

# L-catch and L-escape Differential Game

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**Abstract:** In this work, we investigate a simple motion differential game involving one pursuer and one evader, where the dynamics of the pursuer and the evader are governed by first-order and second-order differential equations, respectively. The control functions for both players are assumed to satisfy generalized integral constraints, which limit their maneuverability over time. Our main focus is on the analysis of pursuit under the  $l$ -capture condition. Specifically, we demonstrate that pursuit is completed when the  $l$ -distance between the positions of the pursuer  $x(t)$  and the evader  $y(t)$  becomes less than or equal to a given threshold  $l$ , that is,  $\|y(t) - x(t)\| \leq l$ , at some finite time  $t$ . We derive an explicit formula for the guaranteed time of pursuit, which holds for all admissible strategies of the evader. Additionally, we show that an optimal escape strategy for the evader, known as  $l$ -escape, can also be realized at this same optimal time. This simultaneous realization of pursuit and escape conditions indicates a balance in the strategies of both players, thus leading to an optimal pursuit time. Our results contribute to the broader understanding of differential games and optimal control theory.

**Keywords:** Optimal Pursuit Time,  $l$ -catch,  $l$ -escape, Players Control Functions

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## 1. Introduction

This paper explores the  $l$ -catch and  $l$ -escape differential game, where one pursuer and one evader are described by first-order and second-order differential equations, respectively. The control functions of both players are ruled by generalized integral constraints in an additional, more realistic manner. There are many ways to deal with these same kinds of issues in the literature. Badakaya et al. [1] study of chase differential games with many pursuers and one evader in the Hilbert space, for example, puts integral and geometric constraints on the control functions, and thus suggests methods for the pursuers to catch evader in  $l$ -catch sense. Likewise, Y.A Hadejia et al. [2] explore short-term pursuit differential games by defining reasonable assumptions for  $l$ -catch under various constraints, and illustrate their results with real examples.

Also B.M Umar and A.B Aihong [3] studied a differential game of one pursuer and one evader, where the motion of the pursuer and evader is described by first order and second order differential equations respectively. Control functions of

pursuers and evader are subject to integral constraints.

Integral constraints are a key element in A. A. Chikrii and A. A. Belousov [4] work, who analyse linear systems with these constraints and create conditions for certain pursuit fulfilment. Gafurjan Ibragimov and Nu'man Satimov [5] apply this result to a group of pursuers and evaders, showing that pursuit can be completed when the total amount of resources for the pursuers exceeds that of the evaders.

Differential Game with Generalized constraints were studied by some researchers (see [6, 7]) B.M Umar et al. [6] studied the pursuit and evasion differential game of many pursuers and one evader in  $\mathbb{R}^n$ , with generalized integral constraints imposed on the Players' control functions. Given some sufficient conditions and a finite time  $\theta$ , they solved the pursuit problem through the construction of an admissible pursuer's strategy and showed that, indeed the strategy guarantees a completion of a pursuit at time  $\theta$ .

Optimal pursuit time differential game is a more challenging game, it involves finding a time in which both pursuit and evasion is possible at that time. Some researchers worked on

this issue (see [8-20]). In [9] G.Ibragimov studied optimal pursuit game problem for the infinite system of DE (1) in the Hilbert space  $l_2$ . Control functions are subjected to integral constraints. The main results of the paper are as follows. (i) Solved a time optimal control problem and found an equation to find the optimal time. Also constructed optimal control. (ii) Studied optimal pursuit problem for the differential game with integral constraints. They have constructed optimal strategies for the players and given an equation to find optimal pursuit time. R. Livermore [11] leveraged the unique attributes of the deviated pure pursuit geometric rule to enforce a specific intercept time or impact angle of a nonmaneuvering target. An optimal control-based guidance law was developed using linear quadratic optimal control theory. J.P Hespanha and M. Prandini [15] addressed the problem of computing optimal policies for probabilistic pursuit games. They showed that, under appropriate assumptions, optimal policies can be computed using value iteration. However, the value iteration procedure can be computationally very costly. We have then considered a greedy solution and determined conditions under which it is optimal. Also in [18] E. Bakolas and P. Tsiotras formulated a new dynamic partitioning problem for a finite set of moving targets, with respect to the minimum time required for a pursuer to intercept each of the moving targets. It was assumed that each moving target employs a time-varying feedback evading control strategy in response to its Player's actions. In the special case when all moving targets adopt the same evading strategy, the problem reduces to the Zermelo-Voronoi diagram problem. They also presented an efficient scheme for the construction of the solution of this partition problem, by exploiting the structure of the solution of a case of Zermelo's navigation problem.

