

Exact Soliton Solutions for New (4+1)-Dimensional Nonlinear Partial Differential Equations by a New $\exp(\phi(\xi))$ -Expansion Method

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Abstract: In this paper, we present a new two equations. The first equation is the $(4 + 1)$ -dimensional Generalized Nonlinear Boussinesq Equation (G-NBE), and the second is the $(4 + 1)$ -dimensional Generalized Camassa–Holm Kadomtsev–Petviashvili Equation (G-CH-KPE). We use a new $\exp(\phi(\xi))$ -expansion method for solve our new equations. We determine a variety of exact solutions for each equation and expressed in terms of hyperbolic functions, trigonometric functions, exponential functions and rational functions.

Keywords: $\exp(\phi(\xi))$ -expansion method, Generalized Nonlinear Boussinesq Equation, Generalized Camassa–Holm Kadomtsev–Petviashvili Equation, Soliton Solution, Traveling Wave Solutions

1. Introduction

Nonlinear (PDEs) are widely used to describe complex phenomena in many fields of science. Therefore, extracting traveling wave solutions to these equations has become an increasingly important topic in nonlinear sciences. Much attention has been paid for this purpose and various powerful methods have been proposed by a lot of researchers such as Homogeneous balance method [1], Hirota's bilinear transformation method [2], the Exp-function method [3], the $\exp(-\Phi(\xi))$ -expansion method [4, 5], the Rational solutions [6], Sinh-Gordon expansion method [7], the multiple-solitons [8], the $\left(\frac{G'}{G}\right)$ -expansion method [9], the (G'/G^2) -Expansion method [10], the mapping method [11], the Laplace-Adomian decomposition method [12], the sine-cosine method [13], the extended hyperbolic function method [14], the Multiple-lump wave solution [15], the modified $\exp(-\vartheta(\sigma))$ -expansion function method [16], and others.

Among the nonlinear equations that have attracted researchers are the (G-NBE) [17]-[23], and the (G-CH-KPE)

[24]- [29], those concerned with studying long waves in shallow water.

The importance of our present work is, in order to generate many exact equation, namely, traveling wave solutions, new approach $\exp(\phi(\xi))$ -expansion method. For illustration of the importance proposed method, we develop two new equations. The $(4 + 1)$ -dimensional (G-NBE) and the $(4 + 1)$ -dimensional (G-CH-KPE).

2. Description of New $\exp(\phi(\xi))$ -expansion Method

Consider the general nonlinear (PDE), say, in two variables,

$$\Psi(v, v_t, v_x, v_{tt}, v_{xt}, v_{xx}, \dots) = 0, \quad (1)$$

where $v = v(x, t)$ is an unknown function, Ψ is a polynomial in $v(x, t)$ and the subscripts stand for the partial derivatives.

We suppose that the combination of real variables x and t by a complex variable ξ

$$v(x, t) = v(\xi), \quad \xi = ax - ct, \quad (2)$$

where a is the wave number and c is the speed of the traveling wave. Now using Eq. (2), Eq. (1) is converted into an ordinary differential equation for $v = v(\xi)$:

$$F(v, -cv', av', c^2v'', -cav'', a^2v'', \dots) = 0, \quad ' \equiv \frac{d}{d\xi}. \quad (3)$$

Suppose the traveling wave solution of Eq. (3) can be expressed by polynomial in $\exp(\phi(\xi))$ as:

$$v(\xi) = \sum_{i=0}^N a_i (\exp(\phi(\xi)))^i, \quad (4)$$

where the coefficients a_i ($0 \leq i \leq N$), a and c are constants to be determined, such that $\phi = \phi(\xi)$ satisfies the following ordinary differential equation:

$$\phi'(\xi) = \sqrt{\mu \exp(\phi(\xi)) + \lambda}, \quad (5)$$

Eq. (5) gives the following solutions:

Case 1. when $\lambda > 0$ and $\mu \neq 0$,

$$\phi(\xi) = \ln \left(\frac{-\lambda}{\mu} \operatorname{sech}^2 \left(\frac{\sqrt{\lambda}}{2} (\xi + k) \right) \right), \quad (6)$$

$$\phi(\xi) = \ln \left(\frac{\lambda}{\mu} \operatorname{csch}^2 \left(\frac{\sqrt{\lambda}}{2} (\xi + k) \right) \right), \quad (7)$$

where k is a constant of integration.

