

A Fractional Parabolic-elliptic Chemotaxis-fluid System

Kerui Jiang*, Zuhan Liu, Ling Zhou

College of Mathematical Science, Yangzhou University, Yangzhou, China

Email address:

krjiang@yzu.edu.cn (Kerui Jiang), zhliu@yzu.edu.cn (Zuhan Liu), zhoul@yzu.edu.cn (Ling Zhou)

*Corresponding author

To cite this article:

Kerui Jiang, Zuhan Liu, Ling Zhou. (2025). A Fractional Parabolic-elliptic Chemotaxis-fluid System. *Applied and Computational Mathematics*, 14(3), 120-163. <https://doi.org/10.11648/j.acm.20251403.13>

Received: 17 April 2025; **Accepted:** 3 May 2025; **Published:** 13 June 2025

Abstract: The fractional diffusion can describe possible singularities and other anomalies, and the non-local system constructed by the fractional chemotaxis-fluid equations can reveal more colorful, realistic and effective biological phenomena. The theoretical research on the fractional chemotaxis-fluid system is still at the initial stage, and new methods and technologies are needed to overcome the difficulties brought by the fractional operator, which has important scientific value. As an exploration, a fractional parabolic-elliptic chemotaxis system coupled with the Navier-Stokes equation is considered in the whole space \mathbb{R}^2 in this paper. Our main objective is to investigate the existence and asymptotic behavior of solutions to system (1). By the aid of L^p - L^q -estimates of the fractional heat semigroup and Kato-Ponce commutator estimate, we show the existence of local solution for large initial data and the existence of global mild solution to system (1) for small initial data in the scale invariant class demonstrating that $n_0 \in L^1 \cap L^{\frac{1}{\alpha}}(\mathbb{R}^2)$ and $u_0 \in L^2_\sigma \cap L^{\frac{2}{2-\alpha-1}}(\mathbb{R}^2)$. Furthermore, under the rest state of the fluid motion, by studying moments $w_\eta = \int_{\mathbb{R}^2} |x|^\eta n(x) dx$ of lower order $\eta \in (1, 2\alpha)$, we establish a blow-up criterion of solution to system (1) with the help of the proof by contradiction.

Keywords: Fractional Chemotaxis, Fractional Navier-Stokes, Blow-up, Mild Solution

1. Introduction

In this paper, for $\alpha \in (0, 1)$, we consider the following fractional chemotaxis-fluid model in \mathbb{R}^2

$$\begin{cases} \partial_t n + (-\Delta)^\alpha n = -u \cdot \nabla n - \nabla \cdot (n \nabla v), & x \in \mathbb{R}^2, \\ -\Delta v = -\gamma v + n, & x \in \mathbb{R}^2, \\ \partial_t u + (-\Delta)^\alpha u = -(u \cdot \nabla) u - \nabla P - |n|^\beta f, & x \in \mathbb{R}^2, \\ \nabla \cdot u = 0, & x \in \mathbb{R}^2, \\ n|_{t=0} = n_0, u|_{t=0} = u_0, & x \in \mathbb{R}^2 \end{cases} \quad (1)$$

with $0 < t < T$, where $n = n(x, t)$ and $v = v(x, t)$ denote the unknown density of amoebae and the unknown concentration of chemical attractant, respectively. The unknown vector $u = u(x, t) = (u_1(x, t), u_2(x, t))$ stands for the fluid velocity field, and the scalar function $P = P(x, t)$ for pressure of the fluid which can be recovered from n and u via Calderón-Zygmund operators. In addition, the gravitational potential ϕ is generally supposed to be sufficiently smooth given function.

Here $n_0 = n_0(x)$, $v_0 = v_0(x)$ and $u_0 = (u_{0,1}(x), u_{0,2}(x))$ denote the given initial data. We denote the Fourier transform of the function z by \hat{z} , then fractional Laplacian is defined by

$$(-\Delta)^\alpha z = |\xi|^{2\alpha} \hat{z}. \quad (2)$$

In the remainder of this introduction, we present the known results on chemotaxis systems in more details.

1.1. Classical Chemotaxis-fluid System

For the parabolic-parabolic case, Tuval et alia (*et al.*) [29] proposed the following chemotaxis system coupled with fluid

$$\begin{cases} \partial_t n + u \cdot \nabla n = \eta \Delta n - \nabla \cdot (\chi(c)n \nabla c), & x \in \Omega, t > 0, \\ \partial_t c + u \cdot \nabla c = \mu \Delta c - n \kappa(c), & x \in \Omega, t > 0, \\ \partial_t u + \varepsilon(u \cdot \nabla)u + \nabla P = \nu \Delta u - n \nabla \phi, & x \in \Omega, t > 0, \\ \nabla \cdot u = 0, & x \in \Omega, t > 0, \\ n|_{t=0} = n_0, c|_{t=0} = c_0, u|_{t=0} = u_0, & x \in \Omega. \end{cases} \quad (3)$$

Positive constants η , μ and ν are the corresponding diffusion coefficients for the amoebae, oxygen and fluid. The constant ε is 0 or 1. Here we present some related work about system (3) over the whole space $\Omega = \mathbb{R}^N$ or in a bounded domain $\Omega \subset \mathbb{R}^N$.

When $\Omega = \mathbb{R}^N$, in 2010, Duan, Lorz and Markowich [14] investigated (3) under certain conditions. Precisely, for the chemotaxis-Navier-Stokes system ($\varepsilon = 1$), they obtained the global existence and rates of convergence on classical solutions to (3) near constant states in \mathbb{R}^3 under the assumption that initial data are small enough, and also proved the global existence of weak solutions to the chemotaxis-Stokes system ($\varepsilon = 0$) in \mathbb{R}^2 under smallness assumption on either $\nabla \phi$ or initial data for oxygen concentration. Tan and Zhang [32] obtained the optimal convergence rates of classical solutions to (3) for small initial perturbation around constant states under the assumption initial data belong to $\dot{H}^{-\sigma} \cap H^N$ ($N \geq 3$, $0 \leq \sigma < \frac{3}{2}$) in 2014. In 2016, Li *et al.* [28] proved the global existence of solutions to (3) in \mathbb{R}^2 . They also proved the approximate solution actually belongs to a class of $C^{2+\theta, 1+\theta/2}$ ($\theta \in (0, 1)$). Recently, Chae, Kang *et al.* [9] presented a new localized regularity criterion and established the temporal decays of classical solutions under the assumption that initial mass of biological cell density is sufficiently small.

When $\Omega \subset \mathbb{R}^N$, In 2010, Lorz [27] showed the local existence of weak solutions to (3) in a bounded domain in \mathbb{R}^N ($N = 2, 3$) with no-flux boundary condition and in \mathbb{R}^2 in the case of inhomogeneous Dirichlet conditions for oxygen. When $\Omega \subset \mathbb{R}^3$, Winkler [36] obtained the global existence, eventually smoothness of weak solutions to (3) and their asymptotic behavior in 2017. Recently, the chemotaxis-competition system coupled with the Navier-Stokes system

was also studied in [11, 22] when $N = 2, 3$.