In this paper we will study pursuit and evasion differential game with more general integral constraints. We called this game,  $l$ -catch and  $l$ -escape differential game because we are considering completion of pursuit when ever the distance between the pursuer and evader is less or equal to some small length  $l$ , and evasion when ever the distance between the pursuer and evader is strictly greater than the same length  $l$ . We will define a time  $T$  at which these instances may occur. If the above situations occurred we called the time  $T$  an optimal pursuit time.

## 2. Statement of the Problem

Consider the space  $\mathbb{R}^n$  with the norm  $\|\cdot\| : \mathbb{R}^n \rightarrow [0, +\infty)$  defined as

$$\|\alpha\| = \left( \sum_{k=1}^n |\alpha_k|^p \right)^{\frac{1}{p}}.$$

$$y(t) = y^0 + \theta y^1 + \int_0^\theta \int_0^r v(s) ds dr = y^0 + \theta y^1 + \int_0^\theta (\theta - t) v(t) dt, \quad (7)$$

$1 < p < \infty$ , respectively.

The dynamic equations of pursuer and the evader are given by

$$\begin{cases} \dot{x}(t) = u(t), & x(0) = x^0, \\ \ddot{y}(t) = v(t), & \dot{y}(0) = y^1, \quad y(0) = y^0, \end{cases} \quad (1)$$

where  $x, x^0, u, y, y^0, v \in \mathbb{R}^n$ ,  $u = (u_1, u_2, \dots, u_n)$  and  $v = (v_1, v_2, \dots, v_n)$  are control parameters of pursuer and evader respectively.

Let  $[0, T]$  be a finite interval of  $\mathbb{R}$  and  $L_p(0, T)$  be the collection of Lebesgue measurable functions defined on  $(0, T)$  respectively, where  $\int_0^T |f(t)|^p dt < \infty$  and  $(1 \leq p < \infty)$ ,  $q$  be such that  $\frac{1}{p} + \frac{1}{q} = 1$ . Let

$$\|f\|_{L_p(0, T)} = \left( \int_0^T \|f(t)\|^p dt \right)^{\frac{1}{p}} < \infty$$

for any  $f \in L_p(0, T)$

**Definition 2.1.** A measurable function  $u : [0, T] \rightarrow \mathbb{R}^n$ , ( $v : [0, T] \rightarrow \mathbb{R}^n$ ) is called an admissible control of the pursuer (evader) if it satisfies the following

$$\int_0^T \|u(s)\|^p ds \leq \rho^p, \quad \left( \int_0^T \|v(t)\|^p dt \leq \sigma^p, \right) \quad (2)$$

respectively, where  $\rho$  and  $\sigma$  are given positive numbers.

If the admissible controls of the pursuer and evader are chosen, then solutions to the dynamic equation (1) are given by

$$x(t) = x_0 + \int_0^t u(s) ds, \quad (3)$$

$$y(t) = y^0 + ty^1 + \int_0^t \int_0^r v(s) ds dr. \quad (4)$$

It is easy to see that

$$\int_0^t \int_0^r v(s) ds dr = \int_0^t (t-s) v(s) ds. \quad (5)$$

We can consider an equivalent differential game with the same control functions described by

$$\begin{cases} \dot{x}(t) = u(t), & x(0) = x^0, \\ \dot{y}(t) = (\theta - t)v(t), & y(0) = y^1\theta + y^0 = y_0, \end{cases} \quad (6)$$

instead of (1). Indeed, if the evader uses an admissible control  $v(t) = (v_1(t), v_2(t), \dots)$ , then according to (1), we have

and the same result can be obtained by (6)

$$y(\theta) = y_0 + \int_0^\theta (\theta - t)v(t)dt = y^0 + \theta y^1 + \int_0^\theta (\theta - t)v(t)dt \quad (8)$$

**Definition 2.2.** Pursuit is said to be completed in l-catch sense in a game if there exist strategies  $u$  of the pursuer such that for any admissible control  $v(\cdot)$  of the evader, the inequality  $\|y(\tau) - x(\tau)\| \leq l$  is satisfied for some  $t \in [0, \theta]$

**Definition 2.3** (Guaranteed Pursuit Time,  $T$ ). Pursuit is said to be completed at time  $T > 0$ , if there exist strategies of the pursuers  $U(t, v(t))$ , such that for any admissible control of the evader  $v(t)$ ,  $0 \leq t \leq T$ ,  $\|y(\tau) - x(\tau)\| \leq l$  at some  $\tau$ ,  $0 \leq \tau \leq T$ . In the sequel, the number  $T$  is called guaranteed pursuit time.