Case 2. when $\lambda < 0$ and $\mu \neq 0$,

$$\phi(\xi) = \ln \left(\frac{-\lambda}{\mu} \sec^2 \left(\frac{\sqrt{-\lambda}}{2} (\xi + k) \right) \right), \quad (8)$$

$$\phi(\xi) = \ln \left(\frac{-\lambda}{\mu} \csc^2 \left(\frac{\sqrt{-\lambda}}{2} (\xi + k) \right) \right). \quad (9)$$

Case 3. when $\lambda = 0$ and $\mu \neq 0$,

$$\phi(\xi) = \ln \left(\frac{4}{\mu (\xi + k)^2} \right), \quad (10)$$

Case 4. when $\lambda \neq 0$ and $\mu = 0$,

$$\phi(\xi) = \sqrt{\lambda} (\xi + k), \quad (11)$$

where A, a_i ($i = 0, 1, 2, \dots, N$), a, c, μ and λ are constants. The positive integer N can be determined by using homogeneous balance between the highest order derivatives and the nonlinear terms appearing in ODE (3). Substituting Eq. (4) into Eq. (3), using Eq. (5) repeatedly, and setting the coefficients of the each order of $(\exp(\phi(\xi)))^i$, $(\exp(\phi(\xi)))^i \sqrt{\mu \exp(\phi(\xi)) + \lambda}$ to zero, we obtain a set of nonlinear algebraic equations for a_i ($i = 0, 1, 2, \dots, N$), a, c, μ and λ . With the aid of the computer program Maple, we can solve the set of nonlinear algebraic equations and obtain all the constants a_i ($i = 0, 1, 2, \dots, N$), a, c . Substituting the values of a_i ($i = 0, 1, 2, \dots, N$), a, c into Eq. (4) along with general solutions of Eq. (5) completes the determination of the solution of Eq. (1).

3. Application of the Method

In this paper, we will present the new $\exp(\phi(\xi))$ -expansion method to the construct the exact solutions of the new (G-NBE) and (G-CH-KPE).

3.1. The (G-NBE)

Consider the (4 + 1)-dimensional (G-NBE) that reads:

$$u_{tt} = \alpha u_{xx} + \beta (u^2)_{xx} + \gamma u_{xxx} + \delta_1 u_{yy} + \delta_2 u_{zz} + \delta_3 u_{ss}. \quad (12)$$

Applying the following wave transformation

$$u(x, y, z, s, t) = u(\xi), \quad \xi = ax + by + cz + ds - et, \quad (13)$$

on equation Eq. (12), we get

$$(e^2 - a^2\alpha - b^2\delta_1 - c^2\delta_2 - d^2\delta_3) u'' - a^2\beta (u^2)'' - a^4\gamma u^{(4)} = 0. \quad (14)$$

Integrating Eq. (14) twice with respect to ξ , and neglecting the constants of integration, we find

$$(e^2 - a^2\alpha - b^2\delta_1 - c^2\delta_2 - d^2\delta_3) u - a^2\beta (u^2) - a^4\gamma u'' = 0. \quad (15)$$