Moreover, when the absorption term $-n\kappa(c)$ in the second equation is replaced by the zero-order term $-c + n$, people considered the following equation

$$\begin{cases} \partial_t n + u \cdot \nabla n = \Delta n - \nabla \cdot (n \nabla c) + rn - \mu n^2, & x \in \Omega, t > 0, \\ \partial_t c + u \cdot \nabla c = \Delta c - c + n, & x \in \Omega, t > 0, \\ \partial_t u + \varepsilon(u \cdot \nabla)u + \nabla P = \Delta u - n \nabla \phi + g, & x \in \Omega, t > 0, \\ \nabla \cdot u = 0, & x \in \Omega, t > 0, \\ n|_{t=0} = n_0, c|_{t=0} = c_0, u|_{t=0} = u_0, & x \in \Omega, \end{cases} \quad (4)$$

where $r \geq 0$ and $\mu > 0$. Here $g = g(x, t)$ is prescribed function. Tao and Winkler [31] showed that the global existence of classical solution in $\Omega \subset \mathbb{R}^3$ ($\varepsilon = 0$) under the explicit condition $\mu \geq 23$ and suitable regularity assumptions on the initial data. When $\Omega \subset \mathbb{R}^2$, [30] proved that if the initial data are sufficiently smooth, then there admits a global bounded classical solution to (4). [35] proved that (4) admits at least one globally defined solution in an appropriate generalized sense in $\Omega \subset \mathbb{R}^3$. Recently, Winkler [33] showed that each of these generalized solutions becomes eventually smooth and classical under some conditions on ρ , μ and ultimate smallness on g . When without logistic source, [34] obtained a globally defined generalized solution to (4) under a smallness condition on the total initial population mass m_0 in $\Omega \subset \mathbb{R}^2$, he further got the eventual regularity and convergence in the large time limit.

To the best of our knowledge, there are few results about the parabolic-elliptic Keller-Segel equations coupled with Stokes equations. In 2012, Lorz [26] investigated the following system in \mathbb{R}^N ($N = 2, 3$)

$$\begin{cases} \partial_t n + u \cdot \nabla n = \Delta n - \nabla \cdot (\chi n \nabla c), & x \in \mathbb{R}^N, t > 0, \\ u \cdot \nabla c = \Delta c + n - a_1 c, & x \in \mathbb{R}^N, t > 0, \\ a_2 \partial_t u = \eta \Delta u - \nabla P - n \nabla \phi, & x \in \mathbb{R}^N, t > 0, \\ \nabla \cdot u = 0, & x \in \mathbb{R}^N, t > 0, \\ n|_{t=0} = n_0, c|_{t=0} = c_0, u|_{t=0} = u_0, & x \in \mathbb{R}^N. \end{cases} \quad (5)$$

The constant $a_1 \geq 0$ measures self-degradation of the chemical and the constants $a_2 \geq 0$, $\eta > 0$ determine the evolution undergone by u . The author showed global in-time existence of solutions for small initial mass in \mathbb{R}^2 . In \mathbb{R}^3 the

author established global existence assuming that the initial $L^{3/2}$ -norm is small. In 2019, Kozono *et al.* [24] considered the following Keller-Segel system, coupled with the Navier-Stokes equations in \mathbb{R}^2

$$\begin{cases} \partial_t n + u \cdot \nabla n = \Delta n - \nabla \cdot (\chi n \nabla c), & x \in \mathbb{R}^2, 0 < t < T, \\ \gamma v = \Delta v + n, & x \in \mathbb{R}^2, 0 < t < T, \\ \partial_t u + u \cdot \nabla u = \Delta u - \nabla \pi - |n|^\beta f, & x \in \mathbb{R}^2, 0 < t < T, \\ \nabla \cdot u = 0, & x \in \mathbb{R}^2, 0 < t < T, \\ n|_{t=0} = n_0, u|_{t=0} = u_0, & x \in \mathbb{R}^2 \end{cases} \quad (6)$$

with $1 < \beta \leq \frac{3}{2}$ and $\gamma \geq 0$. The authors showed the existence of a local mild solution for large initial data and a global mild solution for small initial data in the scale invariant class demonstrating that $n_0 \in L^1(\mathbb{R}^2)$ and $u_0 \in L^2_\sigma(\mathbb{R}^2)$. Indeed, when β is 1, the theorem is difficult to prove however, when β is greater than 1, the theorem is easier to prove despite the initial appearance to contrary.

1.2. Prior Results for the Fractional Chemotaxis System

The classical chemotaxis-fluid models are used with huge success to model phenomena across all scientific and engineering disciplines. However, across an equally wide swath, there exists situations in which the classical models fail

to adequately model observed phenomena, or are not the best available model for that purpose. On the other hand, in many situations, nonlocal models [1, 10, 12, 13, 15, 16] that account for interaction occurring at a distance have been shown to more faithfully and effectively model observed phenomena that involve possible singularities and other anomalies. Escudero [15] extended the results for classical models driven by Brownian motion to those driven by α -stable Lévy process. The author also proved that solutions to a simplification of parabolic-elliptic version are globally bounded in time in one dimension.

Now let us recall certain analytical results of the fractional chemotaxis systems. In 2011, Wu and Zheng [38] studied the nonlinear equations as follows

$$\begin{cases} \partial_t n + (-\Delta)^s n + \nabla \cdot (n \nabla v) = 0, & x \in \mathbb{R}^N, t > 0, \\ \tau \partial_t v + (-\Delta)^s v = n, & x \in \mathbb{R}^N, t > 0, \\ n(x, 0) = n_0(x), v(x, 0) = v_0(x), & x \in \mathbb{R}^N, \end{cases} \quad (7)$$

where n and v have the same fractional diffusion order. They got a local well-posedness result and a global well-posedness result with small initial data in the critical Fourier-Herz spaces. For the case of different fractional diffusion orders, in 2015, Burczak and Granero-Belinchón [8] studied a doubly parabolic Keller-Segel system with logistic source in one spatial dimension. The authors obtained several local and global well-posedness results for the subcritical and critical cases, the latter need certain smallness assumptions. So far, there have been many theoretical development on the study of the fractional chemotaxis systems, for more details one can see [2, 6, 7, 23, 37].

As far as we know, there are few results on the global existence of solutions for the fractional chemotaxis-fluid systems, especially the blow-up criterion. As an exploration, we firstly investigate the global existence and uniqueness of mild solution to (1) in the whole space \mathbb{R}^2 . Secondly, we shall show a blow-up criterion under the rest state of the fluid motion.

Before state our main results, first we introduce several notations for convenience. For a function space $X(\mathbb{R}^2)$, we will often abbreviate it to X . For an interval $I \subseteq \mathbb{R}$ and a Banach space X with the norm $\|\cdot\|_X$, we denote by $L^\theta(I; X)$ the set of all functions $u : I \rightarrow X$ so that $\|u\|_{L^\theta(I; X)} := (\int_I \|u(t)\|_X^\theta dt)^\frac{1}{\theta} < \infty$ for $1 \leq \theta < \infty$ and $\|u\|_{L^\infty(I; X)} := \text{ess sup}_{t \in I} \|u(t)\|_X < \infty$. We denote by $C(I; X)$ the set of all continuous functions on I with values in X .