**Definition 2.4.** A guaranteed pursuit time  $T'$  is called optimal pursuit time if there exists a strategy of the evader  $V$  such that for any admissible control of the pursuer  $\|y(\tau) - x(\tau)\| > l$ ,  $t \in [0, T']$ .

**Lemma 3.1.** When  $T > s$  and  $1 < p, q < \infty$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ , then

1.

$$\left( \int_0^T \|(T-s)\|^p \|v(s)\|^p ds \right)^{\frac{1}{p}} \leq (T-s)\sigma \quad (9)$$

2.

$$\left( \int_0^T \frac{1}{(T-s)^p} ds \right)^{\frac{1}{p}} \leq \frac{1}{\sqrt[p]{T}}$$

3.

$$\left( \int_0^T \frac{1}{\|T-s\|^p} \|u(s)\|^p ds \right)^{\frac{1}{p}} < 2\rho$$

*Proof* The proof come as follows

### 3. Results

This section contain Lemma and Theorems with their proof, also a formula for optimal pursuit time is presented.

1. Taking the L.H.S of (9), i.e

$$\left( \int_0^T \|(T-s)\|^p \|v(s)\|^p ds \right)^{\frac{1}{p}}$$

Using integration by parts

$$\int (mn) ds = m \int nds - \int \left( m' \int nds \right) ds, \quad (10)$$

Let

$$\begin{aligned} m &= (T-s)^p, \quad m' = -p(T-s)^{p-1} \\ n &= \|v(s)\|^p, \quad \int_0^T nds = \int_0^T \|v(s)\|^p ds \leq \sigma^p \end{aligned} \quad (11)$$

then we have

$$\begin{aligned} \int_0^T \|v(s)\|^p (T-s)^p ds &= (T-s)^p \int_0^T \|v(s)\|^p ds - \int_0^T \left( -p(T-s)^{p-1} \int_0^T \|v(s)\|^p ds \right) ds \\ &\leq (T-s)^p \sigma^p - \int_0^T \left( -p(T-s)^{p-1} \sigma^p \right) ds (T-s)^p \sigma^p + p\sigma^p \int_0^T (\theta-s)^{p-1} ds \\ &= (T-s)^p \sigma^p + \sigma^p [(T-T)^p - (T-0)^p], \\ &= (T-s)^p \sigma^p - \sigma^p T^p \leq (T-s)^p \sigma^p \end{aligned}$$

2. Direct integration yield to

$$\begin{aligned} \left( \int_0^T \frac{1}{(T-s)^p} ds \right)^{\frac{1}{p}} &= \left( \int_0^T (T-s)^{-p} ds \right)^{\frac{1}{p}} \\ &= \left( \frac{(T-T)^{-p+1}}{1-p} \right)^{\frac{1}{p}} - \left( \frac{(T-0)^{-p+1}}{1-p} \right)^{\frac{1}{p}} = \left( \frac{(T)^{1-p}}{1-p} \right)^{\frac{1}{p}} < (T^{1-p})^{\frac{1}{p}} = \frac{1}{\sqrt[p]{T}} \end{aligned}$$

3. In this one we use integration by part  $\int(mn)ds = m \int nds - \int(m' \int nds)$  i.e by letting  $m = (T-s)^{-p}$ ,  $n = \|u(s)\|^p$ , then

$$\begin{aligned} \left( \int_0^T \frac{1}{\|T-s\|^p} \|u(s)\|^p ds \right)^{\frac{1}{p}} &\leq \left( (T-s)^{-p} \rho^p - \int_0^T [-p(T-s)^{-p-1} \rho^p] ds \right)^{\frac{1}{p}} \\ &= \left( (T-s)^{-p} \rho^p + p \rho^p \int_0^T [(T-s)^{-p-1}] ds \right)^{\frac{1}{p}} = \left( (T-s)^{-p} \rho^p + p \rho^p \left[ \left[ \frac{(T-T)^{-p}}{-p} \right] - \left[ \frac{(T-0)^{-p}}{-p} \right] \right] \right)^{\frac{1}{p}} \\ &= \left( (T-s)^{-p} \rho^p + p \rho^p \left[ - \left[ \frac{T^{-p}}{-p} \right] \right] \right)^{\frac{1}{p}} = ((T-s)^{-p} \rho^p + \rho^p [T^{-p}])^{\frac{1}{p}} \leq (\rho^p + \rho^p)^{\frac{1}{p}} < 2\rho. \end{aligned}$$

