Mathematics and Computer Science
2024; 9(1): 1-11
http://www.sciencepublishinggroup.com/j/mcs
doi: 10.11648/mcs.20240901.11
ISSN: 2575-6036 (Print); ISSN: 2575-6028 (Online)

Analytical Results of the Motion of Oscillating Dumbbell in a Viscous Fluid

Mohammed Mattar Al-Hatmi[1,*], Anton Purnama[2]

[1]Department of Basic Science, A'Sharqiyah University, Ibra, Oman
[2]Department of Mathematics, Sultan Qaboos University, Muscat, Oman

Email address:
mohammed.alhatmi@asu.edu.om (Mohammed Mattar Al-Hatmi)
[*]Corresponding author

To cite this article:
Mohammed Mattar Al-Hatmi, Anton Purnama. (2024). Analytical Results of the Motion of Oscillating Dumbbell in a Viscous Fluid. *Mathematics and Computer Science, 9*(1), 1-11. https://doi.org/10.11648/mcs.20240901.11

Received: December 31, 2023; **Accepted:** January 24, 2024; **Published:** February 5, 2024

Abstract: The aim of this paper is to investigate analytically the motion of oscillating dumbbell, two micro-spheres connected by a spring, in a viscous incompressible fluid at low Reynolds number. The oscillating dumbbell consists of one conducting sphere and assumed to be actively in motion under the action of an external oscillator field while the other is non-conducting sphere. As result, the oscillating dumbbell moves due to the induced flow oscillation of the surrounding fluid. The fluid flow past the spheres is described by the Stokes equation and the governing equation in the vector form for the oscillating dumbbell is solved asymptotically using the two-timing method. For illustrations, applying a simple oscillatory external field, a systematic description of the average velocity of the oscillating dumbbell is formulated. The trajectory of the oscillating dumbbell was found to be inversely proportional to the frequency of the external field, and the results demonstrated that the oscillating dumbbell moves in a circular path with a speed that decreases inversely with the length of the spring.

Keywords: Fluid Dynamics, Low Reynolds Number, Oscillatory Motion, Stokes Equation, Two-Timing Method

1. Introduction

The study of particle motion in a viscous fluid is a key to understanding the physical processes associated with particle suspensions at low Reynolds number. It has been of interest to scientists for many years and is still an active area of research [2, 3, 7, 8, 24, 25, 27]. The interactions between particles in the viscous fluid plays important roles in many applications in medicine and technology, such as minimizing surgical invasion and controlling drug delivery [13, 14, 16, 18, 20].

To successfully design a micro-robot, the motion of linked two (or more) micro-spheres must be studied and formulated. For example, the motion of dumbbells, in which two particles linked with a thin rod of a fixed length, has been studied by many researchers [1, 6, 12, 22]. In this article, we studied an extended case, in which the length is no longer fixed but varied between two lengths.

A conducting or active micro-sphere suspended in an external oscillatory force tends to accelerate in the direction of the applied external field, while the dumbbell moves due to the local surrounding fluid velocity generated by the motion of the active sphere. There is, however, a secondary effect arising from the fact that the oscillating dumbbell has not solely moved due to the external field, but its motion is also distorted by the presence of the other sphere. The key question is how does the surrounding fluid influence the motion of a oscillating dumbbell; will it also change the direction the active sphere moves, and if so, what trajectory does the oscillating dumbbell follow? Does the size of the micro-spheres or the length of the spring between the spheres have any effects on the motion of the oscillating dumbbell? In an attempt to answer these questions, we develop an analytical framework to study the motion of oscillating dumbbell in Stokes flow driven by an external oscillatory force. This analytical theory serves as a preliminary investigation on the effect of an external oscillatory field on the motion of a suspension of conducting particles.

This paper aims to provide a systematic and explicit description of a two-dimensional motion of oscillating dumbbell in Cartesian coordinates (X, Y) and calculate its average velocities. Two micro-spheres of radii $R_\nu < 10^{-6}$ meter are considered submerged in a viscous fluid at low Reynolds number, $Re \ll 1$, and suppose that one sphere is passive and the other is active which is accelerated by an external oscillatory field such as magnetic field [4, 9, 11, 17], electric field *e.g.* [10] and molecular Brownian force [23].

Formulation of the problem leads us to study the motion of an oscillating dumbbell with time-periodic forces. The problem is described by the Stokes equation, where the fluid inertial effect is neglected, and is solved by employing the two-timing method [11]. The results present in general form and discuss it in detail through an example. Treatment of the problem is simple, but it can be considered as the basis for the development of a full theory of suspension mechanics.

2. Formulation of Problem

Oscillating dumbbell, two solid spherical spheres connected by a spring of stiffness k and length l is considered, see Figure 1. The oscillating dumbbell involved two homogeneous spheres; one of them is active and positively buoyant and the other is passive and negatively buoyant. The active sphere is driven by an external field which oscillates periodically with constant frequency ω. Geometrically, the centers of the spheres and the length of the spring are denoted by

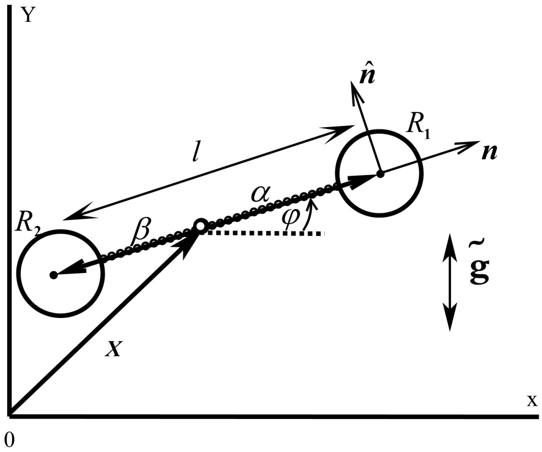


***Figure 1.** Diagram of oscillating dumbbell.*

$$\boldsymbol{x}^{(1)} = \boldsymbol{X} + \boldsymbol{\alpha}, \quad \boldsymbol{x}^{(2)} = \boldsymbol{X} + \boldsymbol{\beta}, \quad \boldsymbol{l} = \boldsymbol{x}^{(1)} - \boldsymbol{x}^{(2)}, \tag{1}$$

where $\boldsymbol{X} = (X, Y)$ is the displacement vector of the system and it is considered a reaction center of the spring's ends. We use the subscript $i, j = 1, 2$ for Cartesian components of vectors and tensors and superscript $\mu, \nu = 1, 2$ to identify the spheres. The vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ are define as

$$\boldsymbol{\alpha} = \alpha\boldsymbol{n} = r_1 l\boldsymbol{n}, \quad \boldsymbol{\beta} = \beta\boldsymbol{n} = -r_2 l\boldsymbol{n}, \quad r_1 = R_2/(R_1 + R_2), \quad r_2 = R_1/(R_1 + R_2) \tag{2}$$

where $\boldsymbol{n}$ is unit vector along $\boldsymbol{l}$ such that $\boldsymbol{n} \equiv \boldsymbol{l}/l$ and $l = \mid \boldsymbol{l} \mid$. The unit vectors $\boldsymbol{n}$, $\hat{\boldsymbol{n}}$ and the angle φ are given by

$$\boldsymbol{n} \equiv \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}, \quad \hat{\boldsymbol{n}} \equiv \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix} \tag{3}$$