Here and what in follows, we denote by $C^\infty_{0,\sigma}(\mathbb{R}^2)$ the set of all $C^\infty(\mathbb{R}^2)$ vector functions $\xi = (\xi_1, \xi_2)$ with compact support of \mathbb{R}^2 , such that $\text{div} \xi = 0$. $L^r_\sigma(\mathbb{R}^2)$ is the closure

of $C^\infty_{0,\sigma}(\mathbb{R}^2)$ with respect to the L^r -norm $\|\cdot\|_{L^r(\mathbb{R}^2)}$; (\cdot, \cdot) denotes the duality pairing between $L^r(\mathbb{R}^2)$ and $L^{r'}(\mathbb{R}^2)$, where $\frac{1}{r} + \frac{1}{r'} = 1$. $L^r(\mathbb{R}^2)$ represents the usual vector-valued L^r -space over \mathbb{R}^2 , $1 < r < \infty$. $H^{1,r}_\sigma(\mathbb{R}^2)$ denotes the closure of $C^\infty_{0,\sigma}(\mathbb{R}^2)$ with respect to the norm:

$$\|\xi\|_{H^{1,r}_\sigma(\mathbb{R}^2)} = \|\xi\|_{L^r(\mathbb{R}^2)} + \|\nabla \xi\|_{L^r(\mathbb{R}^2)},$$

where $\nabla \xi = (\frac{\partial \xi_i}{\partial x_j}; i, j = 1, 2)$.

We will introduce the weak L^p spaces here. Let $1 \leq p < \infty$, we set:

$$\begin{aligned} L^p_w(\mathbb{R}^2) &:= \{g \in L^1_{loc}(\mathbb{R}^2); \|g\|_{L^p_w(\mathbb{R}^2)} \\ &= \sup_{R>0} R \mu\{x \in \mathbb{R}^2; |g(x)| > R\}^\frac{1}{p} < \infty\}. \end{aligned}$$

We will use the homogeneous generalized Sobolev spaces $\dot{H}^{s,p}(\mathbb{R}^2)$, the generalized Sobolev spaces $H^{s,p}(\mathbb{R}^2)$ and the Besov spaces $B^s_{1,r}(\mathbb{R}^2)$.

We denote by $P = \{P_{jk}\}_{j,k=1,2}$ the projection operator onto the solenoidal vector fields with the expression

$$P_{jk} = \delta_{jk} + R_j R_k \quad (R_j \equiv \frac{\partial}{\partial x_j} (-\Delta)^{-1/2} : \text{Riesz operator})$$

for $j, k = 1, 2$.

It is well-known that $L^r_\sigma(\mathbb{R}^2) = PL^r(\mathbb{R}^2)$ for all $1 < r < \infty$.

2. Main Results

In this section, we will state the main result of the existence, uniqueness and blow-up criterion of mild solutions of (1). In view of [5], the original equation (1) can be reduced to the following integral equation:

$$\begin{cases} n(t) = K_t * n_0 - \int_0^t K_{t-\tau} * (u \cdot \nabla n)(\tau) d\tau \\ \quad - \int_0^t K_{t-\tau} * (n \nabla (-\Delta + \gamma)^{-1} n)(\tau) d\tau, \\ u(t) = K_t * u_0 \\ \quad - \int_0^t \nabla \cdot K_{t-\tau} * P(u \otimes u + |n|^\beta f)(\tau) d\tau \end{cases} \quad (8)$$

hold for $0 < t < T$, where K_t is the fractional heat kernel.

We define the norm $\|\cdot\|_Y$ by:

$$\|g\|_Y := \begin{cases} \|g\|_{L_w^\rho(\mathbb{R}^2)} = \|g\|_{L_w^{\frac{2}{4\alpha-1-2\alpha\beta}}(\mathbb{R}^2)} & (1 < \beta < \frac{3}{2}), \\ \|g\|_{L^\infty(\mathbb{R}^2)} & \beta = \frac{3}{2}, \end{cases} \quad (10)$$

and denote G_q and G_p by:

$$G_q := \sup \left\{ \sup_{0 < t < \infty} t^{\frac{1}{\alpha}(\alpha - \frac{1}{q})} \|K_t * g\|_{L^q(\mathbb{R}^2)}; \|g\|_{L^1 \cap L^{\frac{1}{\alpha}}(\mathbb{R}^2)} \leq 1 \right\}, \quad (11)$$

and

$$G_p := \sup \left\{ \sup_{0 < t < \infty} t^{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p})} \|K_t * g\|_{L^p(\mathbb{R}^2)}; \|g\|_{L_\sigma^2 \cap L^{\frac{2}{2\alpha-1}}(\mathbb{R}^2)} \leq 1 \right\}. \quad (12)$$

Definition 2.1. Let the Assumption hold. The measurable functions (n, u) on $\mathbb{R}^2 \times [0, T]$ is called a mild solution of (1)

Theorem 2.1. Let the Assumption hold. Assume that (q, p) satisfies the following condition:

$$\begin{cases} \frac{2\beta}{2\alpha\beta+1-2\alpha} < q < 2 & \text{if } 1 < \beta < \frac{3}{2}, \\ \frac{2\beta}{2\alpha\beta+1-2\alpha} \leq q < 2 & \text{if } \beta = \frac{3}{2}, \\ \frac{q}{\alpha q - 1} \leq p < \infty. \end{cases} \quad (13)$$

Then we have the following properties (i)-(iii):

(i) (local existence). For every $n_0 \in L^1 \cap L^{\frac{1}{\alpha}}(\mathbb{R}^2)$ and $u_0 \in L_\sigma^2 \cap L^{\frac{2}{2\alpha-1}}(\mathbb{R}^2)$, there exist T_0 depending only on $\beta, q, p, \|f\|_Y, n_0$ and u_0 , and a mild solution (n, u) of (1) on $[0, T_0]$ in the class of $M_{q,p}(0, T_0)$ with the properties:

$$\begin{aligned} n &\in C([0, T]; L^1 \cap L^{\frac{1}{\alpha}}(\mathbb{R}^2)), \\ u &\in C([0, T]; L_\sigma^2 \cap L^{\frac{2}{2\alpha-1}}(\mathbb{R}^2)), \end{aligned}$$

and

$$\begin{aligned} \lim_{t \rightarrow 0^+} t^{\frac{1}{\alpha}(\alpha - \frac{1}{q})} \|n(t)\|_{L^q(\mathbb{R}^2)} &= 0, \\ \lim_{t \rightarrow 0^+} t^{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p})} \|u(t)\|_{L^p(\mathbb{R}^2)} &= 0, \end{aligned} \quad (14)$$

Assumption 2.1.

(i) Let $1 < \beta \leq \frac{3}{2}$, $\frac{1}{2} < \alpha < 1$, $\gamma \geq 0$,

$$\frac{4\alpha}{2+\alpha} < q < 2 \text{ and } \frac{2}{\alpha} < p < \infty;$$

(ii) The initial data $(n_0, u_0) \in L^1 \cap L^{\frac{1}{\alpha}}(\mathbb{R}^2) \times L_\sigma^2 \cap L^{\frac{2}{2\alpha-1}}(\mathbb{R}^2)$;

(iii) The external force f satisfies $f \in L_w^\rho(\mathbb{R}^2)$ with ρ so that :

$$\rho = \begin{cases} \frac{2}{4\alpha-1-2\alpha\beta} & (1 < \beta < \frac{3}{2}), \\ \infty & \beta = \frac{3}{2}. \end{cases} \quad (9)$$

In addition, for $\beta = \frac{3}{2}$, the external force f belongs to $L^\infty(\mathbb{R}^2)$.

on $[0, T]$ in the class $M_{q,p}(0, T)$ if:

- (i) $n \in C([0, T]; L^1 \cap L^{\frac{1}{\alpha}}(\mathbb{R}^2))$
and $t^{\frac{1}{\alpha}(\alpha - \frac{1}{q})} n \in BC([0, T]; L^q(\mathbb{R}^2))$,
- (ii) $u \in C([0, T]; L_\sigma^2 \cap L^{\frac{2}{2\alpha-1}}(\mathbb{R}^2))$
and $t^{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p})} u \in BC([0, T]; L^p(\mathbb{R}^2))$,
- (iii) (n, u) satisfies (8) on $(0, T)$.

