
Numerical Solution for One Dimensional Linear Types of Parabolic Partial Differential Equation and Application to Heat Equation

Kedir Aliyi Koroche

Department of Mathematics, College of Natural Sciences, Ambo University, Ambo, Ethiopia

Email address:

kediraliyi39@gmail.com

To cite this article:

Kedir Aliyi Koroche. Numerical Solution for One Dimensional Linear Types of Parabolic Partial Differential Equation and Application to Heat Equation. *Mathematics and Computer Science*. Vol. 5, No. 4, 2020, pp. 76-85. doi: 10.11648/j.mcs.20200504.12

Received: March 10, 2020; **Accepted:** April 7, 2020; **Published:** October 12, 2020

Abstract: In this paper, present solution of one-dimensional linear parabolic differential equation by using Forward difference, backward difference, and Crank Nicholson method. First, the solution domain is discretized using the uniform mesh for step length and time step. Then applying the proposed method, we discretize the linear parabolic equation at each grid point and then rearranging the obtained discretization scheme we obtain the system of equation generated with tri-diagonal coefficient matrix. Now applying inverse matrixes method and writing MATLAB code for this inverse matrixes method we obtain the solution of one-dimensional linear parabolic differential equation. The stability of each scheme analyses by using Von-Neumann stability analysis technique. To validate the applicability of the proposed method, two model example are considered and solved for different values of mesh sizes in both directions. The convergence has been shown in the sense of maximum absolute error (E^{∞}) and Root mean error (E^2). Also, condition number ($K(A)$) and Order of convergence are calculated. The stability of this Three class of numerical method is also guaranteed and also, the comparability of the stability of these three methods is presented by using the graphical and tabular form. The proposed method is validated via the same numerical test example. The present method approximate exact solution very well.

Keywords: Linear Parabolic Equation, Implicit Crank Nicholson Method, Root Mean Square Error, Condition Number, Order of Convergence

1. Introduction

Numerical analysis is a subject that involves computational methods for studying and solving mathematical problems. It is a branch of mathematics and computer the science that creates, analyzes, and implements algorithms for solving mathematical problems numerically [2, 13]. Also, it's widely used by scientists and engineers to solve some problems. Such problems are formulated in terms of an algebraic equation, transcendental equations, ordinary differential equations and partial differential equations [7, 16]. Numerical analysis is also concerned with the theoretical foundation of numerical algorithms for the solution of problems arising in scientific applications [7].

Partial Differential Equations (PDEs) are mathematical equations that are significant in modeling physical phenomena that occur in nature. Applications of PDEs can

be found in physics, engineering, mathematics, and finance. Examples include modeling mechanical vibration, heat, sound vibration, elasticity, and fluid dynamics. Although PDEs have a wide range of applications to real-world problems in science and engineering, the majority of PDEs do not have analytical solutions. It is, therefore, important to be able to obtain an accurate solution numerically. Many computational methods have been developed and implemented to successfully approximate solutions for mathematical modeling in the application of PDEs. In order to make use of mathematical models, it is necessary to have solutions to the model equations. Generally, this requires numerical methods because of the complexity and number of equations [17, 20]. The scientists in the field of computational mathematics are trying to develop more accurate numerical methods by using computers for further application. Same of this

method are forward difference method, backward difference method crank Nicholson method. Due to the wide range of the application of the one dimensional linear parabolic equation, several numerical methods have been developed. Even though many numerical methods were applied to solve these types of equations. Accordingly, more efficient and simpler numerical methods are required to solve linear parabolic equation. Most of the researchers have studied the numerical solutions of 1D heat equation. In [1] presented Numerical solutions of a one-dimensional linear parabolic Equation together with initial condition and Dirichlet boundary conditions. In [4] presented the Numerical Simulation of one-dimensional Heat Equation by using B-Spline Finite Element Method. In [8] they used Chebyshev Wavelets Method for obtaining a numerical solution of the One-Dimensional Heat Equation. In [3] presented the numerical solution of 1D heat with Neumann and Dirichlet boundary conditions. [10] also developed an explicit method for solving inhomogeneous heat equation in free space, following the time evolution of the solution in the Fourier domain. In [9] solved the 1D heat equation by using double interpolation. They used finite difference method for the double interpolation method to solve the 1D heat equation. In [18] solved parabolic partial differential equations using radial basis functions and Application to the heat equation. They used the Gaussian radial basis functions for obtaining the solution of the heat equation. Even though the method is capable of approximating the heat equation, they failed to produce the solution for the relatively small value of shape parameters. Since Gaussian radial basis functions are low accuracy than both Multiquadric Radial Basis Function (MQ-RBF) and Thin Plate Spline Radial Basis Function (TPS-RBF) in an approximation of a function by interpolation process [6].

However, still, the accuracy and stability of the method need attention because of the treatment of the method used to solve the linear type of PDE equation is not trivial distribution. Even though the accuracy and stability of the aforementioned methods need attention, they require large memory and long computational time. So the treatments this method presents severe difficulties that have to be addressed to ensure the accuracy and stability of the solution. To this end, the aim of this paper is to develop the accurate and stable three methods forward difference, Backward difference and Crank Nicholson numerical method that is capable of producing a solution of linear type PDEs equation and approximate the exact solution. The convergence has been shown in the sense of maximum absolute error (L^∞) and Root mean error (L^2). and so that the local behavior of the solution is captured exactly. As well as condition number (K(A)) and Order of convergence are calculated for new

numerical method The stability of those three present methods are also investigated.

2. Description of the Method

Consider the following linear Parabolic type of PDEs equation:

$$\left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\right)(x,t) - \epsilon \frac{\partial^2 u}{\partial x^2}(x,t) = H(x,t), (x,t) \in (0,1) \times (0,T] \quad (1)$$

subject to initial and boundary condition respectively are::

$$u(x,0) = f_1(x), a \leq x \leq b \quad (2)$$

$$u(a,t) = g_1(t), u(b,t) = g_2(t), 0 \leq t \leq T \quad (3)$$

where $\epsilon > 0$ is called the diffusion coefficient, representing the thermal diffuse of the material making up the rod and $f_1(x)$, $g_1(t)$ and $g_2(t)$ are sufficiently smooth function for $(x,t) \in (0,1) \times (0,T]$ The computational domain $(0,1) \times (0,T]$ is partitioned as:

$$0 \leq x_0 < x_1 < x_2 < \dots < x_j < \dots < x_M = 1 \quad (4)$$

where step length is $h = x_{j+1} - x_j, j=1,2,\dots,M$ and M maximum number of grid point in $[0,1]$ and,

$$0 \leq t_0 < t_1 < t_2 < \dots < t_n < \dots < t_N = T \quad (5)$$

where time step $\Delta t = t_{n+1} - t_n, n=1,2,\dots,N$ and N maximum number of grid point in $[0,T]$. Recalling that the one-dimensional linear types of the parabolic equation given in Eq (1), our aim is to approximate the partial derivative of $u(x,t)$ into functional value at each grid point.