The oscillating dumbbell moves due to the fluid flow only; where the forces exerted by the spring on the spheres can be considered negligible, since the spring thickness is smaller in comparison with the radius of spheres. Hence,

$$R_1\boldsymbol{\alpha} + R_2\boldsymbol{\beta} = 0. \tag{4}$$

The length of spring l is changing periodically due to the oscillations of an active sphere which is given by

$$l = L + \epsilon\widetilde{l}(\tau), \quad \tau = \omega t, \quad \epsilon = const \tag{5}$$

where L is a constant (averaged value) and $\widetilde{l}$ is a 2π-periodic function of τ with zero average value. Throughout the paper, a 'tilde' above a function denotes a 2π- periodic function with zero mean value. We consider a high-frequency asymptotic problem of oscillating dumbbell in a viscous incompressible fluid which oscillates as a rigid body due to an external field. The rigid-body oscillations are prescribed as a two-dimensional transitional spatial displacement of fluid particles, $\widetilde{\boldsymbol{\xi}}(\tau) = (\xi_1(\tau), \xi_2(\tau))$, through the related induced acceleration $\widetilde{\boldsymbol{\xi}}_{tt} = \omega^2\widetilde{\boldsymbol{\xi}}_{\tau\tau}$, where the subscripts t, τ stand for related partial time derivatives. This problem is classified as an oscillating (non-inertial) system of reference, in which a fluid at infinity is termed in a state of rest. In this frame, according to Einstein's principle of equivalence, there is an equivalence of gravitational and inertial mass, and so there is no influence of gravity field on the micro-spheres such that the equations of fluid motion are standard [21]. Hence, the micro-spheres float by the buoyancy force

$$\boldsymbol{f}_b^{(\nu)} = -M^{(\nu)}\widetilde{\boldsymbol{g}}(\tau), \tag{6}$$

which produces the potential energy of the spheres [22]

$$\Pi^{(\nu)} = -M^{(\nu)}\widetilde{\boldsymbol{g}} \cdot \boldsymbol{x}^{(\nu)}, \quad \Pi_{sp} = k(l - L)^2/2 \tag{7}$$

where $M^{(\nu)}$ is equal to the difference in the mass of a sphere and that of displaced fluid and $\widetilde{\boldsymbol{g}}(\tau)$ is equivalent to the induced acceleration

$$\widetilde{\boldsymbol{g}}(\tau) = \omega^2 \ \widetilde{\boldsymbol{\xi}}_{\tau\tau}. \tag{8}$$

The potential energy of the spring [11], is defined as

$$\Pi_{sp} = k(l-L)^2/2 \tag{9}$$

The total potential energy of the oscillating dumbbell is given by

$$\begin{aligned} \Pi \quad &= \Pi^{(1)} + \Pi^{(2)} + \Pi_{sp} \\ &= M\,\widetilde{\boldsymbol{g}} \cdot \boldsymbol{l} + k(l-L)^2/2 \end{aligned} \tag{10}$$

where $M \equiv M^{(1)} = -M^{(2)}$.

The micro-spheres experience Stokes force which is described by

$$\boldsymbol{f}_h^{(\nu)} = 6\pi\eta R_\nu(\boldsymbol{x}_t^{(\nu)} - \boldsymbol{u}_{\mu\nu}), \quad \mu \neq \nu \tag{11}$$

where η is a viscosity of the fluid and u is the velocity of the flow due to the movement of other sphere [15] which is defined as

$$\boldsymbol{u}_{\mu\nu} = \frac{3R_\mu}{4}\left[\frac{\boldsymbol{x}_t^{(\mu)} + \boldsymbol{n}(\boldsymbol{x}_t^{(\mu)} \cdot \boldsymbol{n})}{l}\right], \tag{12}$$

such that $\boldsymbol{u}_{12}$ represents the fluid disturbance generated by the movement of the first sphere on the second, while $\boldsymbol{u}_{21}$ represents the fluid disturbance generated by the movement of the second sphere on the first. We assume that the fluid flow past the micro-spheres is solely described by the Stokes equation where all inertial terms are neglected, hence, kinetic energy $\mathcal{K} \equiv 0$.

The geometric configuration of the problem contains two characteristic lengths: the length of the spring L and the radius of the spheres R. Using the reference scales T, G, M and F; characteristic time-scale, constant of gravity, mass and Stokes force where

$$R = (R_1 + R_2)/2, \quad T = 1/\omega, \quad G = max \mid \widetilde{\boldsymbol{g}}(\tau) \mid, \ \ F = 6\pi\eta RL/T \tag{13}$$

The dimensionless variables (marked with asterisks) are chosen as

$$x = Lx^*, l = \qquad Ll^*, R_\nu = RR_\nu^*, t = Tt^*, g = Gg^*, \tag{14}$$

$$f_b^{(\nu)} = FF_b^*, \ \ f_h^{(\nu)} = FF_h^{*(\nu)}. \tag{15}$$

Four independent small parameters of the problem are

$$\epsilon \equiv 1/\omega^*, \quad \delta \equiv 3R/4L, \quad \gamma \equiv MG/F, \quad K = kL/F; \quad \epsilon, \delta \ll 1, \tag{16}$$

where ω^* is the dimensionless frequency of the oscillation.
The substituting (13), (14) and (15) into (11), (6) and (10) yield the dimensionless form (all asterisks are omitted for brevity)

$$\boldsymbol{F}_h^{(\nu)} = -R_\nu \boldsymbol{x}_t^{(\nu)} + \delta R_\mu R_\nu \mathbb{S}\boldsymbol{x}_t^{(\mu)}, \tag{17}$$
$$\boldsymbol{F}_b = -\gamma\,\widetilde{\boldsymbol{g}}, \tag{18}$$
$$\Pi = \gamma\,\widetilde{\boldsymbol{g}} \cdot \boldsymbol{l} + K(l-1)^2/2, \tag{19}$$
$$\mathbb{S} = S_{ij} \equiv \frac{1}{l}(\delta_{ij} + n_i n_j), \tag{20}$$

where δ_{ij} is a Kronecker delta; $\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ when $i \neq j$.

The oscillating dumbbell represents a mechanical system which can be described by choosing generalized coordinates as

$$\boldsymbol{q} = (q_1, q_2, q_3, q_4) \equiv (X, Y, l, \varphi). \tag{21}$$

To describe the motion of the oscillating dumbbell, we use the Lagrangian function $\mathcal{L} = \mathcal{L}(\boldsymbol{q}, \boldsymbol{q}_t)$ which is defined as [11]

$$\mathcal{L}(\boldsymbol{q}, \boldsymbol{q}_t) = \mathcal{K} - \Pi = -\Pi = -\gamma\,\widetilde{\boldsymbol{g}} \cdot \boldsymbol{l} - K(l-1)^2/2. \tag{22}$$

The Lagrange equations are

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial q_{mt}} - \frac{\partial \mathcal{L}}{\partial q_m} = Q_m, \quad Q_m = \sum_{\nu=1}^{2}\sum_{j=1}^{2} F_{jh}^{(\nu)} \frac{\partial x_j^{(\nu)}}{\partial q_m} \tag{23}$$

where $\boldsymbol{Q} = (Q_1, Q_2, Q_3, Q_4)$ is the generalized external viscous force exerted by the fluid on the oscillating dumbbell and subscript $m = 1, 2, 3, 4$ denoted the generalized coordinates.