We first establish the existence of local solution for the arbitrary data (n_0, u_0, f) and the existence of global solution for the small data (n_0, u_0, f) . Besides, we also characterize the behavior of local solutions near the blow-up time.

where $\|\cdot\|_Y$ and G_q are given by (10) and (11), respectively.

(ii) (uniqueness). The mild solution (n, u) of (1) on $[0, T]$ in the class $M_{q,p}(0, T)$ is unique.

(iii) (global existence). For $1 < \beta < \frac{3}{2}$ (for $\beta = \frac{3}{2}$), there is a positive number δ depending only on β, q, p and $\|f\|_{L_w^{\frac{2}{4\alpha-1-2\alpha\beta}}(\mathbb{R}^2)}$ (respectively, β, q, p and $\|f\|_{L^\infty(\mathbb{R}^2)}$) if (n_0, u_0) satisfies:

$$\|n_0\|_{L^1 \cap L^{\frac{1}{\alpha}}(\mathbb{R}^2)} + \|u_0\|_{L_\sigma^2 \cap L^{\frac{2}{2\alpha-1}}(\mathbb{R}^2)} < \delta,$$

then there exists a unique mild solution (n, u) of (1) on $[0, \infty)$ in the class of $M_{q,p}(0, \infty)$ with the property that:

$$\begin{aligned} t^{\frac{1}{\alpha}(\alpha-\frac{1}{q})}n &\in BC([0, \infty); L^q(\mathbb{R}^2)), \\ t^{\frac{1}{\alpha}(\alpha-\frac{1}{2}-\frac{1}{p})}u &\in BC([0, \infty); L^p(\mathbb{R}^2)). \end{aligned}$$

(iv) (local existence time). If $n_0 \in L^1 \cap L^{\hat{q}}(\mathbb{R}^2)$ for $1 < \hat{q} < q$, and $u_0 \in L^2 \cap L^{\hat{p}}(\mathbb{R}^2)$ for some $2 < \hat{p} < p$, then T_0 in Theorem 2.1-(i) is given by:

$$T_0 = \min \left\{ \left(\frac{C_1}{\|n_0\|_{L^{\hat{q}}(\mathbb{R}^2)}} \right)^{\frac{\alpha\hat{q}}{\alpha\hat{q}-1}}, \left(\frac{C_2}{\|u_0\|_{L^{\hat{p}}(\mathbb{R}^2)}} \right)^{\frac{2\alpha\hat{p}}{(2\alpha-1)\hat{p}-2}} \right\}, \quad (15)$$

where $C_1 = C_1(\alpha, \beta, q, p, \hat{q}, \|f\|_Y)$ and $C_2 = C_2(\alpha, \beta, q, p, \hat{p}, \|f\|_Y)$ with (10).

Moreover, if the maximal existence time T_{max} of the above mild solution (n, u) is finite, then it follows that:

$$\|n(t)\|_{L^r(\mathbb{R}^2)} \geq C_*^1(T_{max} - t)^{-\frac{1}{\alpha}(1-\frac{1}{r})} \quad \text{for all } 1 < r \leq q, \quad (16)$$

or

$$\|u(t)\|_{L^{\tilde{r}}(\mathbb{R}^2)} \geq C_*^2(T_{max} - t)^{-\frac{1}{\alpha}(\alpha-\frac{1}{2}-\frac{1}{\tilde{r}})} \quad \text{for all } 1 < \tilde{r} \leq p \quad (17)$$

for all $0 < t < T_{max}$, where $C_*^1 = C_*^1(\alpha, \beta, q, p, r, \|f\|_Y)$ and $C_*^2 = C_*^2(\alpha, \beta, q, p, \tilde{r}, \|f\|_Y)$.

Besides that, there are a positive number ε_0 and a sequence $\{t_j\}_{j=1}^\infty$ with $t_1 \leq t_2 \leq \dots \leq T_{max}$ and with $\lim_{j \rightarrow \infty} t_j = T_{max}$ such that for every $j = 1, 2, \dots$, there are positive integers $k, l \geq j$ satisfying:

$$\|n(t_k) - n(t_l)\|_{L^1(\mathbb{R}^2)} > \varepsilon_0 \quad \text{or} \quad \|u(t_k) - u(t_l)\|_{L^2(\mathbb{R}^2)} > \varepsilon_0. \quad (18)$$

(v) (time continuity). Let (q_*, p_*) hold that:

$$\max \left\{ \frac{2q}{3q-2}, \frac{p}{p-1} \right\} \leq q_* \leq \infty \quad \text{and} \quad \frac{p}{p-1} \leq p_* \leq \infty. \quad (19)$$

In addition to (19), assume that (q_*, p_*) satisfies either one of the following conditions (1)-(4):

- (1) $1 \leq q_* \leq \frac{2\beta}{2\alpha\beta+1-2\alpha}, \frac{2q}{2\beta+q(4\alpha-1-2\alpha\beta)} \leq p_* \leq 2;$
- (2) $\frac{2\beta}{2\alpha\beta+1-2\alpha} < q_* \leq \infty, 1 < p_* < \infty;$
- (3) $\frac{2\beta}{2\alpha\beta+1-2\alpha} < q_* < \infty, p_* = \infty;$
- (4) $q_* = \infty, p_* = \infty.$

If in addition, we suppose that:

$$n_0 \in L^1 \cap L^{q_*}(\mathbb{R}^2), \quad u_0 \in L^2 \cap L^{p_*}(\mathbb{R}^2) \quad \text{and} \quad f \in L^p(\mathbb{R}^2), \quad (20)$$

then there exist \hat{T} and T_0 so that the mild solution (n, u) to (1) satisfies:

$$n \in C([0, \hat{T}); L^1 \cap L^{q_*}(\mathbb{R}^2)) \quad \text{and} \quad u \in C([0, \hat{T}); L^2 \cap L^{p_*}(\mathbb{R}^2)), \quad (21)$$

where \hat{T} is independent of $\alpha, \beta, q, p, q_*, p_*$.

(vi) (additional regularity). Assume that (q_*, p_*) satisfies (19) and:

$$\begin{aligned} \frac{2\beta}{2\alpha\beta + 1 - 2\alpha} &\leq q_* < 2, \quad \frac{q_*}{q_* - 1} \leq p_* < \infty, \\ 0 < k &< \frac{2}{\alpha} \left(\beta \left(\alpha - \frac{1}{q_*} \right) - \left(\alpha - \frac{1}{2} - \frac{1}{p_*} \right) \right). \end{aligned}$$

Moreover, either one of the conditions (1)-(4) in (v) holds. Suppose that (n, u) satisfies (20) and (21). Then there exist \tilde{T} and \hat{T} so that the solution (n, u) to (1) satisfies:

$$\begin{aligned} t^{\frac{k}{2\alpha}} n &\in BC([0, \tilde{T}); \dot{H}^{k, q_*}(\mathbb{R}^2)), \\ t^{\frac{k}{2\alpha}} u &\in BC([0, \tilde{T}); \dot{H}^{k, p_*}(\mathbb{R}^2)), \end{aligned} \quad (22)$$

where \tilde{T} is independent of $\alpha, \beta, q, p, q_*, p_*, k, \|f\|_\rho$.