This mark the end of the proof. The next thing is to present the theorems with their explicit proofs.

following equation.

$$\dot{\eta}(t) = \omega(t), \quad \eta(0) = x_0 \quad (13)$$

### 3.1. L-Catch

This section present completion of pursuit in  $l$ -catch sense. The below Theorem explain how is it possible.

**Theorem 3.1.** . Let  $\rho > \sigma$ , and  $\frac{\|y_0 - x_0\|^q}{\rho^q} \leq \frac{\rho}{2^{q+1}\sigma}$  Then the time define by

$$T := \left( \frac{2\|y_0 - x_0\|\sigma}{\rho} \right)^q \quad (12)$$

is a optimal pursuit time in the game (1)-(2),  $1 \leq p, q < \infty$  and  $\frac{1}{p} + \frac{1}{q} = 1$ .

**Proof** To prove this theorem, we first introduce dummy pursuer with state variable  $\eta$  and motion described by the

with the control function subject to

$$\left( \int_0^T \|\omega(t)\|^p dt \right)^{\frac{1}{p}} \leq \gamma, \quad \gamma = \rho + \frac{l}{\sqrt[p]{T}}. \quad (14)$$

We Construct the dummy pursuer's strategy as follows

$$\omega(t) = \begin{cases} \frac{(y_0 - x_0)}{T} + (T-t)v(t), & 0 \leq t \leq T, \\ 0, & t \geq T, \end{cases} \quad (15)$$

The above constructed strategy (15) is admissibility, i.e it satisfied the given constraint mentioned above, the process is as follows. Using Minkowski's inequality,

$$\begin{aligned} \left( \int_0^T \|\omega(t)\|^p dt \right)^{\frac{1}{p}} &= \left( \int_0^T \left\| \frac{(y_0 - x_0)}{T} + (T-t)v(t) \right\|^p dt \right)^{\frac{1}{p}} \\ &\leq \left( \int_0^T \left\| \frac{(y_0 - x_0)}{T} \right\|^p dt \right)^{\frac{1}{p}} + \left( \int_0^T \|(T-t)v(t)\|^p dt \right)^{\frac{1}{p}} \end{aligned} \quad (16)$$

$$= \left( \int_0^T \left\| \frac{(y_0 - x_0)}{T} \right\|^p dt \right)^{\frac{1}{p}} + \left( \int_0^T \|(T-t)\|^p \|v(t)\|^p dt \right)^{\frac{1}{p}} \quad (17)$$

Looking at (2) and (5), the inequality (17) become

$$\begin{aligned} \left( \int_0^T \|\omega(t)\|^p dt \right)^{\frac{1}{p}} &\leq \left( \int_0^T \left\| \frac{(y_0 - x_0)}{T} \right\|^p dt \right)^{\frac{1}{p}} + (T-t)\sigma = \frac{(\|y_0 - x_0\|)}{T} \left( \int_0^T dt \right)^{\frac{1}{p}} + (T-t)\sigma \\ &= \frac{(\|y_0 - x_0\|)}{T} T^{\frac{1}{p}} + (T-t)\sigma = \frac{(\|y_0 - x_0\|)}{T^{\frac{1}{q}}} + (T-t)\sigma \end{aligned} \quad (18)$$

Substituting the time defined in (12) into (18), we have

$$\left( \int_0^T \|\omega(t)\|^p dt \right)^{\frac{1}{p}} \leq (\|y_0 - x_0\|) \times \left( \frac{\rho}{2\|y_0 - x_0\|\sigma} \right) + \left( \left[ \frac{2\|y_0 - x_0\|\sigma}{\rho} \right]^q - t \right) \sigma \quad (19)$$

$$\leq \frac{\rho}{2\sigma} + \left( 2^q \cdot \frac{\rho}{2^{q+1}\sigma} - t \right) \sigma \leq \frac{\rho}{2\sigma} + \left( \frac{\rho}{2\sigma} \right) \sigma \leq \frac{\rho}{2} + \frac{\rho}{2} = \rho < \gamma, \quad (20)$$

(20) follows from the hypothesis of Theorem 3.1. Hence the strategy is admissible.