Balancing of Eq. (15), give

$$2N = N + 2 \Rightarrow N = 2.$$

By the use of Eq. (4), we present the solution of Eq. (15) as:

$$u(\xi) = a_0 + a_1 \exp(\phi(\xi)) + a_2 (\exp(\phi(\xi)))^2. \quad (16)$$

Substituting Eq. (16) in Eq. (15) and using Eq. (5), collecting the coefficients of each power of $(\exp(\phi(\xi)))^i$, $0 \leq i \leq 4$, setting each coefficients to zero, and solving the algebraic equations by Maple we get,

$$a_0 = 0, a_1 = \frac{-3a^2\gamma\mu}{2\beta}, a_2 = 0, \quad e = \pm\sqrt{a^4\gamma\lambda + a^2\alpha + b^2\delta_1 + c^2\delta_2 + d^2\delta_3}. \quad (17)$$

$$a_0 = \frac{-a^2\gamma\lambda}{\beta}, a_1 = \frac{-3a^2\gamma\mu}{2\beta}, a_2 = 0, \quad e = \pm\sqrt{-a^4\gamma\lambda + a^2\alpha + b^2\delta_1 + c^2\delta_2 + d^2\delta_3}. \quad (18)$$

Substituting Eq. (17) in Eq. (16) and using solutions Eq. (5), we obtain:

When $\lambda > 0$ and $\mu \neq 0$,

$$u_{1,2} = \frac{3a^2\gamma\lambda}{2\beta} \operatorname{sech}^2 \left(\frac{\sqrt{\lambda}}{2} \left(ax + by + cz + ds \pm \sqrt{a^4\gamma\lambda + a^2\alpha + b^2\delta_1 + c^2\delta_2 + d^2\delta_3}t + k \right) \right),$$

$$u_{3,4} = \frac{-3a^2\gamma\lambda}{2\beta} \operatorname{csch}^2 \left(\frac{\sqrt{\lambda}}{2} \left(ax + by + cz + ds \pm \sqrt{a^4\gamma\lambda + a^2\alpha + b^2\delta_1 + c^2\delta_2 + d^2\delta_3}t + k \right) \right).$$

When $\lambda < 0$ and $\mu \neq 0$,

$$u_{5,6} = \frac{3a^2\gamma\lambda}{2\beta} \sec^2 \left(\frac{\sqrt{-\lambda}}{2} \left(ax + by + cz + ds \pm \sqrt{a^4\gamma\lambda + a^2\alpha + b^2\delta_1 + c^2\delta_2 + d^2\delta_3}t + k \right) \right),$$

$$u_{7,8} = \frac{3a^2\gamma\lambda}{2\beta} \csc^2 \left(\frac{\sqrt{-\lambda}}{2} \left(ax + by + cz + ds \pm \sqrt{a^4\gamma\lambda + a^2\alpha + b^2\delta_1 + c^2\delta_2 + d^2\delta_3}t + k \right) \right).$$

When $\lambda = 0$ and $\mu \neq 0$,

$$u_{9,10} = \frac{-6a^2\gamma}{\beta \left(ax + by + cz + ds \pm \sqrt{a^2\alpha + b^2\delta_1 + c^2\delta_2 + d^2\delta_3}t + k \right)^2}.$$

When $\lambda \neq 0$ and $\mu = 0$ is constant.

Substituting Eq. (18) in Eq. (16) and using solutions Eq. (5), we obtain:

When $\lambda > 0$ and $\mu \neq 0$,

$$u_{11,12} = \frac{-a^2\gamma\lambda}{\beta} + \frac{3a^2\gamma\lambda}{2\beta} \operatorname{sech}^2 \left(\frac{\sqrt{\lambda}}{2} \left(ax + by + cz + ds \pm \sqrt{-a^4\gamma\lambda + a^2\alpha + b^2\delta_1 + c^2\delta_2 + d^2\delta_3}t + k \right) \right),$$

$$u_{13,14} = \frac{-a^2\gamma\lambda}{\beta} - \frac{3a^2\gamma\lambda}{2\beta} \operatorname{csch}^2 \left(\frac{\sqrt{\lambda}}{2} \left(ax + by + cz + ds \pm \sqrt{-a^4\gamma\lambda + a^2\alpha + b^2\delta_1 + c^2\delta_2 + d^2\delta_3}t + k \right) \right).$$