Next, we shall discuss a possibility of the existence of a blow-up solution (n, v) under the assumption on the fluid velocity field u . This gives a blow-up criterion on the fluid velocity field u .

Let us define T_{max} by:

$$\begin{aligned} T_{max} &:= \sup\{T > 0; (n, u) \text{ is in the class of } M_{q,p}(0, T), \\ &\quad n \in C([0, T]; L^1 \cap L^{q_*}(\mathbb{R}^2)), \\ &\quad u \in C([0, T]; L^2 \cap L^{p_*}(\mathbb{R}^2))\}. \end{aligned} \quad (23)$$

Theorem 2.2. Let the assumption hold with $\gamma = 0$. We assume that (q, p) satisfies

$$\frac{2\beta}{2\alpha\beta + 1 - 2\alpha} \leq q < 2 \text{ and } \frac{q}{q - 1} \leq p < \infty. \quad (24)$$

Suppose that (q_*, p_*) satisfies (19) and either one of the following conditions (i) or (ii):

(i) $\frac{\beta}{\alpha\beta + 1 - 2\alpha} \leq q_* \leq \infty$ and $\frac{q_*^2}{q_*^2 - 3q_* + 2} < p_* < \infty$;

(ii) $\frac{\beta}{\alpha\beta + 1 - 2\alpha} < q_* < \infty$ and $p_* = \infty$.

Assume that (n_0, u_0, f) satisfies the following additional conditions (a), (b) and (c):

(a) $n_0 \in L^1 \cap L^{q_*}(\mathbb{R}^2)$ with $n_0 \in L^1(\mathbb{R}^2, (1 + |x|^\eta)dx)$ for some $\eta \in (1, 2\alpha)$, and $n_0(x) \geq 0$ for all almost $x \in \mathbb{R}^2$;

(b) $u_0 \in L_\sigma^2 \cap L^{q_*} \cap B_{2,s}^{2(1-\frac{1}{s})}(\mathbb{R}^2)$ for some $1 < s < \infty$;

(c) $f \in L^\rho(\mathbb{R}^2)$.

Let (n, u) be a mild solution of (1) on $[0, T)$ in the class $M_{q,p}(0, T_{max})$ satisfying $u \in L^2(0, T_{max}; L^\infty(\mathbb{R}^2))$. If we choose a suitable positive constant M_0 such that:

$$M_0 > \int_{\mathbb{R}^2} u_0(x) \varphi(x) dx + C_4 \|u\|_{L^2(0, T_{max}; L^\infty(\mathbb{R}^2))}^2, \quad (25)$$

then we deduce

$$T_{max} < \frac{M_0}{C_5 M_0^2 - C_2 M_0 - \varepsilon \|n_0\|_{L^1(\mathbb{R}^2)}^2},$$

and that the mild solution (n, u) on $[0, T_{max})$ behaves like:

$$\limsup_{t \rightarrow T_{max}} (\|n(t)\|_{L^{q_*}(\mathbb{R}^2)} + \|u(t)\|_{L^p(\mathbb{R}^2)} + \|u(t)\|_{L^{p_*}(\mathbb{R}^2)}) = \infty. \quad (26)$$

Here

$$\varphi(x) = (1 + |x|^2)^{\frac{\eta}{2}} - 1, \quad (27)$$

and C_i are independent of t .

Remark 2.1. In the proof of Theorems 2.1-2.2, we assume $\alpha \in (\frac{1}{2}, 1)$. However, it is still unknown whether the assumption of α is optimal or not. In particular, when the term $|n|^\beta f$ is replaced by $n \nabla c$, Guo and He [20] showed that, if the total mass of the cells is strictly less than 8π , classical solutions exist for any finite time. It is worth mentioning that whether the fractional chemotaxis-fluid system has a similar critical mass phenomenon needs to be further explored.

Remark 2.2. From [24], we can see that the range of β guarantees the time local solvability of mild solutions. Actually, when

β is greater than 1, Theorem 2.1 is easier to prove, and when β is 1, the proof of the theorem is still an open problem.

Remark 2.3. The method of proving the nonexistence of global in time nonnegative and nontrivial solutions, used in [24], consists in the study the evolution of the second moment of a solution $w_2(t) = \int_{\mathbb{R}^2} |x|^2 n(x, t) dx$ and to show that $w_2(t)$ vanishes for some $t > 0$ through suitable differential inequalities. However, the second moment of a typical solution to an evolution equation with fractional Laplacian cannot be finite, see [4]. Hence, in Theorem 2.2, motivated by [3], we show the blowup of solution system (1) by studying moments of lower order $\eta \in (1, 2\alpha)$

$$w_\eta(t) = \int_{\mathbb{R}^2} \varphi(x) n(x, t) dx. \quad (28)$$

The rest of this paper is arranged as follows. In Section 3, we give some inequalities and decay estimates which will be fundamental to the arguments. In Section 4, we derive a local solution for large initial data and a global solution for small initial data to system (1). Especially, we employ the L^p - L^q -estimates of the fractional heat semigroup and Kato-Ponce commutator estimate to overcome the difficulties caused by the fractional diffusion $(-\Delta)^\alpha$. We organize the proof of the blow-up criterion of a mild solution in Section 5.

3. Preliminary

In this section, we introduce some lemmas which will be used in the next section. Firstly, we consider the problem

$$\begin{cases} \partial_t n + (-\Delta)^s n = 0, & x \in \mathbb{R}^N, t > 0, \\ n(x, 0) = \phi(x), & x \in \mathbb{R}^N. \end{cases} \quad (29)$$

Using Fourier transformation, the solution of problem (29) can be written as

$$n(t) = K_t * \phi. \quad (30)$$

Here, $K_t(x)$ is the fractional heat kernel, which is denoted by

$$K_t(x) := t^{-\frac{N}{2s}} K(t^{-\frac{1}{2s}} x) \text{ with } K(x) := (2\pi)^{-N} \int_{\mathbb{R}^N} e^{ix \cdot \xi} e^{-|\xi|^{2s}} d\xi. \quad (31)$$

We give some inequalities for $K(x)$.

Lemma 3.1. $K(x)$ satisfies the following properties:

$$|K(x)| \leq C(1 + |x|)^{-N-2s}, \quad K(x) \in L^p(\mathbb{R}^N), \quad \forall p \in [1, \infty] \quad (32)$$

and

$$|\nabla K(x)| \leq C(1 + |x|)^{-N-1}, \quad \nabla K(x) \in L^p(\mathbb{R}^N), \quad \forall p \in [1, \infty], \quad (33)$$

where C is a positive constant.

Proof. One can refer to Lemma 3.1, Lemma 3.2 and Remark 3.1.1 in [19, Chapter 3] for instance.

Next, we give some decay estimates of (29) which will be used in the proof of Theorem 2.1.

