15)

$$\begin{aligned} & \mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ F_A^{(3)} \left[\lambda, \gamma, \sigma, \delta; \mu_1, \mu_2, \mu_3; \frac{x}{1-z}, y, w \right] \right\} \\ &= A F_6^{(4)} [\lambda, \lambda, \lambda, \alpha, \gamma, \sigma, \delta; \mu_1, \mu_2, \mu_3, \beta; x, y, w, z] \end{aligned} \quad (56)$$

$$\begin{aligned} & \mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ F_A^{(3)} \left[\gamma, \lambda, \sigma, \delta; \mu_1, \mu_2, \mu_3; \frac{x}{1-z}, \frac{y}{1-z}, \frac{w}{1-z} \right] \right\} \\ &= A F_A^{(4)} [\gamma, \lambda, \sigma, \delta, \alpha; \mu_1, \mu_2, \mu_3, \beta; x, y, w, z] \end{aligned} \quad (57)$$

$$\mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ F_B^{(3)} \left[\gamma, \lambda_1, \lambda_2, \delta_1, \delta_2, \delta_3; \mu; \frac{x}{1-z}, y, w \right] \right\} \frac{n!}{r!(n-r)!} \quad (58)$$

$$= A F_{76}^{(4)} \left[\gamma, \gamma, \lambda_1, \lambda_2, \delta_1, \alpha, \delta_2, \delta_3; \mu, \beta, \mu, \mu; x, z, y, w \right]$$

$$\mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ F_C^{(3)} \left[\gamma, \lambda; \mu_1, \mu_2, \mu_3; \frac{x}{1-z}, \frac{y}{1-z}, \frac{w}{1-z} \right] \right\} \quad (59)$$

$$= A F_2^{(4)} \left[\lambda, \lambda, \lambda, \alpha, \gamma, \gamma, \gamma, \gamma; \mu_1, \mu_2, \mu_3, \beta; x, y, w, z \right]$$

$$\mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ F_D^{(3)} \left[\gamma, \lambda_1, \lambda_2, \lambda_3; \mu; \frac{x}{1-z}, \frac{y}{1-z}, \frac{w}{1-z} \right] \right\} \quad (60)$$

$$= A F_{60}^{(4)} \left[\lambda_1, \lambda_2, \lambda_3, \alpha, \gamma, \gamma, \gamma, \gamma; \mu, \mu, \mu, \beta; x, y, w, z \right]$$

$$\mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ F_D^{(3)} \left[\lambda, \gamma, \sigma_1, \sigma_2; \mu; \frac{x}{1-z}, y, w \right] \right\} \quad (61)$$

$$= A F_{64}^{(4)} \left[\lambda, \lambda, \lambda, \alpha, \gamma, \sigma_1, \sigma_2, \gamma; \mu, \mu, \mu, \beta; x, y, w, z \right]$$

$$\mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ F_E \left[\gamma, \gamma, \gamma, \sigma, \delta, \delta; \mu_1, \mu_2, \mu_3; \frac{x}{1-z}, \frac{y}{1-z}, \frac{w}{1-z} \right] \right\} \quad (62)$$

$$= A F_3^{(4)} \left[\delta, \delta, \sigma, \alpha, \gamma, \gamma, \gamma, \gamma; \mu_1, \mu_2, \mu_3, \beta; y, w, x, z \right]$$

$$\mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ F_E \left[\lambda, \lambda, \lambda, \gamma, \delta, \delta; \mu_1, \mu_2, \mu_3; \frac{x}{1-z}, y, w \right] \right\} \quad (63)$$

$$= A F_5^{(4)} \left[\lambda, \lambda, \lambda, \alpha, \delta, \delta, \gamma, \gamma; \mu_1, \mu_2, \mu_3, \beta; y, w, x, z \right]$$

$$\mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ F_E \left[\lambda, \lambda, \lambda, \sigma, \gamma, \gamma; \mu_1, \mu_2, \mu_3; x, \frac{y}{1-z}, \frac{w}{1-z} \right] \right\} \quad (64)$$

$$= A F_4^{(4)} \left[\lambda, \lambda, \lambda, \alpha, \gamma, \gamma, \sigma, \gamma; \mu_1, \mu_2, \mu_3, \beta; y, w, x, z \right]$$