When $\lambda < 0$ and $\mu \neq 0$,

$$u_{15,16} = \frac{-a^2\gamma\lambda}{\beta} + \frac{3a^2\gamma\lambda}{2\beta} \sec^2 \left(\frac{\sqrt{-\lambda}}{2} \left(ax + by + cz + ds \pm \sqrt{-a^4\gamma\lambda + a^2\alpha + b^2\delta_1 + c^2\delta_2 + d^2\delta_3}t + k \right) \right),$$

$$u_{17,18} = \frac{-a^2\gamma\lambda}{\beta} + \frac{3a^2\gamma\lambda}{2\beta} \csc^2 \left(\frac{\sqrt{-\lambda}}{2} \left(ax + by + cz + ds \pm \sqrt{-a^4\gamma\lambda + a^2\alpha + b^2\delta_1 + c^2\delta_2 + d^2\delta_3}t + k \right) \right).$$

When $\lambda = 0$ and $\mu \neq 0$, we get the same solution as $u_{9,10}$.

When $\lambda \neq 0$ and $\mu = 0$ is constant.

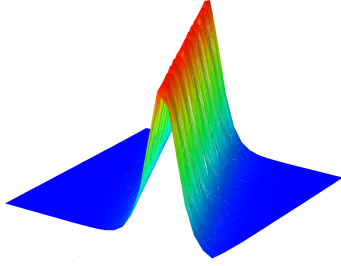


Figure 1. 3D plot a soliton solution $u_{1,2}(x, y, z, s, t)$ when $\lambda > 0$ and $y = z = s = 0$.

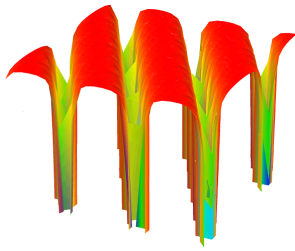


Figure 2. 3D plot singular periodic soliton of solution $u_{5,6}(x, y, z, s, t)$ when $\lambda < 0$ and $y = z = s = 0$.

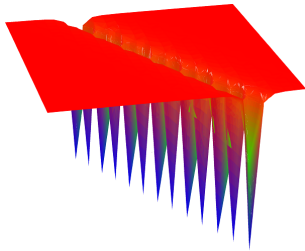


Figure 3. 3D plot singular soliton of exact solution $u_{9,10}(x, y, z, s, t)$ when $\lambda = 0$ and $y = z = s = 0$.

3.2. The (G-CH-KPE)

Consider the $(4 + 1)$ -dimensional (G-CH-KPE) that reads:

$$(u_t + c_1 u_x + \alpha u u_x + \beta u_{xxt})_x + c_2 u_{yy} + c_3 u_{zz} + c_4 u_{ss} = 0. \quad (19)$$

Applying the following wave transformation

$$u(x, y, z, s, t) = u(\xi), \quad \xi = ax + by + cz + ds - et, \quad (20)$$

on equation Eq. (19), we get

$$(-ae + a^2 c_1 + b^2 c_2 + c^2 c_3 + d^2 c_4) u'' + a^2 \alpha (u u')' - e a^3 \beta u^{(4)} = 0. \quad (21)$$

Integrating Eq. (21) twice with respect to ξ , and neglecting the constants of integration, we get

$$(-ae + a^2 c_1 + b^2 c_2 + c^2 c_3 + d^2 c_4) u + \frac{a^2 \alpha}{2} u^2 - e a^3 \beta u'' = 0. \quad (22)$$

Balancing of Eq. (22), give

$$2N = N + 2 \Rightarrow N = 2.$$

By the use of Eq. (4), we present the solution of Eq. (22) as:

$$u(\xi) = a_0 + a_1 \exp(\phi(\xi)) + a_2 (\exp(\phi(\xi)))^2. \quad (23)$$

Substituting Eq. (23) in Eq. (22) and using Eq. (5), collecting the coefficients of each power of $(\exp(\phi(\xi)))^i$, $0 \leq i \leq 4$, setting each coefficients to zero, and solving the algebraic equations by Maple we get,