Next is to show pursuit is possible when the dummy pursuer uses the time defined in Theorem 3.1 above. Solution of  $\dot{\eta}(t)$  is given by

$$\eta(T) = x_0 + \int_0^T \omega(s) ds$$

using the strategy (15) we have

$$\begin{aligned} \eta(T) &= x_0 + \int_0^T \left( \frac{(y_0 - x_0)}{T} + (T-t)v(t) \right) ds = x_0 + \int_0^T \frac{(y_0 - x_0)}{T} ds + \int_0^T (T-t)v(s) ds \\ &= x_0 + \frac{(y_0 - x_0)}{T} \int_0^T ds + \int_0^T (T-t)v(s) ds = y_0 + \int_0^T (T-t)v(s) ds = y(T) \end{aligned}$$

We define the strategy of the real pursuer using the strategy of the dummy pursuer as follows

$$u(t) = \frac{\rho}{\gamma} \omega(t).$$

The strategy is indeed admissible, as shown below

$$\left( \int_0^T \|u(t)\|^p dt \right)^{\frac{1}{p}} = \left( \int_0^T \left\| \frac{\rho}{\gamma} \omega(t) \right\|^p dt \right)^{\frac{1}{p}} = \frac{\rho}{\gamma} \left( \int_0^T \|\omega(t)\|^p dt \right)^{\frac{1}{p}} \leq \frac{\rho}{\gamma} \gamma = \rho.$$

For the completion of pursuit, we show that  $\|y(\tau) - x(\tau)\| \leq l$ . Taking the distance between the players position, we have

$$\begin{aligned} \|y(T) - x(T)\| &= \|\eta(T) - x(T)\| = \left\| x_0 + \int_0^T \omega(s) ds - x_0 - \int_0^T u(s) ds \right\| \\ &= \left\| \int_0^T \omega(s) ds - \int_0^T \frac{\rho}{\gamma} \omega(t) ds \right\| = \left\| \int_0^T \omega(s) ds - \int_0^T \frac{\rho}{\gamma} \omega(t) ds \right\| = \left( 1 - \frac{\rho}{\gamma} \right) \left\| \int_0^T \omega(s) ds \right\| \\ &\leq \left( 1 - \frac{\rho}{\gamma} \right) \int_0^T \|\omega(s)\| ds \leq \left( 1 - \frac{\rho}{\gamma} \right) \left[ \left( \int_0^T 1^q dt \right)^{\frac{1}{q}} \left( \int_0^T \|\omega(s)\|^p ds \right)^{\frac{1}{p}} \right] \\ &= \left( 1 - \frac{\rho}{\gamma} \right) \gamma \sqrt[q]{T} = (\gamma - \rho) \sqrt[q]{T} = l. \end{aligned} \quad (21)$$

Hence pursuit is completed at time  $T$  which is a guaranteed pursuit time.

### 3.2. L-Escape

In this section we present all possible conditions for the evasion to escape catch from the pursuer. This evasion require the use of l-escape idea, where evader's strategy will be constructed in such away that this condition  $\|y(\tau) - x(\tau)\| >$

$l$  is satisfied i.e at some time  $\tau$  the distance between position of the pursuer and evader is greater than some length  $l$ . This will be achieve with the aid of below Theorem.

**Theorem 3.2.** if  $5\rho < \gamma < \sigma$ , then evasion is possible in l-escape sense.

*Proof* We defined evaders strategy as follow

$$v(s) = \frac{1}{(T-t)} \left[ \frac{2(\gamma-\rho)}{\sqrt[p]{T}} \cdot \frac{e}{\|e\|} - \frac{(y_0-x_0)}{T} \cdot \frac{(y_0-x_0)}{\|(y_0-x_0)\|} + u(t) \right]$$

where  $e$  is a unit vector.