$$\mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ F_F \left[\lambda, \lambda, \lambda, \gamma, \delta, \gamma; \mu, \eta, \eta; \frac{x}{1-z}, y, \frac{w}{1-z} \right] \right\} \quad (65)$$

$$= A F_{15}^{(4)} \left[\lambda, \lambda, \lambda, \alpha, \gamma, \delta, \gamma, \gamma; \mu, \eta, \eta, \beta; w, x, y, z \right]$$

$$\mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ F_F \left[\lambda, \lambda, \lambda, \sigma, \gamma, \sigma; \mu, \eta, \eta; x, \frac{y}{1-z}, w \right] \right\} \quad (66)$$

$$= A F_{17}^{(4)} \left[\lambda, \lambda, \lambda, \alpha, \sigma, \sigma, \gamma, \gamma; \eta, \mu, \eta, \beta; w, x, y, z \right]$$

$$\mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ F_F \left[\gamma, \gamma, \gamma, \lambda, \sigma, \lambda; \mu, \eta, \eta; \frac{x}{1-z}, \frac{y}{1-z}, \frac{w}{1-z} \right] \right\} \quad (67)$$

$$= A F_{11}^{(4)} \left[\lambda, \lambda, \sigma, \alpha, \gamma, \gamma, \gamma, \gamma; \eta, \mu, \eta, \beta; w, x, y, z \right]$$

$$\mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ F_G \left[\gamma, \gamma, \gamma, \lambda, \sigma, \delta; \mu, \eta, \eta; \frac{x}{1-z}, \frac{y}{1-z}, \frac{w}{1-z} \right] \right\} \quad (68)$$

$$= A F_{13}^{(4)} \left[\sigma, \delta, \lambda, \alpha, \gamma, \gamma, \gamma, \gamma; \eta, \eta, \mu, \beta; y, w, x, z \right]$$

$$\mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ F_G \left[\lambda, \lambda, \lambda, \sigma, \delta, \gamma; \mu, \eta, \eta; x, y, \frac{w}{1-z} \right] \right\} \quad (69)$$

$$= A F_{22}^{(4)} \left[\lambda, \lambda, \lambda, \alpha, \gamma, \sigma, \delta, \gamma; \eta, \eta, \mu, \beta; w, y, x, z \right]$$

$$\mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ F_G \left[\lambda, \lambda, \lambda, \gamma, \sigma, \delta; \mu, \eta, \eta; \frac{x}{1-z}, y, w \right] \right\} \quad (70)$$

$$= A F_{23}^{(4)} \left[\lambda, \lambda, \lambda, \alpha, \gamma, \sigma, \delta, \gamma; \mu, \eta, \eta, \beta; x, y, w, z \right]$$

$$\begin{aligned} \mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ F_K \left[\lambda, \sigma, \sigma, \gamma, \delta, \gamma; \mu_1, \mu_2, \mu_3; \frac{x}{1-z}, y, \frac{w}{1-z} \right] \right\} \\ = A F_6^{(4)} \left[\gamma, \gamma, \gamma, \delta, \sigma, \lambda, \alpha, \sigma; \mu_3, \mu_1, \beta, \mu_2; w, x, z, y \right] \end{aligned} \quad (71)$$

$$\begin{aligned} \mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ F_K \left[\gamma, \lambda, \lambda, \sigma, \delta, \sigma; \mu_1, \mu_2, \mu_3; \frac{x}{1-z}, y, w \right] \right\} \\ = A F_8^{(4)} \left[\lambda, \lambda, \gamma, \gamma, \sigma, \delta, \sigma, \alpha; \mu_3, \mu_2, \mu_1, \beta; w, y, x, z \right] \end{aligned} \quad (72)$$

$$\begin{aligned} \mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ F_M \left[\gamma, \lambda, \lambda, \sigma, \delta, \sigma; \mu, \eta, \eta; \frac{x}{1-z}, y, w \right] \right\} \\ = A F_{31}^{(4)} \left[\lambda, \lambda, \gamma, \gamma, \sigma, \delta, \sigma, \alpha; \eta, \eta, \mu, \beta; w, y, x, z \right] \end{aligned} \quad (73)$$