2.1. Forward Finite Difference Formula

Recall that one-dimensional parabolic equation in eq (1) and discretization of them is:

$$\frac{\partial u}{\partial x} = \frac{u_{j+1,n} - u_{j-1,n}}{2h}, \quad (6)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{j+1,n} - 2u_{j,n} + u_{j-1,n}}{h^2}, \quad (7)$$

$$\frac{\partial u}{\partial t} = \frac{u_{j,n+1} - u_{j,n}}{\Delta t} \quad (8)$$

Now substituting eqs (6)-(8) into eq. (1) at point (x_j, t_n) we obtain:

$$\left(\frac{u_{j,n+1} - u_{j,n}}{\Delta t} + \frac{u_{j+1,n} - u_{j-1,n}}{2h}\right)(x_j, t_n) - \epsilon \frac{u_{j+1,n} - 2u_{j,n} + u_{j-1,n}}{h^2}(x_j, t_n) = H(x_j, t_n) \quad (9)$$

where $j = 1, 2, 3, \dots, M$ and $n = 1, 2, 3, \dots, N$. Note that initial and boundary conditions give known quantities u_{j0} for $j = 1, 2, 3, \dots, M$ and u_{0n} and u_{Mn} for $n = 0, 1, 2, \dots, N$ which correspond to the bottom and sides of the rectangular domain, Now rearranging Eq. (9) and introducing the vector $U_n = [u_{2n}, u_{3n}, \dots, u_{M-1n}]$, we obtain:

$$u_{j,n+1} = \gamma u_{j+1,n} + (1 - 2\alpha)u_{j,n} + \beta u_{j-1,n} + \Delta t H(x_j, t_n) \quad (10)$$

where $\alpha = \frac{\epsilon \Delta t}{h^2}$, $\beta = \alpha + \frac{\Delta t}{2h}$ and $\gamma = \alpha - \frac{\Delta t}{2h}$. From eq. (10) we obtain matrix form of the system of the equation:

$$U_{n+1} = AU_n + b + b_1 \quad (11)$$

where

$$A = \begin{pmatrix} 1-2\alpha & \gamma & 0 & \dots & 0 \\ \beta & 1-2\alpha & \gamma & \dots & 0 \\ 0 & \beta & 1-2\alpha & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \beta & 1-2\alpha \end{pmatrix}, \quad b = [\gamma u_{0n}, 0, 0, \dots, \gamma u_{Mn}]^T$$

$$\left(\frac{u_{j,n} - u_{j,n-1}}{\Delta t} + \frac{u_{j+1,n} - u_{j-1,n}}{2h} \right) (x_j, t_n) - \epsilon \frac{u_{j+1,n} - 2u_{j,n} + u_{j-1,n}}{h^2} (x_j, t_n) = H(x_j, t_n) \quad (13)$$

Now rearranging (13) and introducing vectors $U_n = [u_{2n}, u_{3n}, \dots, u_{M-1n}]$, we obtain:

$$(1 + 2\alpha)u_{j,n} - \gamma u_{j+1,n} - \beta u_{j-1,n} - u_{j,n-1} = \Delta t H(x_j, t_n) \quad (14)$$

From eq. (14) we obtain matrix form of the system of the equation:

$$BU_n = b + b_1 \quad (15)$$

where

$$B = \begin{pmatrix} 1+2\alpha & -\gamma & 0 & \dots & 0 \\ -\beta & 1+2\alpha & -\gamma & \dots & 0 \\ 0 & -\beta & 1+2\alpha & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & -\beta & 1+2\alpha \end{pmatrix}, \quad b = [\gamma u_{0n}, 0, 0, \dots, \gamma u_{Mn}]^T \quad \text{and} \quad b_1 = \Delta t [H(x_1, t_1), H(x_2, t_2), \dots, H(x_M, t_N)]^T$$

$\alpha = \frac{\epsilon \Delta t}{h^2}$, $\beta = -\left(\alpha + \frac{\Delta t}{2h}\right)$ and $\gamma = -\left(\alpha - \frac{\Delta t}{2h}\right)$. Now writing the MATLAB code for eq. (15) we find the solution.

2.3. By Crank-nicholson Method

To find an accurate solution, we also use the crank Nicholson method. Let us consider the discretization of given linear parabolic partial differential equation by using the crank Nicholson method:

$$\frac{\partial u}{\partial x} = \frac{u_{j+1,n} - u_{j-1,n}}{4h} + \frac{u_{j+1,n-1} - u_{j-1,n-1}}{4h} \quad (16)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{j+1,n} - 2u_{j,n} + u_{j-1,n}}{2h^2} + \frac{u_{j+1,n-1} - 2u_{j,n-1} + u_{j-1,n-1}}{2h^2} \quad (17)$$

$$\frac{\partial u}{\partial t} = \frac{u_{j,n} - u_{j,n-1}}{\Delta t} \quad (18)$$

and $b = \Delta t [H(x_1, t_1), H(x_2, t_2), \dots, H(x_M, t_N)]^T$. Now writing the MATLAB code for eq. (11) we find the solution.