Using the left part of (23) yield

$$Q_1 = 0, \qquad Q_2 = 0, \qquad Q_3 = \gamma\,\widetilde{\boldsymbol{g}} \cdot \boldsymbol{l}_l + K(l-1), \qquad Q_4 = \gamma\,\widetilde{\boldsymbol{g}} \cdot \boldsymbol{l}_\varphi. \tag{24}$$

Using the right part of (23) yield

$$Q_1 = F_{1h}^{(1)} + F_{1h}^{(2)}, \qquad Q_2 = F_{2h}^{(1)} + F_{2h}^{(2)}, \tag{25}$$
$$Q_3 = r_1\, cos\,\varphi\ F_{1h}^{(1)} + r_1\, sin\,\varphi\ F_{2h}^{(1)} - r_2\, cos\,\varphi\ F_{1h}^{(2)} - r_2\, sin\,\varphi\ F_{2h}^{(2)}, \tag{26}$$
$$Q_4 = -r_1\, l\, sin\,\varphi\ F_{1h}^{(1)} + r_1\, l\, cos\,\varphi\ F_{2h}^{(1)} + r_2\, l\, sin\,\varphi\ F_{1h}^{(2)} \tag{27}$$
$$-r_2\, l\, cos\,\varphi\ F_{2h}^{(2)}. \tag{28}$$

Combining (24), (25), (26) and (28) yield:

$$F_{1h}^{(1)} + F_{1h}^{(2)} = 0, \qquad F_{2h}^{(1)} + F_{2h}^{(2)} = 0, \tag{29}$$
$$r_1 cos\,\varphi\, F_{1h}^{(1)} + r_1 sin\,\varphi\, F_{2h}^{(1)} - r_2 cos\,\varphi\, F_{1h}^{(2)} - r_2 sin\,\varphi\, F_{2h}^{(2)} = \gamma\,\widetilde{\boldsymbol{g}} \cdot \boldsymbol{l}_l + K(l-1), \tag{30}$$
$$-r_1\, l\, sin\,\varphi\ F_1^{(1)} + r_1\, l\, cos\,\varphi\ F_{2h}^{(1)} + r_2\, l\, sin\,\varphi\ F_{1h}^{(2)} - r_2\, l\, cos\,\varphi\ F_{2h}^{(2)} = \gamma\,\widetilde{\boldsymbol{g}} \cdot \boldsymbol{l}_\varphi. \tag{31}$$

Integrating (30) with respect to l and then simple simplification yield

$$l\, r_1 \begin{pmatrix} F_{1h}^{(1)} \\ F_{2h}^{(1)} \end{pmatrix} \cdot \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} - l\, r_2 \begin{pmatrix} F_{1h}^{(2)} \\ F_{2h}^{(2)} \end{pmatrix} \cdot \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} = \gamma\,\widetilde{\boldsymbol{g}} \cdot \boldsymbol{l} + K(l-1)^2/2, \tag{32}$$

which can be written in the form

$$\left(r_1 \boldsymbol{F}_h^{(1)} - r_2 \boldsymbol{F}_h^{(1)}\right) \cdot \boldsymbol{n} = \gamma\,\widetilde{\boldsymbol{g}} \cdot \boldsymbol{n} + K(l-1)^2/2l. \tag{33}$$

Similarly, (31) takes the following form

$$\left(r_1 \boldsymbol{F}_h^{(1)} - r_2 \boldsymbol{F}_h^{(1)}\right) \cdot \hat{\boldsymbol{n}} = \gamma\,\widetilde{\boldsymbol{g}} \cdot \hat{\boldsymbol{n}}. \tag{34}$$

For future use, we write (29), (33) and (34) as

$$\boldsymbol{F}_h^{(1)} + \boldsymbol{F}_h^{(2)} = 0 \tag{35}$$
$$\boldsymbol{F}^- \cdot \boldsymbol{n} = \gamma\widetilde{\boldsymbol{g}} \cdot \boldsymbol{n} + K(l-1)^2/2l, \tag{36}$$
$$\boldsymbol{F}^- \cdot \hat{\boldsymbol{n}} = \gamma\widetilde{\boldsymbol{g}} \cdot \hat{\boldsymbol{n}}, \tag{37}$$

where

$$\boldsymbol{F}^{-}=\left(R_2\boldsymbol{F}_h^{(1)}-R_1\boldsymbol{F}_h^{(2)}\right)/2\,R. \tag{38}$$

The use of (17) into (38) yield

$$\boldsymbol{F}^{-}=-\left(R_1R_2\boldsymbol{l}_t+\delta R_1R_2\mathbb{S}\left(\hat{R}\boldsymbol{X}_t+R_1R_2\boldsymbol{l}_t/R\right)\right)/2\,R. \tag{39}$$

The use of (17) into (35) and the use of (39) into (36) and (37) yield:

$$\boldsymbol{X}_t-\delta R_1R_2\mathbb{S}\left(\boldsymbol{X}_t-\hat{R}\boldsymbol{l}_t/4R\right)/\,R=0 \tag{40}$$

$$\left(\boldsymbol{l}_t+\delta\mathbb{S}\left(\hat{R}\boldsymbol{X}_t+R_1R_2\boldsymbol{l}_t/\,R\right)\right)\cdot\hat{\boldsymbol{n}}=-2R\gamma\,\widetilde{\boldsymbol{g}}\cdot\hat{\boldsymbol{n}}/R_1R_2 \tag{41}$$

$$\left(\boldsymbol{l}_t+\delta\mathbb{S}\left(\hat{R}\boldsymbol{X}_t+R_1R_2\boldsymbol{l}_t/\,R\right)\right)\cdot\boldsymbol{n}=-R\left(2\gamma\,\widetilde{\boldsymbol{g}}\cdot\boldsymbol{n}+K(l-1)^2/l\right)/R_1R_2 \tag{42}$$

where $\hat{R}=R_1-R_2$. An approximate solution to the equations (40), (41) and (42) is presented in the next section using an asymptotic procedure containing the two-timing method.

3. Two-timing Method and Asymptotic Procedure

3.1. Definition

Two-timing method constructs an asymptotic solution to the equation of motion (40), (41) and (42) by introducing two dependent time-scales s and τ as mutually independent variables, called slow and fast times, respectively, see Figure 2. This method converts (40), (41) and (42) from an ODE with the only independent variable t into a PDE with two independent variables s, τ.