$$\begin{aligned} \mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ F_M \left[\lambda, \sigma, \sigma, \gamma, \delta, \gamma; \mu, \eta, \eta; \frac{x}{1-z}, y, \frac{w}{1-z} \right] \right\} \\ = A F_{25}^{(4)} \left[\gamma, \gamma, \gamma, \delta, \sigma, \lambda, \alpha, \sigma; \eta, \mu, \beta, \eta; w, y, z, y \right] \end{aligned} \quad (74)$$

$$\begin{aligned} \mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ F_N \left[\gamma, \lambda, \sigma, \delta, \varepsilon, \delta; \mu, \eta, \eta; \frac{x}{1-z}, y, w \right] \right\} \\ = A F_{37}^{(4)} \left[\delta, \delta, \alpha, \lambda, \gamma, \sigma, \gamma, \varepsilon; \mu, \eta, \beta, \eta; x, w, z, y \right] \end{aligned} \quad (75)$$

$$\begin{aligned} \mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ F_N \left[\lambda, \gamma, \sigma, \delta, \varepsilon, \delta; \mu, \eta, \eta; x, \frac{y}{1-z}, w \right] \right\} \\ = A F_{35}^{(4)} \left[\delta, \delta, \gamma, \gamma, \sigma, \lambda, \varepsilon, \alpha; \mu, \mu, \eta, \beta; w, x, y, z \right] \end{aligned} \quad (76)$$

$$\begin{aligned} \mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ F_N \left[\lambda, \sigma, \delta, \gamma, \varepsilon, \gamma; \mu, \eta, \eta; \frac{x}{1-z}, y, \frac{w}{1-z} \right] \right\} \\ = A F_{26}^{(4)} \left[\gamma, \gamma, \gamma, \sigma, \lambda, \delta, \alpha, \varepsilon; \eta, \mu, \beta, \eta; w, x, w, z \right] \end{aligned} \quad (77)$$

$$\begin{aligned} \mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ F_P \left[\lambda, \sigma, \lambda, \gamma, \gamma, \delta; \mu, \eta, \eta; \frac{x}{1-z}, \frac{y}{1-z}, w \right] \right\} \\ = A F_{24}^{(4)} \left[\gamma, \gamma, \gamma, \delta, \lambda, \sigma, \alpha, \lambda; \mu, \eta, \beta, \eta; x, y, z, w \right] \end{aligned} \quad (78)$$

$$\begin{aligned} \mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ F_P \left[\lambda, \gamma, \lambda, \sigma, \sigma, \delta; \mu, \eta, \eta; x, \frac{y}{1-z}, w \right] \right\} \\ = A F_{32}^{(4)} \left[\gamma, \gamma, \lambda, \lambda, \sigma, \alpha, \sigma, \delta; \eta, \beta, \mu, \eta; y, z, x, w \right] \end{aligned} \quad (79)$$

$$\begin{aligned} \mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ F_P \left[\gamma, \lambda, \gamma, \sigma, \sigma, \delta; \mu, \eta, \eta; \frac{x}{1-z}, y, \frac{w}{1-z} \right] \right\} \\ = A F_{24}^{(4)} \left[\gamma, \gamma, \gamma, \lambda, \sigma, \delta, \alpha, \sigma; \mu, \eta, \beta, \eta; x, w, z, y \right] \end{aligned} \quad (80)$$

$$\begin{aligned} \mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ F_R \left[\gamma, \lambda, \gamma, \sigma, \delta, \sigma; \mu, \eta, \eta; \frac{x}{1-z}, y, \frac{w}{1-z} \right] \right\} \\ = A F_{20}^{(4)} \left[\gamma, \gamma, \gamma, \lambda, \sigma, \sigma, \alpha, \delta; \eta, \mu, \beta, \eta; w, x, z, y \right] \end{aligned} \quad (81)$$

$$\begin{aligned} \mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ F_R \left[\lambda, \gamma, \lambda, \sigma, \delta, \sigma; \mu, \eta, \eta; x, \frac{y}{1-z}, w \right] \right\} \\ = A F_{30}^{(4)} \left[\lambda, \lambda, \gamma, \gamma, \sigma, \sigma, \delta, \alpha; \eta, \mu, \eta, \beta; w, x, y, z \right] \end{aligned} \quad (82)$$

$$\begin{aligned} \mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ F_S \left[\lambda, \gamma, \gamma, \sigma, \delta, \varepsilon; \mu, \mu, \mu; x, \frac{y}{1-z}, \frac{w}{1-z} \right] \right\} \\ = A F_{67}^{(4)} \left[\gamma, \gamma, \gamma, \lambda, \delta, \varepsilon, \alpha, \sigma; \mu, \mu, \beta, \mu; y, w, z, x \right] \end{aligned} \quad (83)$$