Lemma 3.2. Let $N \geq 0$ be an integer. Assume $n = K_t * \phi$ is a solution of problem (29). Then the following estimates hold for any $1 < q \leq p < \infty$ and multi-index α

$$\|K_t * \phi\|_{L^p(\mathbb{R}^N)} \leq C t^{-\frac{N}{2s}(\frac{1}{q} - \frac{1}{p})} \|\phi\|_{L^q(\mathbb{R}^N)}, \quad \forall t > 0 \quad (34)$$

and

$$\|\nabla^\alpha (K_t * \phi)\|_{L^p(\mathbb{R}^N)} \leq C t^{-\frac{1}{2s}(|\alpha| + \frac{N}{q} - \frac{N}{p})} \|\phi\|_{L^q(\mathbb{R}^N)}, \quad \forall t > 0, \quad (35)$$

where α is a multiindex with $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and $\nabla^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$. Here, C is a positive constant, not depending on t .

Proof. See Lemma 2.4 and Lemma 2.6 in [37].

Lemma 3.3. Let $N \geq 0$ be an integer. Assume $n = K_t * \phi$ is a solution of problem (29). Then the following estimates hold for any $1 < q \leq p < \infty$ and multi-index α

$$\|K_t * \phi\|_{L^p(\mathbb{R}^N)} \leq C t^{-\frac{N}{2s}(\frac{1}{q} - \frac{1}{p})} \|\phi\|_{L^q_w(\mathbb{R}^N)}, \quad \forall t > 0 \quad (36)$$

and

$$\|\nabla^\alpha(K_t * \phi)\|_{L^p(\mathbb{R}^N)} \leq C t^{-\frac{1}{2s}(|\alpha| + \frac{N}{q} - \frac{N}{p})} \|\phi\|_{L_w^q(\mathbb{R}^N)}, \quad \forall t > 0, \quad (37)$$

where α is a multiindex with $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and $\nabla^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$. Here, C is a positive constant, not depending on t .

Proof. Let us recall the generalized Young's inequality. Assume

$$A(x) \in L^p(\mathbb{R}^N), \quad \phi(x) \in L_w^q(\mathbb{R}^N) \quad (38)$$

and

$$1 < q < \rho', \quad \frac{1}{\rho} + \frac{1}{\rho'} = 1. \quad (39)$$

Then

$$\|A(x) * \phi(x)\|_{L^p(\mathbb{R}^N)} \leq \|A(x)\|_{L^p(\mathbb{R}^N)} \|\phi(x)\|_{L_w^q(\mathbb{R}^N)} \quad (40)$$

where

$$\frac{1}{\rho'} = 1 - \frac{1}{\rho} = \frac{1}{q} - \frac{1}{p}. \quad (41)$$

From formulas (31) and (32), changing the variable $\xi = t^{-\frac{1}{2s}}x$, we have

$$\|K_t(x)\|_{L^p(\mathbb{R}^N)} = t^{-\frac{N}{2s}} \left(\int_{\mathbb{R}^N} K^\rho(t^{-\frac{1}{2s}}x) dx \right)^{\frac{1}{\rho}} t^{-\frac{N}{2s}} (t^{\frac{1}{2s}})^{\frac{N}{\rho}} \left(\int_{\mathbb{R}^N} K^\rho(\xi) d\xi \right)^{\frac{1}{\rho}} \leq C t^{-\frac{N}{2s}(1-\frac{1}{\rho})}. \quad (42)$$

Inserting (41) into (42) yields

$$\|K_t(x)\|_{L^p(\mathbb{R}^N)} \leq C t^{-\frac{N}{2s}(\frac{1}{q} - \frac{1}{p})}. \quad (43)$$

Combining (40) with (43), we obtain (36). For multiindex α , based on (30), we have

$$\nabla^\alpha(n(t, x)) = \nabla^\alpha(K_t(x) * \phi(x)) = \nabla^\alpha(K_t(x)) * \phi(x). \quad (44)$$

Similarly, from formulas (31) and (33), we can observe that

$$\|\nabla^\alpha K_t(x)\|_{L^p(\mathbb{R}^N)} \leq C t^{-\frac{1}{2s}(|\alpha| + \frac{N}{q} - \frac{N}{p})}. \quad (45)$$

Plugging (40), (45) into (44), we can easily get (37).

Lemma 3.4. (Kato-Ponce commutator estimate.) Let $s > 0$ and $1 < p < \infty$. Then

$$\|(-\Delta)^{s/2}(fg)\|_{L^p} \leq C \|f\|_{L^{p_1}} \|(-\Delta)^{s/2}g\|_{L^{p_2}} + C \|(-\Delta)^{s/2}f\|_{L^{p_3}} \|g\|_{L^{p_4}} \quad (46)$$

with $1 < p_1, p_4 \leq \infty$ and $1 < p_2, p_3 < \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$.

Proof. See [21] for more details.

We shall estimate the non-linear term $n\nabla(-\Delta + \gamma)^{-1}n$ in (1).

Lemma 3.5. Let $\frac{4\alpha}{2+\alpha} < q < 2$. If $n, w \in L^q(\mathbb{R}^2)$, then it holds that $n\nabla(-\Delta + \gamma)^{-1}w \in L^r(\mathbb{R}^2)$ for r with $\frac{1}{r} = \frac{2}{q} - \frac{1}{2}$ with estimate

$$\|n\nabla(-\Delta + \gamma)^{-1}w\|_{L^r(\mathbb{R}^2)} \leq C \|n\|_{L^q(\mathbb{R}^2)} \|w\|_{L^q(\mathbb{R}^2)}, \quad (47)$$

where C is a positive constant independent of n and w .

Proof. See Lemma 3.1 in [25] for more details.

We investigate the following Navier-Stokes equation and demonstrate the $L^s(0, T; L^\infty)$ -estimate:

$$\begin{cases} \partial_t u + (-\Delta)^\alpha u + (u \cdot \nabla)u + \nabla P = g, & x \in \mathbb{R}^2, 0 < t < T, \\ \nabla \cdot u = 0, & x \in \mathbb{R}^2, 0 < t < T, \\ u|_{t=0} = u_0, & x \in \mathbb{R}^2. \end{cases} \quad (48)$$

Lemma 3.6. Let $1 < s < \infty$. For every $u_0 \in L_\sigma^2 \cap B_{2,s}^{2(1-\frac{1}{s})}(\mathbb{R}^2)$ and $g \in L^s(0, T; L^2(\mathbb{R}^2))$, there exists a unique weak solution u of (48) in the Leray-Hopf class,

$$u \in L^\infty(0, T; L_\sigma^2(\mathbb{R}^2)) \cap L^2(0, T; H_\sigma^{\alpha,2}(\mathbb{R}^2)).$$

Moreover, u has an additional regularity that:

$$u \in H^{1,s}(0, T; L^2_\sigma(\mathbb{R}^2)) \cap L^s(0, T; H^{2\alpha,2}_\sigma(\mathbb{R}^2))$$

with the estimates as follows:

$$\begin{aligned} \text{(i)} \quad & \|u(t)\|_{L^2(\mathbb{R}^2)}^2 + 2 \int_0^t \|\nabla^\alpha u(\tau)\|_{L^2(\mathbb{R}^2)}^2 \\ & \leq \|u_0\|_{L^2(\mathbb{R}^2)}^2 + (1 + \|u_0\|_{L^2(\mathbb{R}^2)}^2) \\ & \quad \exp\left(\int_0^T \|g(\tau)\|_{L^2(\mathbb{R}^2)} d\tau\right) \int_0^t \|g(\tau)\|_{L^2(\mathbb{R}^2)} d\tau; \\ \text{(ii)} \quad & \|u_t\|_{L^s(0,T;L^2(\mathbb{R}^2))} + \sum_{k=0}^2 \|\nabla^{k\alpha} u\|_{L^s(0,T;L^2(\mathbb{R}^2))} \leq C, \end{aligned}$$

where $C = C(s, \|u_0\|_{L^2(\mathbb{R}^2)}, \|u_0\|_{B_{2,s}^{2(1-\frac{1}{s})}(\mathbb{R}^2)}, \|g\|_{L^s(0,T;L^2(\mathbb{R}^2))})$.