After construction of evader's strategy, we insure that the strategy satisfied the restriction imposed on it (the constraint), and its not beyond the resource of the evader. The process is shown below. Putting the strategy in the given constraint we have

$$\begin{aligned} \left( \int_0^T \|v(s)\|^p ds \right)^{\frac{1}{p}} &= \left( \int_0^T \frac{1}{\|(T-t)\|^p} \left\| \left( \frac{2(\gamma-\rho)}{\sqrt[p]{T}} \cdot \frac{e}{\|e\|} - \frac{(y_0-x_0)}{T} \cdot \frac{(y_0-x_0)}{\|(y_0-x_0)\|} \right) + u(s) \right\|^p ds \right)^{\frac{1}{p}} \\ &\leq \left( \int_0^T \frac{1}{\|(T-t)\|^p} \left\| \left( \frac{2(\gamma-\rho)}{\sqrt[p]{T}} \cdot \frac{e}{\|e\|} - \frac{(y_0-x_0)}{T} \cdot \frac{(y_0-x_0)}{\|(y_0-x_0)\|} \right) \right\|^p ds \right)^{\frac{1}{p}} + \left( \int_0^T \frac{1}{\|(T-t)\|^p} \|u(s)\|^p ds \right)^{\frac{1}{p}} \\ &= \left( \int_0^T \frac{1}{\|(T-t)\|^p} \left\| \left( \frac{2(\rho-\gamma)}{\sqrt[p]{T}} \cdot \frac{e}{\|e\|} + \frac{(y_0-x_0)}{T} \cdot \frac{(y_0-x_0)}{\|(y_0-x_0)\|} \right) \right\|^p ds \right)^{\frac{1}{p}} + \left( \int_0^T \frac{1}{\|(T-t)\|^p} \|u(s)\|^p ds \right)^{\frac{1}{p}} \\ &\leq \left( \int_0^T \frac{1}{\|(T-t)\|^p} \left\| \frac{2(\rho-\gamma)}{\sqrt[p]{T}} \cdot \frac{e}{\|e\|} \right\|^p ds \right)^{\frac{1}{p}} + \left( \int_0^T \frac{1}{\|(T-t)\|^p} \left\| \frac{(y_0-x_0)}{T} \cdot \frac{(y_0-x_0)}{\|(y_0-x_0)\|} \right\|^p ds \right)^{\frac{1}{p}} \\ &\quad + \left( \int_0^T \frac{1}{\|(T-t)\|^p} \|u(s)\|^p ds \right)^{\frac{1}{p}} \\ &= \left\| \frac{2(\rho-\gamma)}{\sqrt[p]{T}} \right\| \left( \int_0^T \frac{1}{\|(T-t)\|^p} ds \right)^{\frac{1}{p}} + \frac{\|y_0-x_0\|}{T} \left( \int_0^T \frac{1}{\|(T-t)\|^p} ds \right)^{\frac{1}{p}} + \left( \int_0^T \frac{1}{\|(T-t)\|^p} \|u(s)\|^p ds \right)^{\frac{1}{p}} \end{aligned}$$

Using Lemma 3.1 number 2&3 we obtain the following

$$\left( \int_0^T \|v(s)\|^p ds \right)^{\frac{1}{p}} < \frac{2(\rho-\gamma)}{\sqrt[p]{T}} \cdot \frac{1}{\sqrt[p]{T}} + \frac{\|y_0-x_0\|}{T} \cdot \frac{1}{\sqrt[p]{T}} + 2\rho = \frac{2(\rho-\gamma)}{T} + \frac{\|y_0-x_0\|}{T} \cdot \frac{1}{\sqrt[p]{T}} + 2\rho,$$

at this point we insert the time defined in Theorem 3.1 and as a result we have

$$\begin{aligned} \left( \int_0^T \|v(s)\|^p ds \right)^{\frac{1}{p}} &\leq \frac{2(\rho-\gamma)}{T} + \frac{\|y_0-x_0\|}{T} \cdot \frac{1}{\sqrt[q]{\left( \frac{2\|y_0-x_0\|\sigma}{\rho} \right)^q}} + 2\rho \\ &= \frac{2(\rho-\gamma)}{T} + \frac{\rho}{2T\sigma} + 2\rho \leq 2(\rho-\gamma) + \rho + 2\rho = 5\rho - 2\gamma < \sigma. \end{aligned}$$

This indicate that the constructed evader's strategy is admissible, because it satisfied the constraint and everything is less than resource of the evader i.e  $\sigma$ . Next is to show evasion under the stated condition in Theorem 3.2. This will be done in l-escape sense as follows.