2.2. By Backward Difference Method

As an alternative, the finite difference approach can be redone with better error magnification properties by using an implicit method. As before, we replace u_x and u_{xx} in eq (1) by using a centered-difference formula, but we use the backward-difference formula for u_t ,

$$\frac{\partial u}{\partial t} = \frac{u_{j,n} - u_{j,n-1}}{\Delta t} \quad (12)$$

Thus substituting eqs (6), (7) and (12) into eq. (1) at the point (x_j, t_n) , we obtain:

Now substituting eqs (16)-(18) into eq (1) at the point (x_j, t_n) we obtain:

$$\begin{aligned} & \left(\frac{u_{j,n} - u_{j,n-1}}{\Delta t} + \frac{u_{j+1,n} - u_{j-1,n}}{4h} + \frac{u_{j+1,n-1} - u_{j-1,n-1}}{4h} \right) (x_j, t_n) - \epsilon \left(\frac{u_{j+1,n} - 2u_{j,n} + u_{j-1,n}}{2h^2} \right) (x_j, t_n) \\ & - \epsilon \left(\frac{u_{j+1,n-1} - 2u_{j,n-1} + u_{j-1,n-1}}{2h^2} \right) (x_j, t_n) = H(x_j, t_n) \end{aligned} \quad (19)$$

Now multiplying both side of eq (19) by Δt and rearranging it we obtain:

$$\begin{aligned} (1+2\alpha)u_{j,n} - (\alpha+\beta)u_{j-1,n} + (\beta-\alpha)u_{j+1,n} &= (\alpha+\beta)u_{j-1,n-1} + (2-2\alpha)u_{j,n-1} \\ &+ (\alpha-\beta)u_{j+1,n-1} + \Delta t H(x_j, t_n) \end{aligned} \quad (20)$$

where $\alpha = \frac{\epsilon \Delta t}{h^2}$, $\beta = \frac{\Delta t}{2h}$. Now by introducing the vector

$U_n = [u_{2n}, u_{3n}, \dots, u_{M-1n}]$, in eq (20), we obtain the matrix form of the system of equation is:

$$Au_n = Bu_{n-1} + b + C_{-1} + b_1 \quad (21)$$

where

$$A = \begin{pmatrix} 2+2\alpha & (\beta-\alpha) & 0 & \dots & 0 \\ -(\beta+\alpha) & 2+2\alpha & (\beta-\alpha) & \dots & 0 \\ 0 & -(\beta+\alpha) & 2+2\alpha & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & -(\beta+\alpha) & 2+2\alpha \end{pmatrix},$$

$$B = \begin{pmatrix} 2-2\alpha & -(\beta-\alpha) & 0 & \dots & 0 \\ (\beta+\alpha) & 2-2\alpha & -(\beta-\alpha) & \dots & 0 \\ 0 & \beta+\alpha & 2-2\alpha & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \beta+\alpha & 2-2\alpha \end{pmatrix}$$

$$b = [-(\beta+\alpha)u_{0,n}, 0, 0, \dots, -(\beta-\alpha)u_{M,n}]^t$$

$$b_1 = \Delta t [H(x_1, t_1), H(x_2, t_2), \dots, H(x_M, t_N)]^t$$

$$C_{-1} = [(\beta+\alpha)u_{0,n-1}, 0, 0, \dots, -(\beta+\alpha)u_{M,n-1}]^t$$

Then write the MATLAB code for all scheme and we finding the solution of the given linear types of the parabolic partial differential equation.

3. Stability and Convergent Analysis

The Von-Neumann stability analysis technique is applied to investigate the stability of the proposed method. such an approach has been used by many researchers like [11, 12, 15, 19]. Now assume that the solution of the given problem at the point (x_j, t_n) is

$$u_{j,n} = \lambda^n e^{ij\theta} \quad (22)$$

where $i = \sqrt{-1}$, $\theta = \frac{j\pi}{M}$ a real number and λ is a complex number. For analysis of the stability of those numerical

methods that we proposed above, we substitute eq. (22) into eqs (10), (14), and, (20). From eqs. (22) and (10) we have:

$$\lambda^{n+1} e^{ij\theta} = (1-2\alpha)\lambda^n e^{ij\theta} + \beta\lambda^n e^{i(j-1)\theta} + \gamma\lambda^n e^{i(j+1)\theta} \quad (23)$$

Divided eq (23) by $\lambda^n e^{ij\theta}$ we obtain:

$$\begin{aligned} \lambda &= (1-2\alpha) + \beta e^{i\theta} + \gamma e^{-i\theta} = \\ &= 1-2\alpha + \beta(\cos(\theta) + i\sin(\theta)) + \gamma(\cos(\theta) - i\sin(\theta)) \\ &= 1-2\alpha + (\beta+\gamma)\cos(\theta) + i(\beta-\gamma)\sin(\theta) \end{aligned} \quad (24)$$

Thus from this equation, we get the criteria of stability of the forward difference method. Again substituting eq. (22) into eq. (14) the stability analysis for back ward difference method is:

$$\lambda^{n-1} e^{ij\theta} = (1+2\alpha)\lambda^n e^{ij\theta} - \gamma\lambda^n e^{i(j+1)\theta} - \beta\lambda^n e^{i(j-1)\theta} \quad (25)$$

Divided eq (25) by $\lambda^n e^{ij\theta}$ we obtain:

$$\begin{aligned} \lambda^{-1} &= (1+2\alpha) - \gamma e^{i\theta} - \beta e^{-i\theta} \\ &= (1+2\alpha) - \gamma(\cos(\theta) + i\sin(\theta)) - \beta(\cos(\theta) - i\sin(\theta)) \\ &= (1+2\alpha) - (\beta+\gamma)\cos(\theta) - i(\gamma-\beta)\sin(\theta) \end{aligned}$$

This implies that:

$$\begin{aligned} \lambda &= \frac{1}{(1+2\alpha) - (\beta+\gamma)\cos(\theta) - i(\gamma-\beta)\sin(\theta)} \\ \lambda &= \frac{X}{X^2 + Y^2} - \frac{Y}{X^2 + Y^2} \end{aligned} \quad (26)$$

where $X = (1+2\alpha) - (\beta+\gamma)\cos(\theta)$ and $Y = i(\gamma-\beta)\sin(\theta)$. Thus from eq. (26) we obtain the criteria that the Backward difference method is stable. We also analyze the stability of our third method which is the crank Nicholson method. Substitute eq (22) into Eq. (20), we obtain:

$$(2+2\alpha)\lambda^n e^{ij\theta} - (\alpha+\beta)\lambda^n e^{i(j-1)\theta} + (\beta-\alpha)\lambda^n e^{i(j+1)\theta} =$$

$$(\alpha + \beta)\lambda^{n-1}e^{i(j-1)\theta} + (2-2\alpha)\lambda^{n-1}e^{ij\theta} - (\alpha - \beta)\lambda^{n-1}e^{i(j+1)\theta} \tag{27}$$