Proof. In order to prove Lemma 3.6, we investigate the following perturbed Stokes equation:

$$\begin{cases} \partial_t w + (-\Delta)^\alpha w + (h \cdot \nabla)w + \nabla P = g, \\ \quad \quad \quad x \in \mathbb{R}^2, 0 < t < T, \\ \nabla \cdot w = 0, x \in \mathbb{R}^2, 0 < t < T, \\ w|_{t=0} = w_0, x \in \mathbb{R}^2, \end{cases} \quad (49)$$

with $\operatorname{div} h = 0$.

A key lemma is introduced as follows:

Lemma 3.7. Let $h \in L^{\frac{\theta}{\alpha}}(0, T; L^r_\sigma(\mathbb{R}^2))$ for $\frac{2\alpha}{\theta} + \frac{2}{r} = 1$ with $2 < r < \infty$. For every $w_0 \in L^2_\sigma \cap B_{2,s}^{2(1-\frac{1}{s})}(\mathbb{R}^2)$ and every $g \in L^s(0, T; L^2(\mathbb{R}^2))$ with $1 < s < \theta$, there is a unique weak solution w of (49) in the Leray-Hopf class,

$$w \in L^\infty(0, T; L^2_\sigma(\mathbb{R}^2)) \cap L^2(0, T; H^{\alpha,2}_\sigma(\mathbb{R}^2)).$$

Furthermore, the weak solution w also has an additional regularity:

$$w \in H^{1,s}(0, T; L^2_\sigma(\mathbb{R}^2)) \cap L^s(0, T; H^{2\alpha,2}_\sigma(\mathbb{R}^2))$$

with the following estimates:

$$\begin{aligned} \text{(i)} \quad & \|w(t)\|_{L^2(\mathbb{R}^2)}^2 + 2 \int_0^t \|\nabla^\alpha w(\tau)\|_{L^2(\mathbb{R}^2)}^2 \\ & \leq \|w_0\|_{L^2(\mathbb{R}^2)}^2 + (1 + \|w_0\|_{L^2(\mathbb{R}^2)}^2 + \int_0^T \|g(\tau)\|_{L^2(\mathbb{R}^2)} d\tau) \\ & \quad \exp\left(\int_0^T \|g(\tau)\|_{L^2(\mathbb{R}^2)} d\tau\right) \int_0^t \|g(\tau)\|_{L^2(\mathbb{R}^2)} d\tau; \\ \text{(ii)} \quad & \|w_t\|_{L^s(0,T;L^2(\mathbb{R}^2))} + \sum_{k=0}^2 \|\nabla^{k\alpha} w\|_{L^s(0,T;L^2(\mathbb{R}^2))} \\ & \leq C(\|w_0\|_{B_{2,s}^{2(1-\frac{1}{s})}(\mathbb{R}^2)} + \|g\|_{L^s(0,T;L^2(\mathbb{R}^2))}) \\ & \text{for all } 0 < t < T, \text{ where } C = C(s, \|h\|_{L^{\frac{\theta}{\alpha}}(0,T;L^r(\mathbb{R}^2))}). \end{aligned}$$

Proof. The use of variable $w(t) = e^t v(t)$ is necessary for solving (49), and it is evident that (49) is equivalent to the following problem:

$$\begin{cases} \partial_t v + \tilde{A}v + P(h \cdot \nabla v) = \tilde{g} \text{ in } L^2_\sigma \text{ for a.a. } t \in (0, T), \\ v|_{t=0} = w_0, \end{cases} \quad (50)$$

where $\tilde{A} = A + 1$ with $A = P(-\Delta)^\alpha$ denoting the Stokes operator in L^2_σ and $\tilde{g} = e^{-t}Pg$.

We shall solve (50) by the following successive approximation $\{v_j\}_{j=1}^\infty$:

$$\begin{cases} \partial_t v_j + \tilde{A}v_j + P(h \cdot \nabla v_{j-1}) = \tilde{g} \text{ in } L^2_\sigma \\ \text{for a.a. } t \in (0, T), \\ v_j|_{t=0} = w_0, \end{cases} \quad (51)$$

for $j = 2, 3, \dots$. Here v_1 is expressed as:

$$v_1(t) = e^{-t} (K_t * w_0)(x) + \int_0^t e^{-(t-\tau)} (K_{t-\tau} * \tilde{g})(\tau) d\tau.$$

Since $w_0 \in B_{2,s}^{2(1-\frac{1}{s})}(\mathbb{R}^2)$ and $g \in L^s(0, T; L^2(\mathbb{R}^2))$, we find that:

$$v_1 \in H^{1,s}(0, T; L_\sigma^2(\mathbb{R}^2)) \cap L^s(0, T; H^{2\alpha,2}(\mathbb{R}^2)).$$

Suppose that $v_{j-1} \in H^{1,s}(0, T; L_\sigma^2(\mathbb{R}^2)) \cap L^s(0, T; H^{2\alpha,2}(\mathbb{R}^2))$. Take s_1 and r_1 so that:

$$\frac{1}{s_1} = \frac{1}{s} - \frac{\alpha}{\theta}, \quad \frac{1}{r_1} = \frac{1}{2} - \frac{1}{r}$$

and using the Hölder inequality, we find that:

$$\begin{aligned} & \|P(h \cdot \nabla v_{j-1})\|_{L^s(0,T;L_\sigma^2(\mathbb{R}^2))} \\ & \leq \|h\|_{L_\alpha^\theta(0,T;L^r(\mathbb{R}^2))} \|\nabla v_{j-1}\|_{L^{s_1}(0,T;L^{r_1}(\mathbb{R}^2))}. \end{aligned}$$

Since it holds that

$$\begin{aligned} \frac{2}{s_1} + \frac{2}{r_1} &= \frac{2}{s} - \frac{2\alpha}{\theta} + 1 - \frac{2}{r} = \frac{2}{s} + \frac{2}{2} - \left(\frac{2\alpha}{\theta} + \frac{2}{r}\right) \\ &= \frac{2}{s} + \frac{2}{2} - 1, \end{aligned}$$

it follows from Giga-Sohr [17, Lemma 3.2] that:

$$\begin{aligned} & \|\nabla v_{j-1}\|_{L^{s_1}(0,T;L^{r_1}(\mathbb{R}^2))} \\ & \leq C \left(\left\| \frac{\partial v_{j-1}}{\partial t} \right\|_{L^s(0,T;L_\sigma^2(\mathbb{R}^2))} + \|\tilde{A}v_{j-1}\|_{L^s(0,T;L_\sigma^2(\mathbb{R}^2))} \right. \\ & \quad \left. + \|w_0\|_{B_{2,s}^{2(1-\frac{1}{s})}(\mathbb{R}^2)} \right) \end{aligned}$$

with $C = C(s_1, r_1, s)$ independent of T , which yields that:

$$\begin{aligned} & \|P(h \cdot \nabla v_{j-1})\|_{L^s(0,T;L_\sigma^2(\mathbb{R}^2))} \\ & \leq C \|h\|_{L_\alpha^\theta(0,T;L^r(\mathbb{R}^2))} \left(\left\| \frac{\partial v_{j-1}}{\partial t} \right\|_{L^s(0,T;L_\sigma^2(\mathbb{R}^2))} \right. \\ & \quad \left. + \|\tilde{A}v_{j-1}\|_{L^s(0,T;L_\sigma^2(\mathbb{R}^2))} + \|w_0\|_{B_{2,s}^{2(1-\frac{1}{s})}(\mathbb{R}^2)} \right). \end{aligned}$$