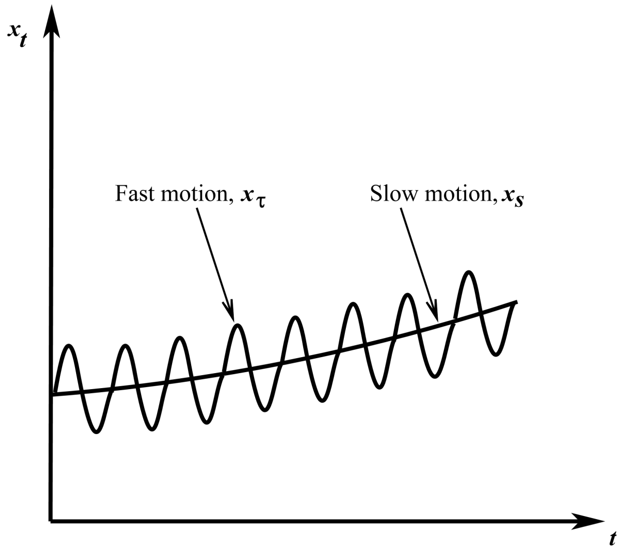


Figure 2. *Diagram of the fast and slow motions.*

The proper relations between s, τ and $t$is defined by [19]

$$\tau=\omega t,\quad s=t/\omega^{\alpha};\ \alpha>-1. \tag{43}$$

For different values of α there are multi paths for the asymptotic solution. In a rigorous asymptotic procedure $\omega\longrightarrow\infty$, there is a unique path can be lead to valid solution. If such a limit brings valid an asymptotic result, then it is called a distinguished limit. In this paper, we have chosen $\alpha=1$; hence,

$$\tau=\omega t,\quad s=t/\omega, \tag{44}$$

such that they lead to a valid asymptotic procedure of successive approximations [5, 26].

3.2. Functions and Notation

For making further progress analytically, we introduce a few convenient notation. We assume that any dimensionless functions $f(s,\tau)$ (which could be scalar, vectorial, or tensorial one) [2], has the following properties:

1. Subscripts τ and s stand for related partial time derivatives.
2. $f=f(s,\tau)$ belongs to class $O(1)$ such that $f=O(1)$, and all partial s and τ derivatives of f (required for our consideration) are also $O(1)$.
3. (iii) We consider only *periodic function in* τ $\{f\in\mathcal{P}: f(s,\tau)=f(s,\tau+2\pi)\}$, where s-dependence is not specified.
4. (iv) For arbitrary $f\in\mathcal{P}$, the *averaging-operation* is

$$\langle f\rangle\equiv\frac{1}{2\pi}\int_{\tau_0}^{\tau_0+2\pi}f(s,\tau)\,d\tau\equiv\overline{f}(s),\qquad\forall\,\tau_0. \tag{45}$$

The *bar-function* $\overline{f}=\overline{f}(s)$ (or mean-function) does not depend on τ.

5. (v) *The tilde-function*, $\widetilde{f}(\tau)$(or purely oscillating function) represents a special case of $\mathcal{P}$-function with

zero average

$$\langle \widetilde{f} \rangle = 0. \tag{46}$$

A unique decomposition is valid

$$f = \overline{f} + \widetilde{f}. \tag{47}$$

6. (vi) A special notation $\widetilde{f}^{\tau}$ is the *tilde-integration* of $\widetilde{f}(\tau)$,

$$\widetilde{f}^{\tau} = \int_0^{\tau} \widetilde{f}(s,\sigma)\, d\sigma - \frac{1}{2\pi}\int_0^{2\pi}\left(\int_0^{\mu} \widetilde{f}(s,\sigma)\, d\sigma\right) d\mu. \tag{48}$$

The *tilde-integration* is inverse to the τ-differentiation

$$\left(\widetilde{f}^{\tau}\right)_{\tau} \equiv \left(\widetilde{f}_{\tau}\right)^{\tau} \equiv \widetilde{f}. \tag{49}$$

3.3. Successive Approximations

The choices $\tau = \omega t$ and $s = t/\omega$ lead to the following chain rule:

$$\begin{aligned} d/dt &= \omega\, \partial/\partial\tau + 1/\omega\ \partial/\partial s \\ &= \omega \left(\partial/\partial\tau + \epsilon^2\ \partial/\partial s\right); \quad \epsilon = 1/\omega. \end{aligned} \tag{50}$$

However, we consider series expansions in ϵ, at most $O(\epsilon^2)$ and keep at most linear in δ terms. The unknown $\boldsymbol{X}$ is written as series in ϵ

$$\boldsymbol{X}(s,\tau) = \left(\overline{\boldsymbol{X}}_0 + \widetilde{\boldsymbol{X}_0}\right) + \varepsilon\left(\overline{\boldsymbol{X}}_1 + \widetilde{\boldsymbol{X}_1}\right) + \varepsilon^2\left(\overline{\boldsymbol{X}}_2 + \widetilde{\boldsymbol{X}_2}\right) + O(\epsilon^2) \tag{51}$$

Differentiate (51) with respect to t gives

$$\boldsymbol{X}_t = \left(\overline{\boldsymbol{X}}_{0t} + \widetilde{\boldsymbol{X}_{0t}}\right) + \varepsilon\left(\overline{\boldsymbol{X}}_{1t} + \widetilde{\boldsymbol{X}_{1t}}\right) + \varepsilon^2\left(\overline{\boldsymbol{X}}_{2t} + \widetilde{\boldsymbol{X}_{2t}}\right) + O(\epsilon^2) \tag{52}$$

The use of (50) into (52) yield

$$\begin{aligned} \boldsymbol{X}_t = \quad & \omega\left(\left(\overline{\boldsymbol{X}}_{0\tau} + \epsilon^2 \overline{\boldsymbol{X}}_{0s}\right) + \left(\widetilde{\boldsymbol{X}_{0\tau}} + \epsilon^2 \widetilde{\boldsymbol{X}_{0s}}\right)\right) + \omega\epsilon\left(\left(\overline{\boldsymbol{X}}_{1\tau} + ...\right) + \left(\widetilde{\boldsymbol{X}_{1\tau}} + ...\right)\right) \\ & + \omega\epsilon^2\left(\left(\overline{\boldsymbol{X}}_{2\tau} + ...\right) + \left(\widetilde{\boldsymbol{X}_{2\tau}} + ...\right)\right) + ... \end{aligned} \tag{53}$$

and with similar expressions for $\boldsymbol{l}$. The two-timing method study only the class of solutions with

$$\widetilde{\boldsymbol{X}_0}(s,\tau) = 0,\ \widetilde{\boldsymbol{l}}_0 = 0 \quad \text{while} \quad \overline{\boldsymbol{X}}_0(s,\tau) \neq 0 \text{ and } \bar{\boldsymbol{l}}_0(s,\tau) \neq 0. \tag{54}$$

Physically, this constraint means that the amplitude of oscillations is small compared with the amplitude of the averaged solution such that when $\widetilde{\boldsymbol{X}_0}(s,\tau) \neq 0$, the main term of velocity grows to infinity as $\omega \longrightarrow \infty$.