*Proof* Taking the difference between the position of the players we have

$$\|y(T) - x(T)\| = \left\| y_0 + \int_0^T (T-t)v(s)ds - x_0 - \int_0^T u(s)ds \right\| = \left\| y_0 - x_0 + \int_0^T ((T-t)v(s) - u(s)) ds \right\|,$$

inserting the admissible evader's strategy, we end of having

$$\begin{aligned}
\|y(T) - x(T)\| &= \left\| y_0 - x_0 + \int_0^T \left( \frac{2(\gamma - \rho)}{\sqrt[q]{T}} \cdot \frac{e}{\|e\|} - \frac{(y_0 - x_0)}{T} \cdot \frac{(y_0 - x_0)}{\|(y_0 - x_0)\|} + u(s) - u(s) \right) ds \right\| \\
&= \left\| y_0 - x_0 - \int_0^T \left( \frac{2(\rho - \gamma)}{\sqrt[q]{T}} \cdot \frac{e}{\|e\|} + \frac{(y_0 - x_0)}{T} \cdot \frac{(y_0 - x_0)}{\|(y_0 - x_0)\|} \right) ds \right\| \\
&\geq \|y_0 - x_0\| - \left\| \int_0^T \left( \frac{2(\rho - \gamma)}{\sqrt[q]{T}} \cdot \frac{e}{\|e\|} + \frac{(y_0 - x_0)}{T} \cdot \frac{(y_0 - x_0)}{\|(y_0 - x_0)\|} \right) ds \right\| \\
&\geq \|y_0 - x_0\| - \left( \left\| \int_0^T \frac{2(\rho - \gamma)}{\sqrt[q]{T}} \cdot \frac{e}{\|e\|} ds \right\| + \left\| \int_0^T \frac{(y_0 - x_0)}{T} \cdot \frac{(y_0 - x_0)}{\|(y_0 - x_0)\|} ds \right\| \right) \\
&\geq \|y_0 - x_0\| - \int_0^T \left\| \frac{2(\rho - \gamma)}{\sqrt[q]{T}} \cdot \frac{e}{\|e\|} \right\| ds - \int_0^T \left\| \frac{(y_0 - x_0)}{T} \cdot \frac{(y_0 - x_0)}{\|(y_0 - x_0)\|} \right\| ds \\
&= \|y_0 - x_0\| - \frac{2(\rho - \gamma)}{\sqrt[q]{T}} T - \|(y_0 - x_0)\| = -2(\rho - \gamma)T^{\frac{1}{q}} = 2(\gamma - \rho)T^{\frac{1}{q}} > l.
\end{aligned}$$

This guarantee evasion, since the distance between the players (pursuer and evader) exceeded  $l$  at some time  $\tau \in [0, T]$ , which further clearly show that  $T$  is an optimal pursuit time.

## 4. Numerical Example

Consider a differential game described by (1) in  $\mathbb{R}^3$ , with the initial positions of the pursuer and evader are given by  $x_0 = (0, 0, 0)$  and  $y_0 = (1, 0, 0)$  respectively. Let  $p = 3$  and  $q = \frac{3}{2}$ , the control function of the pursuer and evader are subject to

$$\int_0^T \|u(s)\|^3 ds \leq 10^3 \text{ and } \int_0^T \|v(t)\|^3 dt \leq 2^3, \quad (22)$$

respectively.

Therefore

$$\begin{aligned}
\|y_0 - x_0\|^3 &= \|(1, 0, 0) - (0, 0, 0)\|^3 \\
&= \|(1, 0, 0)\|^3 = 1. \\
\frac{\|y_0 - x_0\|^p}{\rho^p} &= \frac{1}{10^3}
\end{aligned}$$

And

$$\frac{\rho}{2^{q+1}\sigma} = \frac{10}{2^{\frac{5}{2}} \cdot 2} = \frac{10}{11.3} = 0.9. \quad (23)$$

Thus

$$\frac{\|y_0 - x_0\|^p}{\rho^p} < \frac{\rho}{2^{q+1}\sigma}.$$

Since the hypothesis of the Theorem is satisfied, then pursuit can be completed.

## 5. Conclusion

We have studied a differential game of one pursuer and one evader, where the motion of the pursuer and evader is

described by first order and second order differential equations respectively. Control functions of pursuers and evader are subject to integral constraints. We defined time  $T$  and proved that it is indeed an optimal pursuit time. Pursuit and evasion were shown in  $l$ -catch and  $l$ -escape sense respectively. We further gave an example to demonstrate our result.

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## Abbreviations

L.H.S Left Hand Side

## Conflicts of Interest

The authors declare no conflicts of interest.

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