Divided both side of eq. (27) by $\lambda^n e^{ij\theta}$ we obtain:

$$(2+2\alpha) - (\alpha + \beta)e^{-i\theta} + (\beta - \alpha)e^{i\theta} = (\alpha + \beta)\lambda^{-1}e^{-i\theta} + (2-2\alpha)\lambda^{-1} - (\alpha - \beta)e^{i\theta}$$

$$(2+2\alpha) - (\alpha + \beta)(\cos(\theta) - i\sin(\theta)) + (-\alpha + \beta)(\cos(\theta) + i\sin(\theta)) =$$

$$\lambda^{-1}(\alpha + \beta)(\cos(\theta) - i\sin(\theta)) + (2-2\alpha) - (\alpha - \beta)(\cos(\theta) + i\sin(\theta))$$

$$\lambda(4\alpha - 2(\alpha \cos(\theta) + i\beta \sin(\theta))) = 2(\alpha \cos(\theta) - i\beta \sin(\theta))$$

$$\lambda = \frac{2(\alpha \cos(\theta) - i\beta \sin(\theta))}{(4\alpha - 2(\alpha \cos(\theta) + i\beta \sin(\theta)))}$$

$$\lambda = \frac{X(4\alpha - X)}{Y^2 + Z^2} \tag{28}$$

where $X = 2(\alpha \cos(\theta) - i\beta \sin(\theta))$, $Y = 4\alpha - 2\alpha \cos(\theta)$ and $Z = -i2\beta \sin(\theta)$. Therefore from Eqs (24) (26) and (28), we obtain the required eigenvalues. The maximum eigenvalue is less than one (i.e. $|\lambda| < 1$). Therefore the obtained system of the equation is stable.

such that λ of the system matrix say matrix 'A' satisfy $Re al(\lambda) \leq 0$.
proof: Assuming that the system matrix is a diagonal matrix. Let 'P' be invertible matrix. Then, $A = p\lambda p^{-1}$ where λ are eigenvalues of matrix A and

Theorem 1:-The obtained system of the equation is stable

$$\lambda = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \lambda_{N-1} \end{pmatrix}$$

for all $n=1,2,3,\dots,N-1$. Then we have:

$$e^{At} = \sum_{n=1}^{\infty} \frac{1}{n!} A^n t^n = \sum_{n=1}^{\infty} \frac{1}{n!} (p\lambda p^{-1})^n t^n = \sum_{n=1}^{\infty} \frac{1}{n!} p\lambda^n p^{-1} t^n = p \sum_{n=1}^{\infty} \frac{1}{n!} (\lambda^n t^n) p^{-1} = p e^{\lambda t} p^{-1} = p \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \lambda_{N-1} \end{pmatrix} p^{-1} \tag{29}$$

theorem Hence $e^{\lambda t} \rightarrow 0$ if and only if the real part of the eigenvalue of "A" is less than zero ($Re al(\lambda) \leq 0$). This follows that $|e^{\lambda t}| \rightarrow 0$ if and only if $Re al(\lambda) \rightarrow 0$.

Therefore the obtained system of the equation is stable

$$E^2 = \sqrt{\frac{1}{N} \sum_{k=1}^N |U(x_i, t_j) - u(x_i, t_j)|^2}, i = 1(1)N, \tag{30}$$

$$E^\infty = \max_{1 \leq i \leq N} |U(x_i, t_M) - u(x_i, t_M)|, \tag{31}$$

4. Criteria for Investigating the Accuracy of the Method

In this section, we investigate the accuracy of the present method. To show the accuracy of the present method, the Root Mean Square (RMS) error (E^2), maximum absolute error (E^∞) are used to measure the accuracy of the method. The RMS error and maximum absolute error are calculated as follows (Tatari M., 2010).

Here, $U(x_i, t_M)$ and $u(x_i, t_M)$ are the exact and approximation solutions of Eqs. (1), (2), and (3), respectively. The condition number $K(A)$ is obtained by using the formula

$$k(A) = |A|_2 |A^{-1}|_2 \tag{32}$$

We also report the corresponding order of convergence. The order of convergence is calculated by:

$$Ord = \log_2 \left(\frac{E^\infty}{E^2} \right) \tag{33}$$

Example 2. Consider the parabolic equation considered by (Hikmet, 2008):

$$u_t(x,t) = \alpha u_{xx}(x,t), (x,t) \in (0,1) \times (0,1]$$

with initial condition and boundary conditions

$$u(x_i, 0) = \cos\left(\frac{\pi}{x} x_i\right), 0 \leq x_i \leq 1$$

$$u(0,t) = e^{-\frac{\pi^2}{4}t}, u(1,t) = \frac{2}{\pi} e^{-\frac{\pi^2}{4}t}, 0 \leq t \leq 1,$$

5. Numerical Experiments

In order to test the validity of the proposed method, we have considered the following model problem.

Example1: Consider the classical heat equation considered by (Tatari M., 2010) given by

$$u_t(x,t) = \epsilon u_{xx}(x,t), (x,t) \in (0,1) \times (0,T]$$

with initial condition and boundary conditions

$$u(x_i, 0) = \sin(\pi x_i), 0 \leq x_i \leq 1$$

$$u(0,t) = u(1,t) = 0, 0 \leq t \leq T,$$

The unique exact solution of the above IBVP one-dimensional heat equation is given by:

$$U(x_i) = \sin(\pi x_i) e^{(-\pi^2 t)},$$

The unique exact solution of the above IBVP one-dimensional heat equation is given by: $U(x_i) = \cos\left(\frac{\pi}{x} x_i\right) e^{-\frac{\pi^2}{4}t}$,

The numerical results are presented in tables in terms of E^2 , E^∞ and $K(A)$ as the means for measuring the accuracy of the present method.

Table 1. Maximum Absolute error E^∞ , Root mean square error E^2 , condition number ($K(A)$) and rate of convergence, Or dN of the solution for Example 1 with a uniform. mesh by using three present methods $\Delta t = 0.01$.