$$\begin{aligned} \mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ F_S \left[\gamma, \lambda, \lambda, \sigma, \delta, \varepsilon; \mu, \mu, \mu; \frac{x}{1-z}, y, w \right] \right\} \\ = A F_{72}^{(4)} \left[\lambda, \lambda, \gamma, \gamma, \delta, \varepsilon, \sigma, \alpha; \mu, \mu, \mu, \beta; y, w, x, z \right] \end{aligned} \quad (84)$$

$$\begin{aligned} \mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ F_S \left[\lambda, \sigma, \sigma, \delta, \gamma, \varepsilon; \mu, \mu, \mu; x, \frac{y}{1-z}, w \right] \right\} \\ = A F_{74}^{(4)} \left[\sigma, \sigma, \alpha, \lambda, \gamma, \varepsilon, \gamma, \delta; \mu, \mu, \beta, \mu; y, w, z, x \right] \end{aligned} \quad (85)$$

$$\begin{aligned} \mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ F_T \left[\gamma, \lambda, \lambda, \sigma, \delta, \sigma; \mu, \mu, \mu; \frac{x}{1-z}, y, w \right] \right\} \\ = A F_{70}^{(4)} \left[\lambda, \lambda, \gamma, \gamma, \sigma, \delta, \sigma, \alpha; \mu, \mu, \mu, \beta; y, w, x, z \right] \end{aligned} \quad (86)$$

$$\begin{aligned} \mathcal{M}_z^{\alpha, \beta, \gamma} \left\{ F_T \left[\lambda, \sigma, \sigma, \gamma, \delta, \gamma; \mu, \mu, \mu; \frac{x}{1-z}, y, \frac{w}{1-z} \right] \right\} \\ = A F_{65}^{(4)} \left[\gamma, \gamma, \gamma, \delta, \sigma, \lambda, \alpha, \sigma; \mu, \mu, \beta, \mu; y, x, z, w \right] \end{aligned} \quad (87)$$

4. Multivariable N-fractional Operator

Definition 5. Let $\mathcal{M}_{z_1, \dots, z_n}^{\alpha_1, \beta_1, \dots, \alpha_n, \beta_n; \gamma}$ be a multivariable N-

Motivated by the results of the previous sections, we aim here at presenting a multivariable generalization of the N-fractional operator (5) as follows.

fractional operator defined by

$$\begin{aligned} \mathcal{M}_{z_1, \dots, z_n}^{\alpha_1, \beta_1, \dots, \alpha_n, \beta_n; \gamma} &= \left(\dots \left(\prod_{j=1}^n \{ z_j^{\alpha_j - 1} \} (1 - z_1 - \dots - z_n)^{-\gamma} \right)_{\alpha_1 - \beta_1} \dots \right)_{\alpha_n - \beta_n} \\ &= \frac{\prod_{j=1}^n \{ \Gamma(\alpha_j - \beta_j) \}}{(2\pi i)^n} \int_c \dots \int_c \prod_{j=1}^n \left\{ \frac{\zeta_j^{\alpha_j - 1}}{(\zeta_j - z_j)^{\alpha_j - \beta_j}} \right\} (1 - \zeta_1 - \dots - \zeta_n)^{-\gamma} d\zeta_1 \dots d\zeta_n, \end{aligned} \quad (88)$$

where $\gamma_j, \alpha_j, \beta_j \in \mathbb{C}$ and $(\alpha_j - \beta_j) \notin \mathbb{Z}^-, j=1, 2, \dots, n$.