Hence, (53) is given in the form

$$\boldsymbol{X}_t = \quad \omega\left(\epsilon\widetilde{\boldsymbol{X}_{1\tau}} + \epsilon^2\left(\widetilde{\boldsymbol{X}_{2\tau}} + \overline{\boldsymbol{X}}_{0s}\right) + O(\epsilon^2)\right). \tag{55}$$

Taylor series expansion of the tensor $\mathbb{S}_{ij}$ (20) about $\boldsymbol{l} = \boldsymbol{l}_0$ takes the form

$$\mathbb{S}_{ij}(\boldsymbol{l}) = \mathbb{S}_{ij}(\boldsymbol{l}_0) + \mathbb{S}'_{ij}(\boldsymbol{l}_0)\,(\boldsymbol{l} - \boldsymbol{l}_0) + O(\epsilon^2), \tag{56}$$

where $\boldsymbol{l} - \boldsymbol{l}_0 = \epsilon \boldsymbol{l}_1 + \epsilon^2 \boldsymbol{l}_2 + O(\epsilon^2)$. Hence, (56) can be written as

$$\mathbb{S}_{ij}(\boldsymbol{l}) = \mathbb{S}_{ij}(\boldsymbol{l}_0) + \epsilon \boldsymbol{l}_{1k}\frac{\partial \mathbb{S}_{ij}(\boldsymbol{l}_0)}{\partial x_k} + O(\epsilon^2), \tag{57}$$

where $\boldsymbol{l}_0, \boldsymbol{l}_{1k}, \mathbb{S}_{ij}(\boldsymbol{l}_0)$ and $\partial \mathbb{S}_{ij}(\boldsymbol{l}_0)/\partial x_k$ are given by

$$\boldsymbol{l}_0 = \boldsymbol{x}_0^{(1)} - \boldsymbol{x}_0^{(2)}, \quad \boldsymbol{l}_{1k} = \boldsymbol{x}_{1k}^{(1)} - \boldsymbol{x}_{1k}^{(2)}, \tag{58}$$

$$\mathbb{S}_{ij}(\boldsymbol{l}_0)=\frac{1}{l_0}[\delta_{ij}+\boldsymbol{n}_{0i}\boldsymbol{n}_{0j}],\quad \boldsymbol{n}_0=\boldsymbol{l}_0/l_0;\;\; l_0=\mid\boldsymbol{l}_0\mid, \tag{59}$$

$$\frac{\partial\mathbb{S}_{ij}(\boldsymbol{l}_0)}{\partial x_k}=\frac{1}{l_0^2}\left(-\delta_{ij}\boldsymbol{n}_{0k}+\delta_{ik}\boldsymbol{n}_{0j}+\delta_{jk}\boldsymbol{n}_{0i}-3\boldsymbol{n}_{0i}\boldsymbol{n}_{0j}\boldsymbol{n}_{0k}\right). \tag{60}$$

For future use, we write the first and second terms in the right-hand side of (57) as

$$\mathbb{S}_0=\mathbb{S}_{ij}(\boldsymbol{l}_0),\quad \mathbb{S}_1=\mathbb{S}_{1ij}(\boldsymbol{l}_0)=\boldsymbol{l}_{1k}\frac{\partial\mathbb{S}_{ij}(\boldsymbol{l}_0)}{\partial x_k}, \tag{61}$$

The use of (55) and (57) into the system (40), (41) and (42) yield

$$\begin{aligned}&\epsilon\widetilde{\boldsymbol{X}}_{1\tau}+\epsilon^2(\widetilde{\boldsymbol{X}}_{2\tau}+\overline{\boldsymbol{X}}_{0s})+O(\epsilon^2)-\delta\frac{R_1R_2}{R}(\mathbb{S}_0+\epsilon\mathbb{S}_1+O(\epsilon^2))(\epsilon\widetilde{\boldsymbol{X}}_{1\tau}\\&+\epsilon^2(\widetilde{\boldsymbol{X}}_{2\tau}+\overline{\boldsymbol{X}}_{0s})+O(\epsilon^2))-\frac{\hat{R}}{4R}(\epsilon\widetilde{\boldsymbol{l}}_{1\tau}+\epsilon^2(\widetilde{\boldsymbol{l}}_{2\tau}+\bar{\boldsymbol{l}}_{0s})+O(\epsilon^2))=0\end{aligned} \tag{62}$$

$$\begin{aligned}&[\epsilon\widetilde{\boldsymbol{l}}_{1\tau}+\epsilon^2(\widetilde{\boldsymbol{l}}_{2\tau}+\bar{\boldsymbol{l}}_{0s})+O(\epsilon^2)+\delta(\mathbb{S}_0+\epsilon\mathbb{S}_1+O(\epsilon^2))(\hat{R}(\epsilon\widetilde{\boldsymbol{X}}_{1\tau}+\epsilon^2(\widetilde{\boldsymbol{X}}_{2\tau}+\overline{\boldsymbol{X}}_{0s})+O(\epsilon^2))\\&+\frac{R_1R_2}{R}(\epsilon\widetilde{\boldsymbol{l}}_{1\tau}+\epsilon^2(\widetilde{\boldsymbol{l}}_{2\tau}+\bar{\boldsymbol{l}}_{0s})+O(\epsilon^2)))]\cdot\hat{\boldsymbol{n}}_0=\epsilon\frac{-2R\gamma}{R_1R_2}\;\widetilde{\boldsymbol{g}}\cdot\hat{\boldsymbol{n}}_0\end{aligned} \tag{63}$$

$$\begin{aligned}&[\epsilon\widetilde{\boldsymbol{l}}_{1\tau}+\epsilon^2(\widetilde{\boldsymbol{l}}_{2\tau}+\bar{\boldsymbol{l}}_{0s})+O(\epsilon^2)+\delta(\mathbb{S}_0+\epsilon\mathbb{S}_1+O(\epsilon^2))(\hat{R}(\epsilon\widetilde{\boldsymbol{X}}_{1\tau}+\epsilon^2(\widetilde{\boldsymbol{X}}_{2\tau}+\overline{\boldsymbol{X}}_{0s})+O(\epsilon^2))\\&+\frac{R_1R_2}{R}(\epsilon\widetilde{\boldsymbol{l}}_{1\tau}+\epsilon^2(\widetilde{\boldsymbol{l}}_{2\tau}+\bar{\boldsymbol{l}}_{0s})+O(\epsilon^2)))]\cdot\boldsymbol{n}_0=\epsilon\frac{-R}{R_1R_2}\left(2\,\gamma\,\widetilde{\boldsymbol{g}}\cdot\boldsymbol{n}_0+K(l-1)^2/l\right)\end{aligned} \tag{64}$$

The successive approximations of (62), (63) and (64) lead to:
terms of order ϵ^0 give the identities 0=0. Terms of order ϵ yield

$$\widetilde{X}_{1\tau}=\delta R_1R_2\overline{\mathbb{S}}_0(\widetilde{X}_{1\tau}-\hat{R}\widetilde{\boldsymbol{l}}_{1\tau}/4R)/R, \tag{65}$$

$$\left[\widetilde{\boldsymbol{l}}_{1\tau}+\delta\mathbb{S}_0(\hat{R}\widetilde{X}_{1\tau}+R_1R_2\widetilde{\boldsymbol{l}}_{1\tau}/R)\right]\cdot\hat{\boldsymbol{n}}_0=\frac{-2R\gamma}{R_1R_2}\;\widetilde{\boldsymbol{g}}\cdot\hat{\boldsymbol{n}}_0, \tag{66}$$

$$\left[\widetilde{\boldsymbol{l}}_{1\tau}+\delta\mathbb{S}_0(\hat{R}\widetilde{X}_{1\tau}+R_1R_2\widetilde{\boldsymbol{l}}_{1\tau}/R)\right]\cdot\boldsymbol{n}_0=\frac{-2R}{R_1R_2}\left(\gamma\,\widetilde{\boldsymbol{g}}\cdot\boldsymbol{n}_0+K(l-1)^2/2l\right). \tag{67}$$