By: Present Method for any Shape parameter c,				
M ↓	E^∞ ↓	E^2 ↓	$\kappa(A)s$ ↓	Ord ↓
Foreword difference method				
10	3: 8324E-04	1.2119 E-4	1.7994E+03	1.6610
Backward difference method				
10	7.1981 E-06	7.1980E-07	4.4650	3.3219
20	5.1696E-06	5.1695E-07	15.3861	3.3219
40	2.78166E-06	2.7816E-07	59.0740	3.3219
80	1.4154E-06	1.41546E-07	233.8265	3.3219
Crank Nicholson method				
10	1.80723 E-05	1.8072 E-06	2.8134	3.3219
20	1.04313E-05	1.0431E-06	8.5306	3.32198
40	5.39091E-06	5: 39093E-07	31.4019	3.32198
80	2.71745E-06	2.7174E-07	122.8872	3.32197
Tatari, M. and Dehghan, M., 2010				

Shape parameter c ↓	E^2 ↓	E^∞ ↓	$\kappa(A)s$ ↓
0.5	9.0651E+39	7.0000E+73	1.9676E+19
1	3.3136E+17	2.0000E+29	2.0434E+17
2	5.0227E+1	5.0001E+0	9.3751E+17
3	1.0196E-01	7.1266E-03	6.6616E+17
4	1.0384E-01	7.4346E-03	4.0811E+17
5	1.0400E-03	7.6029E-03	1.1529E+16
6	1.0150E-03	7.5571E-03	1.4832E+16
7	9.5715E-03	7.3241E-03	3.0267E+15

Table 2. Maximum Absolute error E^∞ , Root mean square error E^2 , condition number ($K(A)$) and rate of convergence, Ord of the solution for Example 2 with a uniform. mesh by using three present methods $\Delta t = 0.025$.

By: Present Method				
M ↓	E^∞ ↓	E^2 ↓	$\kappa(A)s$ ↓	Ord ↓
Backward difference method				
11	1.4491 E-06	2.2631 E-07	8.4903	2.6788
21	3.1474 E-06	4: 9154 E-07	35.3020	2.6788
41	3.9270 E-06	6.1329 E-07	132.8058	2.6788
61	4.1981 E-06	6.5563 E-07	293.2158	2.6788
Crank Nicholson method				
11	1.7561 E-06	2.7426 E-07	5.1445	2.6788
21	3.4602E-06	5.4039E-7	19.9905	2.6788
41	4.2425E-06	6.6257E-07	73.9798	2.6788

By: Present Method				
M ↓	E^∞ ↓	E^2 ↓	$\kappa(A)s$ ↓	Ord ↓
61	4.5146E-06	7.0507E-07	162.8013	2.6788
Hikmet C et. at, (2008)				
21			E^∞ ↓	
41			6.1575E-03	
61			1.751E-03	
			9.5842E-04	

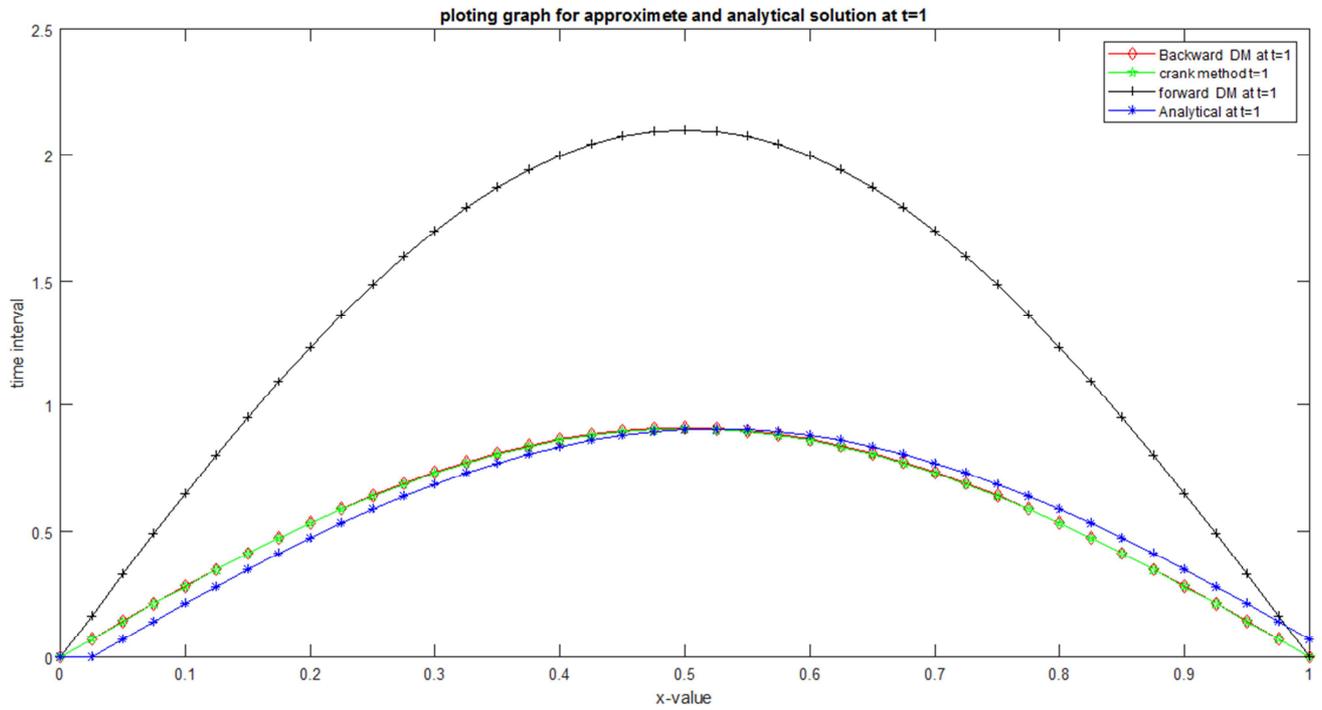


Figure 1. Solution profile and graph of the exact solution for Example 1 with $M = 40$ and $\Delta t = 0.01$.