Clearly, for $n=1$, (88) reduces to (5). Moreover, in view of lemma 1 and the multinomial theorem [12, p. 329 (220)]:

$$(1 - z_1 - \dots - z_n)^{-\gamma} = \sum_{m_1, \dots, m_n=0}^{\infty} (\gamma)_{m_1 + \dots + m_n} \frac{z_1^{m_1} \dots z_n^{m_n}}{m_1! \dots m_n!}, \quad (89)$$

the operators $\mathcal{M}_{z_1, \dots, z_n}^{\alpha_1, \beta_1, \dots, \alpha_n, \beta_n; \gamma}$, can be written in the form

$$\mathcal{M}_{z_1, \dots, z_n}^{\alpha_1, \beta_1, \dots, \alpha_n, \beta_n; \gamma} = A_{(1)}^{(n)} F_A^{(n)} [\gamma, \alpha_1, \dots, \alpha_n; \beta_1, \dots, \beta_n; z_1, \dots, z_n], \quad (90)$$

where $F_A^{(n)}$ is Lauricella function of n -variables (cf. [12, p. 33 (1)]) and throughout this work

$$A_{(1)}^{(n)} = \prod_{j=1}^n \left\{ e^{-i\pi(\alpha_j - \beta_j)} z_j^{\beta_j - 1} \frac{\Gamma(1 - \beta_j)}{\Gamma(1 - \alpha_j)} \right\}. \quad (91)$$

Next, in [3] we have established hyper-geometric function in several variables and denoted it by ${}^{(k)}E_{AD}^{(n)}$, $k \leq n$, which provide multivariable generalization of a number of known hyper-geometric functions. The function ${}^{(k)}E_{AD}^{(n)}$ is defined as below

$${}^{(k)}E_{AD}^{(n)} [\lambda, \alpha_1, \dots, \alpha_n; \beta, \gamma_{k+1}, \dots, \gamma_n; x_1, \dots, x_n] = \sum_{m_1, \dots, m_n=0}^{\infty} \frac{(\lambda)_{m_1+\dots+m_n}}{(\beta)_{m_1+\dots+m_k}} \frac{(\alpha_1)_{m_1} \dots (\alpha_n)_{m_n}}{(\gamma_{k+1})_{m_{k+1}} \dots (\gamma_n)_{m_n}} \frac{x_1^{m_1}}{m_1!} \dots \frac{x_n^{m_n}}{m_n!} \quad (92)$$

By the general theory of convergence of multiple hypergeometric functions (see e.g [8, section 2.9]; also [21, Chapter 9]), it follows that the region of convergence for ${}^{(k)}E_{AD}^{(n)}$ is $\max \{|x_j| + |x_{k+1}| + \dots + |x_n|\} < 1$, $j = 1, 2, \dots, n$. Now, we consider only four interesting multivariable applications of the operator (88). Considering

$I = \mathcal{M}_{z_{k+1}, \dots, z_n}^{\alpha_{k+1}, \beta_{k+1}, \dots, \alpha_n, \beta_n; \gamma} \left\{ F_c^{(n-k)} \left[\frac{\gamma}{2}, \frac{\gamma+1}{2}; \beta_1, \dots, \beta_k; \frac{4z_1}{(z_{k+1}, \dots, z_n)^2}, \dots, \frac{4z_k}{(z_{k+1}, \dots, z_n)^2} \right] \right\}$, On expressing $F_c^{(k)}$ in series form [22] and employing (89) and the Legendre duplication formula [22, p. 23 (24)]:

$$(\lambda)_{2m} = 2^{2m} \left(\frac{\lambda}{2}\right)_m \left(\frac{\lambda+1}{2}\right)_m,$$

$$\begin{aligned} \text{we find } I &= \sum_{m_1, \dots, m_n=0}^{\infty} \frac{(\gamma)_{m_1+\dots+m_n}}{(\beta_1)_{m_1} \dots (\beta_n)_{m_n}} \frac{(\alpha_1)_{m_{k+1}} \dots (\alpha_n)_{m_n}}{m_1! \dots m_k! m_{k+1}! \dots m_n!} \\ &\times \mathcal{M}_{z_{k+1}, \dots, z_n}^{\alpha_{k+1}, \beta_{k+1}, \dots, \alpha_n, \beta_n; \gamma} \left\{ \prod_{j=k+1}^n \{z_j^{\alpha_j+m_j-1}\} \right\}, \end{aligned}$$

which on using lemma 1 gives us the desired result:

$$\begin{aligned} &\mathcal{M}_{z_{k+1}, \dots, z_n}^{\alpha_{k+1}, \beta_{k+1}, \dots, \alpha_n, \beta_n; \gamma} \left\{ F_c^{(n-k)} \left[\frac{\gamma}{2}, \frac{\gamma+1}{2}; \beta_1, \dots, \beta_k; \frac{4z_1}{(z_{k+1}, \dots, z_n)^2}, \dots, \frac{4z_k}{(z_{k+1}, \dots, z_n)^2} \right] \right\} \\ &= A_{(k+1)}^{(n)} {}^{(k)}H_4^{(n)} [\gamma, \alpha_{k+1}, \dots, \alpha_n; \beta_1, \dots, \beta_n; z_1, \dots, z_n]. \end{aligned} \quad (93)$$