$$a_0 = 0, a_1 = \frac{3\beta\mu(a^2 c_1 + b^2 c_2 + c^2 c_3 + d^2 c_4)}{\alpha(a^2 \beta \lambda + 1)}, a_2 = 0, \quad e = \frac{a^2 c_1 + b^2 c_2 + c^2 c_3 + d^2 c_4}{a(a^2 \beta \lambda + 1)}, \quad (24)$$

$$a_0 = \frac{-2\beta\lambda(a^2 c_1 + b^2 c_2 + c^2 c_3 + d^2 c_4)}{\alpha(a^2 \beta \lambda - 1)}, a_2 = 0, \quad (25)$$

$$a_1 = \frac{-3\beta\mu(a^2 c_1 + b^2 c_2 + c^2 c_3 + d^2 c_4)}{\alpha(a^2 \beta \lambda - 1)}, e = -\frac{a^2 c_1 + b^2 c_2 + c^2 c_3 + d^2 c_4}{a(a^2 \beta \lambda - 1)}.$$

Substituting Eq. (24) in Eq. (23) and using solutions Eq. (5), we obtain:

When $\lambda > 0$ and $\mu \neq 0$,

$$u_1 = \frac{-3\beta\lambda (a^2c_1 + b^2c_2 + c^2c_3 + d^2c_4)}{\alpha (a^2\beta\lambda + 1)} \operatorname{sech}^2 \left(\frac{\sqrt{\lambda}}{2} \left(ax + by + cz + ds - \frac{(a^2c_1 + b^2c_2 + c^2c_3 + d^2c_4)}{a (a^2\beta\lambda + 1)} t + k \right) \right),$$

$$u_2 = \frac{3\beta\lambda (a^2c_1 + b^2c_2 + c^2c_3 + d^2c_4)}{\alpha (a^2\beta\lambda + 1)} \operatorname{csch}^2 \left(\frac{\sqrt{\lambda}}{2} \left(ax + by + cz + ds - \frac{(a^2c_1 + b^2c_2 + c^2c_3 + d^2c_4)}{a (a^2\beta\lambda + 1)} t + k \right) \right).$$

When $\lambda < 0$ and $\mu \neq 0$,

$$u_3 = \frac{-3\beta\lambda (a^2c_1 + b^2c_2 + c^2c_3 + d^2c_4)}{\alpha (a^2\beta\lambda + 1)} \sec^2 \left(\frac{\sqrt{-\lambda}}{2} \left(ax + by + cz + ds - \frac{(a^2c_1 + b^2c_2 + c^2c_3 + d^2c_4)}{a (a^2\beta\lambda + 1)} t + k \right) \right),$$

$$u_4 = \frac{-3\beta\lambda (a^2c_1 + b^2c_2 + c^2c_3 + d^2c_4)}{\alpha (a^2\beta\lambda + 1)} \csc^2 \left(\frac{\sqrt{-\lambda}}{2} \left(ax + by + cz + ds - \frac{(a^2c_1 + b^2c_2 + c^2c_3 + d^2c_4)}{a (a^2\beta\lambda + 1)} t + k \right) \right).$$

When $\lambda = 0$ and $\mu \neq 0$,

$$u_5 = \frac{12\beta (a^2c_1 + b^2c_2 + c^2c_3 + d^2c_4)}{\alpha \left(ax + by + cz + ds - \frac{(a^2c_1 + b^2c_2 + c^2c_3 + d^2c_4)}{a} t + k \right)^2}.$$

When $\lambda \neq 0$ and $\mu = 0$ is constant.

Substituting Eq. (25) in Eq. (23) and using solutions Eq. (5), we obtain:

When $\lambda > 0$ and $\mu \neq 0$,

$$u_6 = \frac{-\beta\lambda (a^2c_1 + b^2c_2 + c^2c_3 + d^2c_4)}{\alpha (a^2\beta\lambda - 1)} \left(2 - 3 \operatorname{sech}^2 \left(\frac{\sqrt{\lambda}}{2} \left(ax + by + cz + ds + \frac{(a^2c_1 + b^2c_2 + c^2c_3 + d^2c_4)}{a (a^2\beta\lambda - 1)} t + k \right) \right) \right),$$

$$u_7 = \frac{-\beta\lambda (a^2c_1 + b^2c_2 + c^2c_3 + d^2c_4)}{\alpha (a^2\beta\lambda - 1)} \left(2 + 3 \operatorname{csch}^2 \left(\frac{\sqrt{\lambda}}{2} \left(ax + by + cz + ds + \frac{(a^2c_1 + b^2c_2 + c^2c_3 + d^2c_4)}{a (a^2\beta\lambda - 1)} t + k \right) \right) \right).$$

When $\lambda < 0$ and $\mu \neq 0$,

$$u_8 = \frac{-\beta\lambda (a^2c_1 + b^2c_2 + c^2c_3 + d^2c_4)}{\alpha (a^2\beta\lambda - 1)} \left(2 - 3 \sec^2 \left(\frac{\sqrt{-\lambda}}{2} \left(ax + by + cz + ds + \frac{(a^2c_1 + b^2c_2 + c^2c_3 + d^2c_4)}{a (a^2\beta\lambda - 1)} t + k \right) \right) \right),$$

$$u_9 = \frac{-\beta\lambda(a^2c_1 + b^2c_2 + c^2c_3 + d^2c_4)}{\alpha(a^2\beta\lambda - 1)} \left(2 - 3 \csc^2 \left(\frac{\sqrt{-\lambda}}{2} \left(ax + by + cz + ds + \frac{(a^2c_1 + b^2c_2 + c^2c_3 + d^2c_4)}{a(a^2\beta\lambda - 1)}t + k \right) \right) \right).$$

When $\lambda = 0$ and $\mu \neq 0$, we get the same solution as u_5 .

When $\lambda \neq 0$ and $\mu = 0$ is constant.

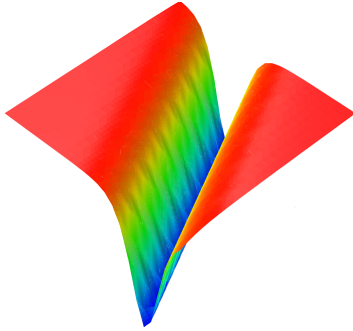


Figure 4. 3D plot a soliton solution $u_1(x, y, z, s, t)$ when $\lambda > 0$ and $y = z = s = 0$.

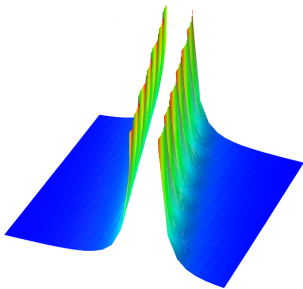


Figure 5. 3D plot singular soliton of exact solution $u_5(x, y, z, s, t)$ when $\lambda = 0$ and $y = z = s = 0$.

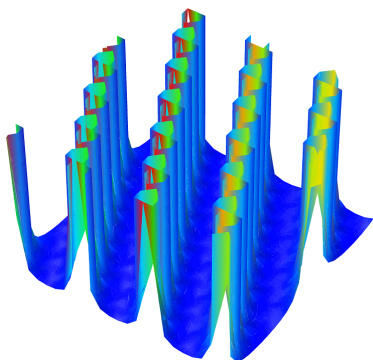


Figure 6. 3D plot singular periodic soliton of solution $u_8(x, y, z, s, t)$ when $\lambda < 0$ and $y = z = s = 0$.

4. Conclusion

In this paper, we presented a new the $(4 + 1)$ -dimensional (G-NBE) and new the $(4 + 1)$ -dimensional (G-CH-KPE). A new $\exp(\phi(\xi))$ -expansion method has been successfully implemented to find new traveling wave solutions for our equations. We expressed to some our new solutions graphical.

Abbreviations

G-NBE	Generalized Nonlinear Boussinesq Equation
G-CH-KPE	Generalized Camassa–Holm Kadomtsev–Petviashvili Equation
PDE	Partial Differential Equation

Conflicts of Interest

The authors declare no conflicts of interest.

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