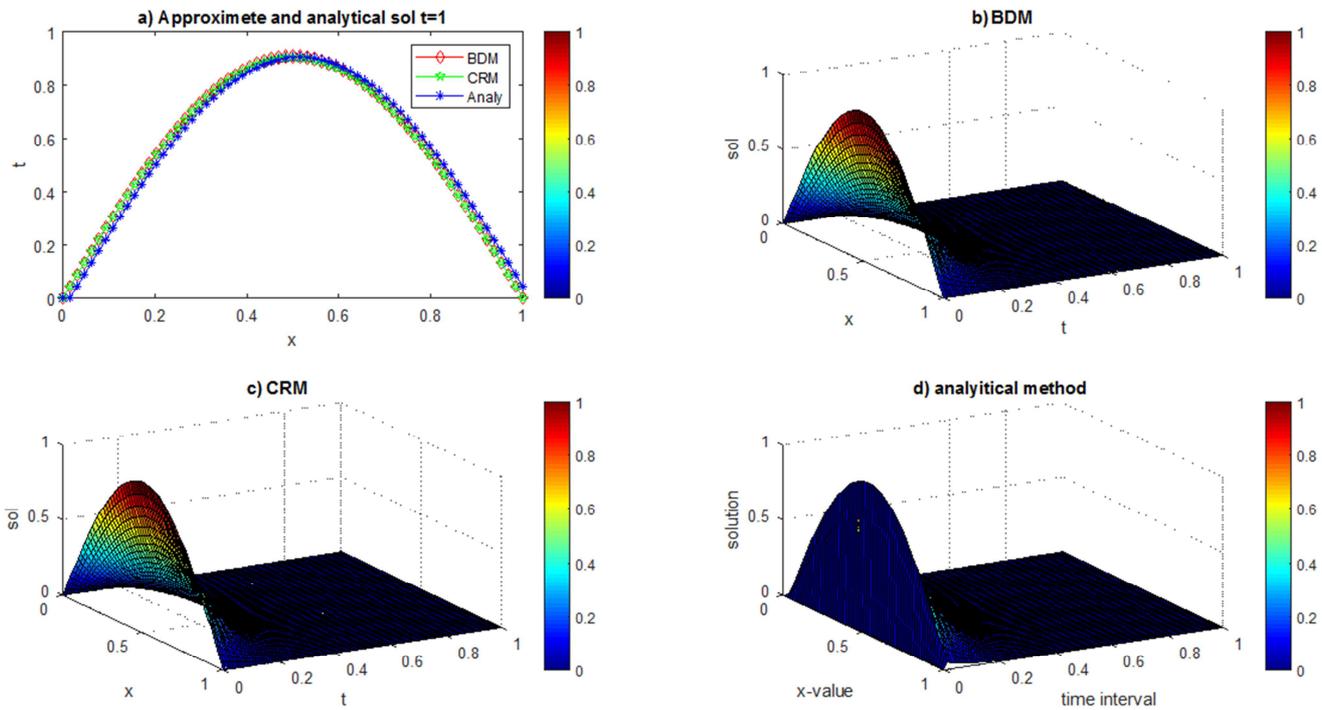


Figure 2. Surface graphs and Solution profile for the numerical solution of Example 1 with $M = 64$ and $\Delta t = 0.01$.

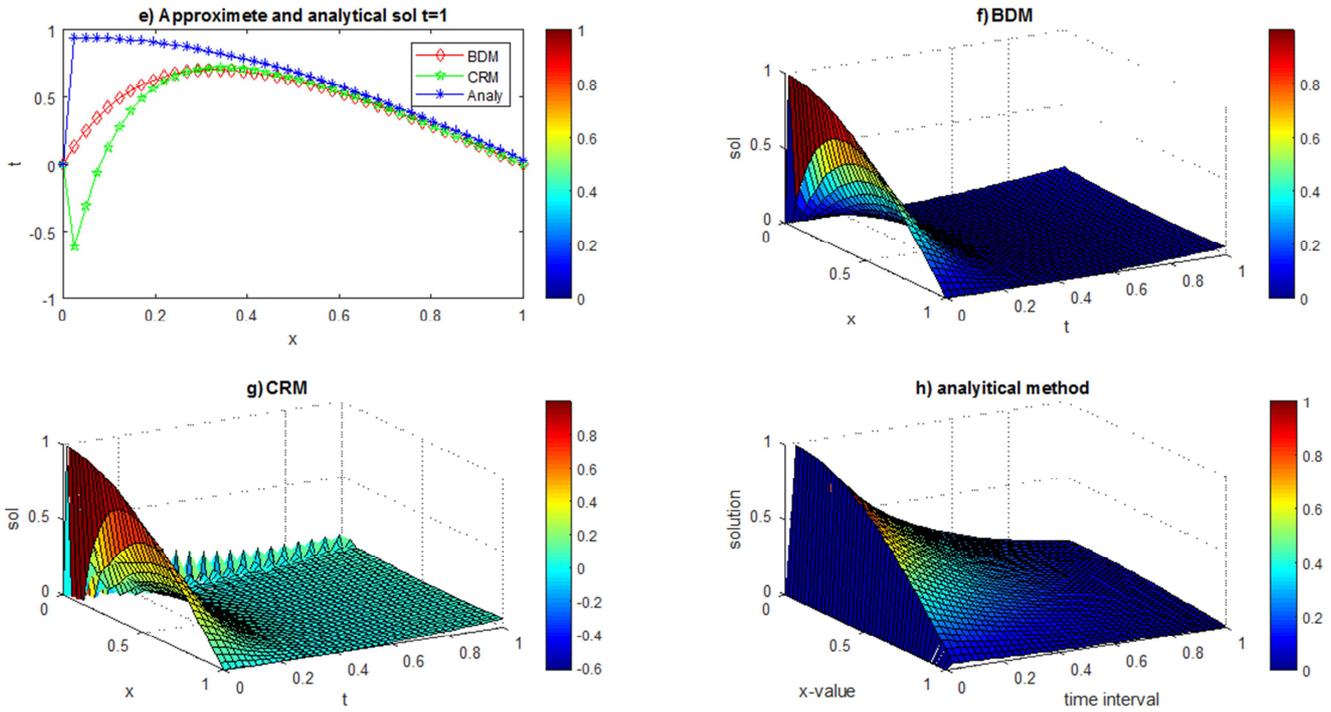


Figure 3. Surface graphs and Solution profile for the numerical solution of Example 2 with $M = 41$ and $\Delta t = 0.025$.