For the other three Lauricella functions $F_A^{(n)}$, $F_B^{(n)}$ and $F_D^{(n)}$ [21, p. 33] the operator (88) [in conjunction with (89) and lemma 1], similarly yields the following results:

$$\begin{aligned} &\mathcal{M}_{z_1, \dots, z_n}^{\alpha_1, \beta_1, \dots, \alpha_n, \beta_n; \gamma} \left\{ F_A^{(k)} \left[\gamma, \lambda_1, \dots, \lambda_k; \sigma_1, \dots, \sigma_k; \frac{x_1}{(z_1+\dots+z_n)}, \dots, \frac{x_k}{(z_1+\dots+z_n)} \right] \right\} \\ &= A_{(1)}^{(n)} F_A^{(n+k)} [\gamma, \alpha_1, \dots, \alpha_n, \lambda_1, \dots, \lambda_k; \beta_1, \dots, \beta_n, \sigma_1, \dots, \sigma_n; z_1, \dots, z_n, x_1, \dots, x_k]; \end{aligned} \quad (94)$$

$$\begin{aligned} &\mathcal{M}_{z_1, \dots, z_k}^{\alpha_1, \beta_1, \dots, \alpha_k, \beta_k; \gamma} \left\{ F_B^{(k)} \left[\alpha_{k+1}, \dots, \alpha_n, \lambda_{k+1}, \dots, \lambda_n; 1-\gamma; \frac{-z_{k+1}}{(z_1+\dots+z_k)^{-1}}, \dots, \frac{-z_n}{(z_1+\dots+z_k)^{-1}} \right] \right\} \\ &= A_{(1)}^{(k)} {}^{(k)}H_2^{(n)} [\gamma, \lambda_{k+1}, \dots, \lambda_n, \alpha_1, \dots, \alpha_n; \beta_1, \dots, \beta_k; z_1, \dots, z_n] \end{aligned} \quad (95)$$

$$\begin{aligned} &\mathcal{M}_{z_{k+1}, \dots, z_n}^{\alpha_{k+1}, \beta_{k+1}, \dots, \alpha_n, \beta_n; \gamma} \left\{ F_c^{(n-k)} \left[\frac{\gamma}{2}, \frac{\gamma+1}{2}; \beta_1, \dots, \beta_k; \frac{4z_1}{(z_{k+1}, \dots, z_n)^2}, \dots, \frac{4z_k}{(z_{k+1}, \dots, z_n)^2} \right] \right\} \\ &= A_{(k+1)}^{(n)} {}^{(k)}H_4^{(n)} [\gamma, \alpha_{k+1}, \dots, \alpha_n; \beta_1, \dots, \beta_n; z_1, \dots, z_n] \end{aligned} \quad (96)$$

where ${}^{(k)}H_2^{(n)}$ and ${}^{(k)}H_4^{(n)}$ are Exton's generalized Horn's functions [8, p. 97 (3.5)],

$$\mathcal{M}_{z_{k+1}, \dots, z_n}^{\alpha_{k+1}, \beta_{k+1}, \dots, \alpha_n, \beta_n; \gamma} \left\{ F_D^{(k)} \left[\gamma; \alpha_1, \dots, \alpha_k; \lambda; \frac{z_1}{(z_{k+1}+\dots+z_n)}, \dots, \frac{z_k}{(z_{k+1}+\dots+z_n)} \right] \right\}$$

$$= A_{(k+1)}^{(n)} {}^{(K)}E_{AD}^{(n)} [\gamma, \alpha_1, \dots, \alpha_n; \lambda, \beta_{k+1}, \dots, \beta_n; z_1, \dots, z_n]. \quad (97)$$

Finally, let us stress that the schema suggested in Sections 3 and 4 can be applied to find N-fractional relations for other generalized hypergeometric functions. In a forthcoming papers we will consider the problems of establishing N-fractional relations for other generalized hypergeometric functions by following the technique discussed in this paper.

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