The use of (65) into (66) and (67), keep at most linear in δ, yield

$$\left[\widetilde{\boldsymbol{l}}_{1\tau}+\delta\mathbb{S}_0R_1R_2\widetilde{\boldsymbol{l}}_{1\tau}/R\right]\cdot\hat{\boldsymbol{n}}_0=\frac{-2R\gamma}{R_1R_2}\;\widetilde{\boldsymbol{g}}\cdot\hat{\boldsymbol{n}}_0, \tag{68}$$

$$\left[\widetilde{\boldsymbol{l}}_{1\tau}+\delta\mathbb{S}_0R_1R_2\widetilde{\boldsymbol{l}}_{1\tau}/R\right]\cdot\boldsymbol{n}_0=\frac{-2R}{R_1R_2}\left(\gamma\,\widetilde{\boldsymbol{g}}\cdot\boldsymbol{n}_0+K(l-1)^2/2l\right). \tag{69}$$

Terms of order ϵ^2 yield

$$\widetilde{\boldsymbol{X}}_{2\tau}+\overline{\boldsymbol{X}}_{0s}-\delta\frac{R_1R_2}{R}\mathbb{S}_0(\widetilde{\boldsymbol{X}}_{2\tau}+\overline{\boldsymbol{X}}_{0s}-\frac{\hat{R}}{4R}(\widetilde{\boldsymbol{l}}_{2\tau}+\bar{\boldsymbol{l}}_{0s}))-\delta\frac{R_1R_2}{R}\mathbb{S}_1(\widetilde{\boldsymbol{X}}_{1\tau}-\frac{\hat{R}}{4R}\widetilde{\boldsymbol{l}}_{1\tau})=0 \tag{70}$$

$$[\widetilde{\boldsymbol{l}}_{2\tau} + \bar{\boldsymbol{l}}_{0s} + \delta\mathbb{S}_0(\hat{R}(\widetilde{\boldsymbol{X}}_{2\tau} + \overline{\boldsymbol{X}}_{0s}) + \frac{R_1R_2}{R}(\widetilde{\boldsymbol{l}}_{2\tau} + \bar{\boldsymbol{l}}_{0s})) + \delta\mathbb{S}_1(\hat{R}\widetilde{\boldsymbol{X}}_{1\tau} + \frac{R_1R_2}{R}\widetilde{\boldsymbol{l}}_{1\tau})] \cdot \hat{\boldsymbol{n}}_0 = 0 \tag{71}$$

$$[\widetilde{\boldsymbol{l}}_{2\tau} + \bar{\boldsymbol{l}}_{0s} + \delta\mathbb{S}_0(\hat{R}(\widetilde{\boldsymbol{X}}_{2\tau} + \overline{\boldsymbol{X}}_{0s}) + \frac{R_1R_2}{R}(\widetilde{\boldsymbol{l}}_{2\tau} + \bar{\boldsymbol{l}}_{0s})) + \delta\mathbb{S}_1(\hat{R}\widetilde{\boldsymbol{X}}_{1\tau} + \frac{R_1R_2}{R}\widetilde{\boldsymbol{l}}_{1\tau})] \cdot \boldsymbol{n}_0 = 0 \tag{72}$$

The use of averaging procedure (45) and (46) into (70), (71) and (72) yield

$$\overline{\boldsymbol{X}}_{0s} - \delta\frac{R_1R_2}{R}\mathbb{S}_0\overline{\boldsymbol{X}}_{0s} - \delta\frac{R_1R_2\hat{R}}{4R^2}\ \mathbb{S}_0\bar{\boldsymbol{l}}_{0s} + \delta\frac{R_1R_2\hat{R}}{4R^2}\ \langle\mathbb{S}_1\widetilde{\boldsymbol{l}}_{1\tau}\rangle = 0 \tag{73}$$

$$\left(\bar{\boldsymbol{l}}_{0s} + \delta\hat{R}\mathbb{S}_0\overline{\boldsymbol{X}}_{0s} + \delta\frac{R_1R_2}{R}\ \mathbb{S}_0\bar{\boldsymbol{l}}_{0s} + \delta\hat{R}\langle\mathbb{S}_1\widetilde{\boldsymbol{X}}_{1\tau}\rangle + \delta\frac{R_1R_2}{R}\ \langle\mathbb{S}_1\widetilde{\boldsymbol{l}}_{1\tau}\rangle\right) \cdot \hat{\boldsymbol{n}}_0 = 0 \tag{74}$$

$$\left(\bar{\boldsymbol{l}}_{0s} + \delta\hat{R}\mathbb{S}_0\overline{\boldsymbol{X}}_{0s} + \delta\frac{R_1R_2}{R}\ \mathbb{S}_0\bar{\boldsymbol{l}}_{0s} + \delta\hat{R}\langle\mathbb{S}_1\widetilde{\boldsymbol{X}}_{1\tau}\rangle + \delta\frac{R_1R_2}{R}\ \langle\mathbb{S}_1\widetilde{\boldsymbol{l}}_{1\tau}\rangle\right) \cdot \boldsymbol{n}_0 = 0 \tag{75}$$

The equation (73) can be written as

$$\overline{\boldsymbol{X}}_{0s} = \delta\frac{R_1R_2}{R}\mathbb{S}_0\overline{\boldsymbol{X}}_{0s} + \delta\frac{R_1R_2\hat{R}}{4R^2}\ \mathbb{S}_0\bar{\boldsymbol{l}}_{0s} - \delta\frac{R_1R_2\hat{R}}{4R^2}\ \langle\mathbb{S}_1\widetilde{\boldsymbol{l}}_{1\tau}\rangle \tag{76}$$

The use of (65) and (76) into (74) and (75), keep at most linear in δ, yield

$$\left(\bar{\boldsymbol{l}}_{0s} + \delta\frac{R_1R_2}{R}\ \mathbb{S}_0\bar{\boldsymbol{l}}_{0s} + \delta\frac{R_1R_2}{R}\ \langle\mathbb{S}_1\widetilde{\boldsymbol{l}}_{1\tau}\rangle\right) \cdot \hat{\boldsymbol{n}}_0 = 0 \tag{77}$$

$$\left(\bar{\boldsymbol{l}}_{0s} + \delta\frac{R_1R_2}{R}\ \mathbb{S}_0\bar{\boldsymbol{l}}_{0s} + \delta\frac{R_1R_2}{R}\ \langle\mathbb{S}_1\widetilde{\boldsymbol{l}}_{1\tau}\rangle\right) \cdot \boldsymbol{n}_0 = 0 \tag{78}$$