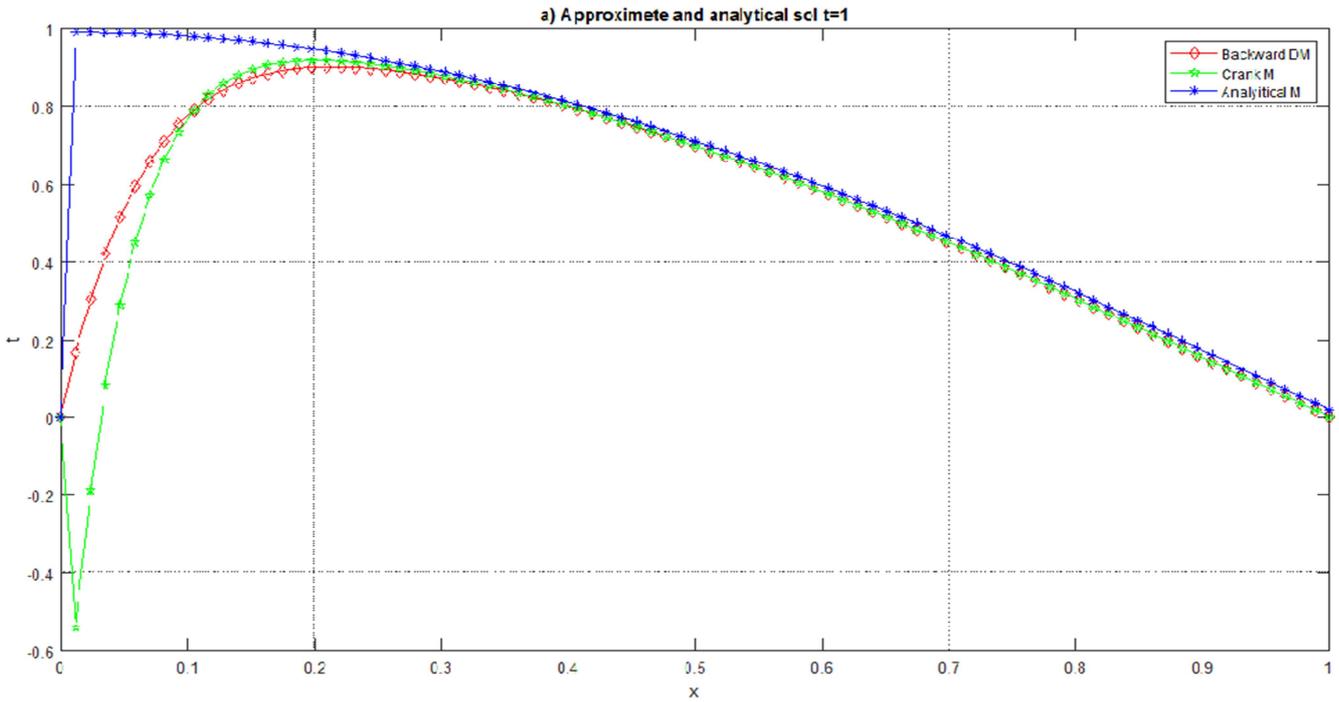


Figure 4. Solution profile of Example 2 with $M = 64$ and $\Delta t = 0.01$.

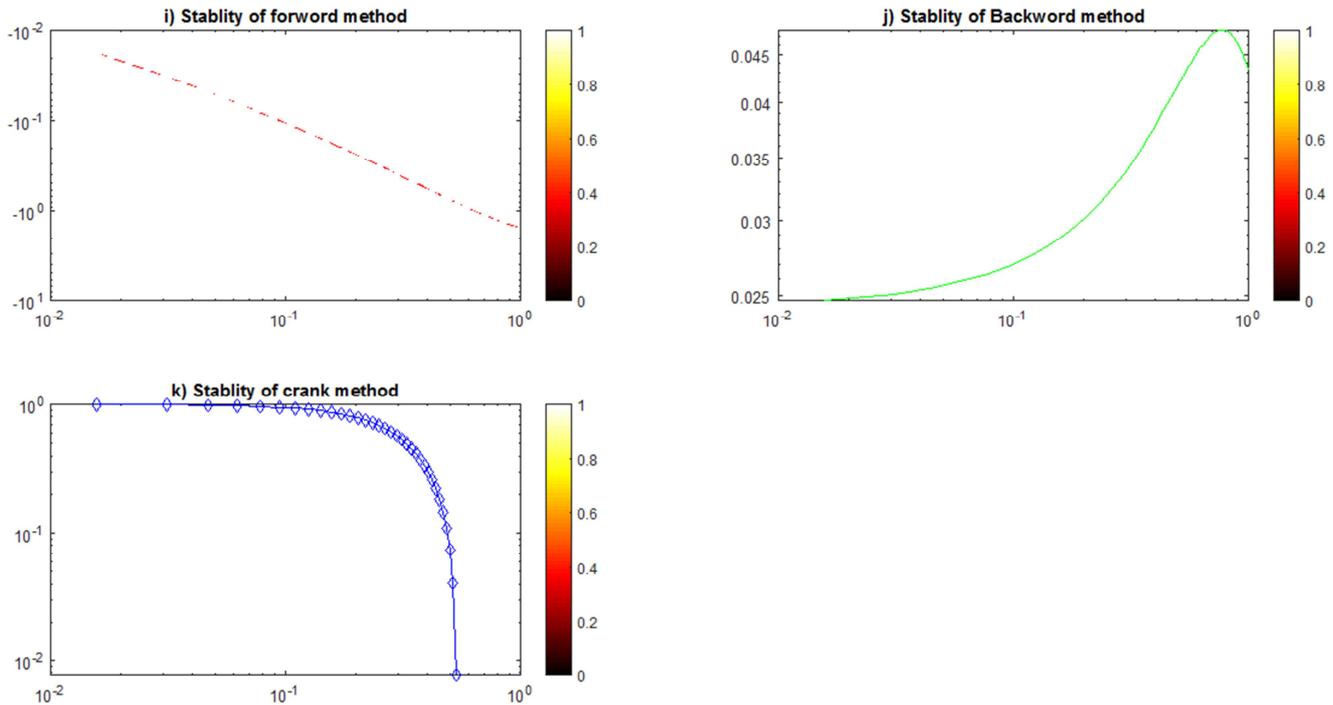


Figure 5. Stability profile of the present method for Example 1 with $M = 64$ and $\Delta t = 0.01$.

6. Discussion and Conclusion

6.1. Discussion

In these three methods, Forward difference, Backward difference, and Crank Nicholson is used to obtaining the scheme to solve one-dimensional linear parabolic differential equation. First, the domain is discretized using the uniform mesh and then discretizing partial derivative at each grid point. Then, the transformed system of equations can be solved by matrix inverse method. The stability and consistency of the method is well established. To validate the applicability of the method, two the model example has been considered and solved by varying the value step-length h and time-step Δt . As can be seen from the numerical results and predicted in tables 1 and 2 the present method is superior to the method developed in [18] and approximate the exact solution very well. Since as depicted in Table 1, the present method is able to generate a convergent numerical solution at which the method presented by Tatari and Dehghan, 2010 fails to produce the convergent solution. The condition number of the system matrix of the present method is in the range $2.8134 \leq \kappa(A) \leq 233.8265$ whereas the condition number of the system matrix presented by Tatari and Dehghan, 2010 is $103.0267E+15 \leq \kappa(A) \leq 1.9676E+19$. Thus, the effect of the condition number on the accuracy of the numerical solution is more significant on the method presented by Tatari and Dehghan, 2010 than on the numerical solution of the present method. The value E^∞ in Table 1 confirms this issue. That is the smaller the value of E^∞ the less the effect of the condition number on the accuracy of the approximate solution. As can also be seen from Tables 1-

Table 2, the Order of convergence is kept constant for the same values of the mesh-size in each table. This is because the condition number depends only on the step length of the spatial variable. From table 1 shows as the values of mesh sizes decrease, the maximum absolute error, root mean square error also decreases. But Condition number increases. This is formed as a trade-off or uncertainty principle in [5, 14]. Again figure 2 shows, the surface plot of approximate solution of Example 1 is well established with an analytical solution. Again Figure 3 and 4 shows, the solution obtained by the present method for Example 2 is good agreement with the analytical solution. So the series solution of the 1D parabolic equation is a good approximation compared to the existing solution. Therefore we can conclude that a small number of arguments are sufficient to provide an accurate solution present method. Figure 5 shows the Stability profile of the present method for Example 1 and then Crank Nicholson is more stable. Therefore, the present scheme that obtained from the finite difference methods and Crank Nicholson are more accurate and convergent method for solving the second order one-dimensional linear parabolic equation.

6.2. Conclusion

In this paper three methods, Forward difference, Backward difference, and Crank Nicholson is used to obtaining the scheme to solve one-dimensional linear parabolic differential equation. First, the domain is discretized using the uniform mesh and then discretizing partial derivative at each grid point. Then, the transformed system of equations can be solved by matrix inverse method. The stability and consistency of the method is well established. To validate the applicability of the method, two a model example has been

considered and solved by varying the value step-length h and time-step Δt . Generally As can be seen from the numerical results presented in tables and graphs, the present method is superior over the method pre-existing method and approximates the exact solution very well.

Acknowledgements

The authors would like to thanks Ambo University for financial and material support.

References

- [1] Benyam Mabrate (2015). Numerical solution of a one-dimensional Heat Equation with Dirichlet boundary conditions. *American Journal of Applied Mathematics*, 3 (4): 305-311.
- [2] Bernatz, R. (2010). Fourier series and numerical methods for partial differential equations. John Wiley and Sons.
- [3] Cheninguael, A. (2014, March). Numerical Method for the Heat Equation with Dirichlet and Neumann Conditions. *In Proceedings of the International Multiconference of Engineering and computer scientists* (vol. 1).
- [4] Dabral, V., Kapoor, S. and Dhawan, S. (Apr-May 2011). Numerical Simulation of one dimensional Heat Equation: B-Spline Finite Element Method, *Indian Journal of Computer Science and Engineering*. 2 (2), 222-235.
- [5] Fasshauer, G. (2007). Meshfree Application Method with Matlab. *Interdisciplinary Mathematical Sciences*.
- [6] Franke, R. (1982). Scattered data interpolation: tests of some methods. *Mathematics of computation*, 38 (157): 181-200.
- [7] Francis B. Hildebrand (1987). Introduction to numerical analysis, *Second-edition, Dover Publications-Inc., Canada*.
- [8] Hooshmandasl M. R Haidari, M., and MaalekGhaini F. M. (2012). Numerical solution of one dimensional heat equation by using the Chebyshev Wavelets method. *JACM an open-access journal*, 1 (6), 1-7.
- [9] Kalyanil, P., and Rao (2013). Numerical solution of heat equation through double Interpolation. *IOSR Journal. of math.* 6 (6): 58-62.
- [10] Li, J. R., and Greengard, L. (2007). On the numerical solution of the heat equation I: Fast solvers in free space. *Journal of Computational Physics*, 226 (2): 1891-1901.
- [11] Muluneh Dingeta, Gemechis File and Tesfaye Aga (2018) Numerical Solution of Second Numerical Solution of Second-Order One Dimensional Linear Hyperbolic Telegraph Equation *Ethiop. J. Educ. & Sc. Vol. 14 No 1*.
- [12] Rashidinia J. Esfahani F. and Jamalzadeh S., (2013), B-spline Collocation Approach for Solution of Klein-Gordon Equation. *International Journal of Mathematical Modeling and Computations* 3, 25-33.
- [13] Ray S, S., (2016) Numerical analysis with algorithms and programming, *Taylor & Francis Group, LLC, USA*.
- [14] Schaback, R. (1995). Error estimates and condition numbers for radial basis function interpolation. *Advances in Computational Mathematics*, 3 (3), 251-264.
- [15] Shokofeh S. and Rashidinia J. (2016), Numerical solution of the hyperbolic telegraph the equation by cubic B-spline collocation method. *Applied Mathematics and Computation* 281, 28–38.
- [16] Sastry, S. S. (2006). Introductory method of numerical analysis, Fourth-edition, Asoke Ghash, prentice Hall of India.
- [17] Schiesser, W. E., and Griffiths, G. W. (2009). A compendium of partial differential equation models: method of lines analysis with Matlab. Cambridge University Press.
- [18] Tatari, M. and Dehghan, M., (2010). A method for solving partial differential equations via radial basis functions: Application to the heat equation *Engineering Analysis with Boundary element* 34 (3), 206-212.
- [19] Terefe Asrat, Gemechis File and Tesfaye Aga (2016), Fourth-order stable central difference method for selfadjoint singular perturbation problems, *Ethiop. J. Sci. & Technol.* 9 53-68.
- [20] Hikmet, ag̃lar, Mehmet O. zer, Nazan, ag̃lar (2008) The numerical solution of the one-dimensional heat equation by using third degree B-spline functions.