The system of equations, (77) and (78) lead to

$$\bar{\boldsymbol{l}}_{0s} + \delta\frac{R_1R_2}{R}\ \mathbb{S}_0\bar{\boldsymbol{l}}_{0s} + \delta\frac{R_1R_2}{R}\ \langle\mathbb{S}_1\widetilde{\boldsymbol{l}}_{1\tau}\rangle = 0 \tag{79}$$

which can be written as

$$\bar{\boldsymbol{l}}_{0s} = -A^{-1}\delta\frac{R_1R_2}{R}\ \langle\mathbb{S}_1\widetilde{\boldsymbol{l}}_{1\tau}\rangle \tag{80}$$

where the matrix $A = I + \delta R_1R_2\mathbb{S}_0/R$ such that I is the identity matrix. Using simple algebraic transformation, the inverse matrix of A is given by

$$A^{-1}_{ik} = (1 - \delta\frac{R_1R_2}{Rl})\delta_{ik} - \delta\frac{R_1R_2}{Rl}\boldsymbol{n}_{0i}\boldsymbol{n}_{0k} \tag{81}$$

The use of (81) into (80), keep at most linear in δ, yield

$$\bar{\boldsymbol{l}}_{0s} = -\delta\frac{R_1R_2}{R}\langle\mathbb{S}_1\widetilde{\boldsymbol{l}}_{1\tau}\rangle \tag{82}$$

The substituting (82) into (76), keep at most linear in δ, yield

$$\overline{\boldsymbol{X}}_{0s} = -B^{-1}\,\delta\frac{R_1R_2\hat{R}}{4R^2}\ \langle\mathbb{S}_1\widetilde{\boldsymbol{l}}_{1\tau}\rangle \tag{83}$$

where the matrix $B = I - \delta R_1R_2\mathbb{S}_0/R$. The inverse matrix of B is given by

$$B^{-1}_{ik} = (1 + \delta\frac{R_1R_2}{Rl})\delta_{ik} + \delta\frac{R_1R_2}{Rl}\boldsymbol{n}_{0i}\boldsymbol{n}_{0k} \tag{84}$$

The use of (84) into (83), keep at most linear in δ, yield

$$\overline{\boldsymbol{X}}_{0s} = -\delta\frac{R_1R_2\hat{R}}{4R^2}\ \langle\mathbb{S}_1\widetilde{\boldsymbol{l}}_{1\tau}\rangle \tag{85}$$

Expressions (85) still contain unknown functions $\langle\mathbb{S}_1\widetilde{\boldsymbol{l}}_{1\tau}\rangle$, which can be determined from (60) and (61). Hence,

$$\overline{\boldsymbol{X}}_{0s} = -\delta\frac{\hat{R}\gamma^2}{l^2R_1R_2}\ \langle\widetilde{\boldsymbol{g}}^{\tau}\widetilde{\boldsymbol{g}}\rangle\,\boldsymbol{n}_0 \tag{86}$$

which represents the main result of this paper.

Equation (86) shows that the oscillating dumbbell moves with a constant velocity in the fixed direction $\boldsymbol{n}_0$. It is clear that $\overline{\boldsymbol{X}}_{0s}$ is inversely proportional to the length of the spring l and it does not depend on the spring stiffness k; and in the limit as length $l \longrightarrow \infty$, then $\overline{\boldsymbol{X}}_{0s} \longrightarrow 0$.

Integrating of (86) lead to the displacement vector

$$\overline{\boldsymbol{X}}_0 = -\delta\, U\, \hat{\boldsymbol{n}}_0 + \boldsymbol{C} \quad \text{or} \quad (\overline{\boldsymbol{X}}_0 - \boldsymbol{C})^2 = \delta^2 U^2 \tag{87}$$

where $U = \hat{R}\,\gamma^2\, \langle \widetilde{\boldsymbol{g}}^\tau \widetilde{\boldsymbol{g}} \rangle / l^2 R_1 R_2$ and $\boldsymbol{C}$ is a vectorial constant of integration. The equality (87) represents that $\overline{\boldsymbol{X}}_0$ changes along a circular path of radius $\delta\, U$.

It is interesting that the average velocity $\overline{\boldsymbol{X}}_{0s}$ can be arranged by appropriate choice of the induced acceleration, $\widetilde{\boldsymbol{g}}(\tau)$.

3.4. An Illustrative Example

Let us consider a particular example

$$\widetilde{\boldsymbol{g}}(\tau) = \begin{pmatrix} a \sin \tau \\ b \cos \tau \end{pmatrix} \tag{88}$$

where a and b are constants. The use of the integral (48) gives

$$\widetilde{\boldsymbol{g}}^\tau(\tau) = \begin{pmatrix} -a \cos \tau \\ b \sin \tau \end{pmatrix}, \tag{89}$$

and thus

$$\widetilde{\boldsymbol{g}}_i^\tau \widetilde{\boldsymbol{g}}_k = \begin{pmatrix} -a^2 \sin \tau \cos \tau & -ab \cos^2 \tau \\ ab \sin^2 \tau & b^2 \sin \tau \cos \tau \end{pmatrix}. \tag{90}$$

Calculating $\langle \widetilde{\boldsymbol{g}}_i^\tau \widetilde{\boldsymbol{g}}_k \rangle$ using the definition of average (45) yields

$$\langle \widetilde{\boldsymbol{g}}_i^\tau \widetilde{\boldsymbol{g}}_k \rangle = \frac{ab}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{91}$$

Substituting (91) into (86) gives

$$\overline{\boldsymbol{X}}_{0s} = -\delta \frac{\hat{R}\,\gamma^2\, a\, b}{2 R_1 R_2 l^2}\, \boldsymbol{n}_0 \tag{92}$$

Integrating of (92) leads to

$$\overline{\boldsymbol{X}}_0 = -\delta \lambda\, \hat{\boldsymbol{n}}_0 + \boldsymbol{C} \tag{93}$$

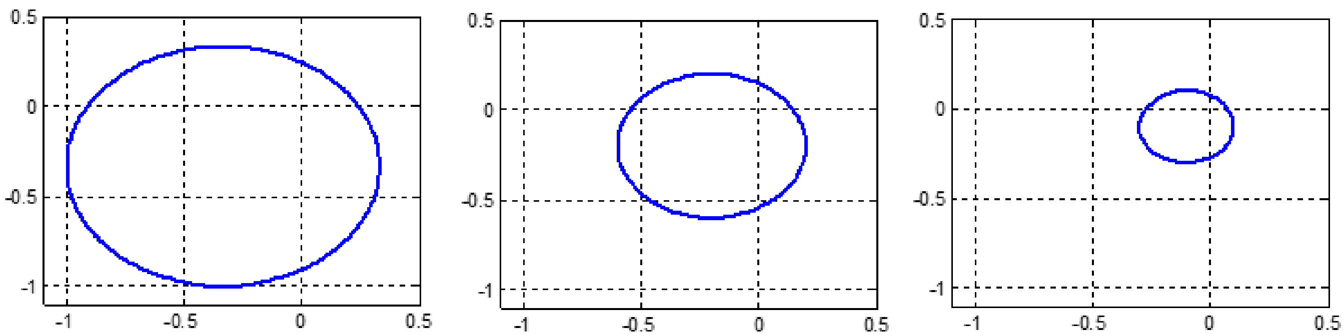
where $\lambda = U a\, b/2$.

In this paper, we consider motions (40) with large ω where all functions and its derivatives belongs to class $O(1)$. Hence,

$$\boldsymbol{C} = (a - \delta P, b - \delta P), \quad \text{where} \quad p = \hat{R}\gamma^2 ab/2R_1 R_2 l^2. \tag{94}$$

The use of (58) and (59) lead to

$$\hat{\boldsymbol{n}}_0 = (-\sin \phi, \cos \phi) \quad \text{where} \quad \phi = \varphi_0. \tag{95}$$



(a) $\omega = 3$ *(b)* $\omega = 5$ *(c)* $\omega = 10$

Figure 3. *Trajectory of the oscillating dumbbell for different* ω.

The slow time scale $s = t/\omega$ implies that in order to obtain dimensionless trajectory of the oscillating dumbbell we have to multiply $\overline{\boldsymbol{X}}_0$ by $1/\omega$.

The trajectory of oscillating dumbbell for different frequencies is plotted in Figure 3. It is clearly shown that the circular path of the oscillating dumbbell is decreased as the frequency increased which a small oscillation leads to a large motion of the surrounding fluid. This result agrees with the classical studies of the oscillatory motion of particle suspensions at low Reynolds numbers [12, 13].

4. Conclusion

In this paper, we investigated analytically the dynamics of two micro-spheres, that are elastically connected by a spring, one of them is magnetized and driven by an external oscillator field. We constructed an asymptotic procedure with the dimensionless inverse frequency $\epsilon = 1/\omega$ and derived the average velocity of the system using the two-timing method. Our choice of slow time $s = \epsilon t$ and fast time $\tau = t/\epsilon$

lead to a result that agrees with the experimental studies of an oscillating sphere in a viscous fluid [7, 12]. It shows the strength and analytical simplicity to describe the oscillatory motion of the dumbbell at low Reynolds number. It is worth noting that the average velocities of the oscillating dumbbell is given in the most general form which could be applicable to study a full three-dimensional problem.

Conflicts of Interest

The authors declare no conflicts of interest.

References

[1] Alexander, G. P., and Yeomans, J. M. (2008). Dumb-bell swimmers. *Europhysics Letters*, 83(3), 34006.

[2] Al-Hatmi, M. M., and Purnama, A. (2021). On the motion of two micro-spheres in a Stokes flow driven by an external oscillator field. *Journal of Mathematics and Mathematical Sciences*, 2021: 9211272.

[3] Amaratunga, M., Rabenjafimanantsoa, H. A., and Time, R. W. (2021). Influence of low-frequency oscillatory motion on particle settling in Newtonian and shear-thinning non-Newtonian fluids. *Journal of Petroleum Science and Engineering*, 196: 107786.

[4] Belovs, M. and Cebers, A. (2009). Ferromagnetic microswimmer. *Physical Review E*, 79(5): 051503.

[5] Bogoliubov, N. N. and Mitropolskii, Y. A. (1961). Asymptotic Methods in the Theory of Nonlinear Oscillations, 10. *CRC Press*.

[6] Box, F., Han, E., Tipton, C.R., and Mullin, T. (2017). On the motion of linked spheres in a Stokes flow. *Experiments in Fluids*, 58: 1-10.

[7] Box, F., Singh, K., and Mullin, T. (2018). The interaction between rotationally oscillating spheres and solid boundaries in a Stokes flow. *Journal of Fluid Mechanics*, 849: 834-859.

[8] Castilla, R. (2022). Dynamics of a microsphere inside a spherical cavity with Newtonian fluid subjected to periodic contractions. *Physics of Fluids*, 34: 071901.

[9] Dogangil, G., Ergeneman, O., Abbott, J. J., Pan, S., Hall, H., Muntwyler, S., and Nelson, B. J. (2008). Toward targeted retinal drug delivery with wireless magnetic microrobots. *In 2008 IEEE/RSJ International Conference on Intelligent Robots and Systems*, 1921-1926.

[10] Fusco, S., Chatzipirpiridis, G., Sivaraman, K. M., Ergeneman, O., Nelson, B. J., and Pan, S. (2013). Chitosan electrodeposition for microrobotic drug delivery. *Advanced Healthcare Materials*, 2(7): 1037-1044.

[11] Gilbert, A. D., Ogrin, F. Y., Petrov, P. G., and Wimlove, C. P. (2010). Theory of ferromagnetic microswimmers. *Quarterly Journal of Mechanics and Applied Mathematics*, 64(3): 239-263.

[12] Grosjean, G., Hubert, M., Lagubeau, G., and Vandewalle, N. (2016). Realization of the Najafi-Golestanian microswimmer. *Physical Review E*, 94(2): 021101.

[13] Happel, J. and Brenner, H. (2012). Low Reynolds Number Hydrodynamics with Special Applications to Particulate Media, 1. *Springer Science & Business Media*.

[14] Ijspeert, A. J. (2014). Biorobotics: Using robots to emulate and investigate agile locomotion. *Science*, 346(6206): 196-203.

[15] Landau, L., and Lifshitz, E. (1987). Course of Theoretical Physics. *Fluid Mechanics*, 6. Elsevier.

[16] Lyubimova, T., Lyubimov, D., and Shardin, M. (2011). The interaction of rigid cylinders in a low Reynolds number pulsational flow. *Microgravity Science and Technology*, 23(3): 305-309.

[17] Mathieu, J. B., Beaudoin, G., and Martel, S. (2006). Method of propulsion of a ferromagnetic core in the cardiovascular system through magnetic gradients generated by an MRI system. *IEEE Transactions on Biomedical Engineering*, 53(2): 292-299.

[18] Mullin, T., Li, Y., Del Pino, C., and Ashmore, J. (2005). An experimental study of fixed points and chaos in the motion of spheres in a Stokes flow. *Journal of Applied Mathematics*, 70(5): 666-676.

[19] Nayfeh, A. (1973). Perturbation Methods. *John Wiley and Sons*, New York.

[20] Nelson, B. J., Kaliakatsos, I. K., and Abbott, J. J. (2010). Microrobots for minimally invasive medicine. *Annual Review of Biomedical Engineering*, 12: 55-85.

[21] Norton, J. (1985). What was Einstein's principle of equivalence?. *Studies in History and Philosophy of Science Part A*, 16(3): 203-246.

[22] Ogrin, F. Y., Petrov, P. G., and Winlove, C. P. (2008). Ferromagnetic microswimmers. *Physical Review Letters*, 100(21): 218102.

[23] Romanczuk, P., Bär, M., Ebeling, W., Lindner, B., and Schimansky-Geier, L. (2012). Active Brownian Particles. *The European Physical Journal Special Topics*, 202(1): 1-162.

[24] Shapere, A., and Wilczek, F. (1989). Efficiencies of self-propulsion at low Reynolds number. *Journal of Fluid Mechanics*, 198:587-599.

[25] Taylor, G. I. (1951). Analysis of the swimming of microscopic organisms. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, 209(1099): 447-461.

[26] Verhulst, F. (2007). Singular perturbation methods for slowfast dynamics. *Journal of Nonlinear Dynamics*, 50(4):747-753.

[27] Wilson, H. J. (2005). An analytic form for the pair distribution function and rheology of a dilute suspension of rough spheres in plane strain flow. *Journal of Fluid Mechanics*, 534: 97-114.