Employing the maximal regularity theorem of \tilde{A} , we deduce a unique solution v_{j+1} of (51) with the estimate:

$$\begin{aligned} & \left\| \frac{\partial v_j}{\partial t} \right\|_{L^s(0,T;L_\sigma^2(\mathbb{R}^2))} + \|\tilde{A}v_j\|_{L^s(0,T;L_\sigma^2(\mathbb{R}^2))} \\ & \leq C \left(\|w_0\|_{B_{2,s}^{2(1-\frac{1}{s})}(\mathbb{R}^2)} + \|g\|_{L^s(0,T;L_\sigma^2(\mathbb{R}^2))} \right. \\ & \quad + \|h\|_{L_\alpha^\theta(0,T;L^r(\mathbb{R}^2))} \left(\left\| \frac{\partial v_{j-1}}{\partial t} \right\|_{L^s(0,T;L_\sigma^2(\mathbb{R}^2))} \right. \\ & \quad \left. \left. + \|\tilde{A}v_{j-1}\|_{L^s(0,T;L_\sigma^2(\mathbb{R}^2))} + \|w_0\|_{B_{2,s}^{2(1-\frac{1}{s})}(\mathbb{R}^2)} \right) \right) \end{aligned} \tag{52}$$

with $C = C(\theta, r, s)$ is independent of T .

Define

$$X^s(T) := H^{1,s}(0, T; L_\sigma^2(\mathbb{R}^2)) \cap L^s(0, T; H^{2\alpha,2}(\mathbb{R}^2))$$

with the norm:

$$\begin{aligned} \|v\|_{X^s(T)} &:= \|v_t\|_{L^s(0,T;L_\sigma^2(\mathbb{R}^2))} + \|\tilde{A}v\|_{L^s(0,T;L_\sigma^2(\mathbb{R}^2))} \\ &\quad + \|v_0\|_{B_{2,s}^{2(1-\frac{1}{s})}(\mathbb{R}^2)}. \end{aligned}$$

From (52), we arrive at:

$$\begin{aligned} \|v_j\|_{X^s(T)} &\leq C(\|w_0\|_{B_{2,s}^{2(1-\frac{1}{s})}(\mathbb{R}^2)} + \|g\|_{L^s(0,T;L_\sigma^2(\mathbb{R}^2))}) \\ &\quad + \|h\|_{L_{\frac{\theta}{\alpha}}(0,T;L^r(\mathbb{R}^2))} \|v_{j-1}\|_{X^s(T)} \end{aligned} \quad (53)$$

with $C = C(\theta, r, s)$ independent of T . Now choosing T_* sufficiently small so that:

$$C\|h\|_{L_{\frac{\theta}{\alpha}}(t,T_*;L^r(\mathbb{R}^2))} \leq \frac{1}{2} \quad \text{for all } 0 < t < T. \quad (54)$$

Applying (53), we deduce

$$\begin{aligned} \|v_j\|_{X^s(T_*)} &\leq C(\|w_0\|_{B_{2,s}^{2(1-\frac{1}{s})}(\mathbb{R}^2)} + \|g\|_{L^s(0,T_*;L_\sigma^2(\mathbb{R}^2))}) \\ &\quad + \frac{1}{2} \|v_{j-1}\|_{X^s(T_*)}, \end{aligned}$$

which yield that:

$$\|v_j\|_{X^s(T_*)} \leq 2C(\|w_0\|_{B_{2,s}^{2(1-\frac{1}{s})}(\mathbb{R}^2)} + \|g\|_{L^s(0,T_*;L_\sigma^2(\mathbb{R}^2))})$$

for all $j = 2, 3, \dots$. Next, we define \bar{v}_j by:

$$\bar{v}_j := v_j(t) - v_{j-1}(t), \quad v_0(t) \equiv 0$$

for $0 < t \leq T_*$ and it holds that:

$$\begin{cases} \partial_t \bar{v}_j + \tilde{A} \bar{v}_j + P(h \cdot \nabla \bar{v}_{j-1}) = 0 & \text{in } L_\sigma^2 \\ \text{for a.a. } t \in (0, T_*), \\ \bar{v}_j|_{t=0} = 0. \end{cases} \quad (55)$$

Similarly, based on the condition (54), we find that:

$$\|\bar{v}_j\|_{X^s(T_*)} \leq \frac{1}{2} \|\bar{v}_{j-1}\|_{X^s(T_*)} \quad \text{for } j = 2, 3, \dots,$$

which yields that:

$$\|\bar{v}_j\|_{X^s(T_*)} \leq \left(\frac{1}{2}\right)^{j-1} (\|w_0\|_{B_{2,s}^{2(1-\frac{1}{s})}(\mathbb{R}^2)} + \|\tilde{g}\|_{L^s(0,T_*;L_\sigma^2(\mathbb{R}^2))})$$

for $j = 2, 3, \dots$. Since $v_j = \sum_{k=1}^{j-1} \bar{v}_k$, there exists $v \in X^s(T_*)$ such that $v_j \rightarrow v$ in $X^s(T_*)$ as $j \rightarrow \infty$. It is easy to obtain that v is a unique solution of (50) on $[0, T_*]$. According to the trace theorem, we have that $v(T_*) \in B_{2,s}^{2(1-\frac{1}{s})} \cap L_\sigma^2(\mathbb{R}^2)$, and therefore the condition (54) makes it possible to deduce the solution v of (50) on $[T_*, 2T_*]$ with w_0

replaced by $v(T_*)$. Repeating such an argument on $[2T_*, \infty)$, within finitely many steps, we obtain the solution $v \in X^s(T)$ of (50) with the estimates (ii) of Lemma 3.7.

We indicate that the solution w of (49) belongs to the Leray-Hopf class $L^\infty(0, T; L_\sigma^2) \cap L^2(0, T; H_{\sigma}^{\alpha,2}(\mathbb{R}^2))$ with the estimate (i) of Lemma 3.7. Since $w \in H^{1,s}(0, T; L_\sigma^2(\mathbb{R}^2))$, applying (49) and integration by parts, we find that:

$$\frac{1}{2} \frac{d}{dt} \|w(t)\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla^\alpha w(t)\|_{L^2(\mathbb{R}^2)}^2 + (h \cdot \nabla w, w) = (g, w) \quad (56)$$

for almost all $t \in (0, T)$. Since $h(t) \in L^r(\mathbb{R}^2)$, $\nabla w(t) \in L^{r_1}(\mathbb{R}^2)$, $w(t) \in L^2(\mathbb{R}^2)$ with $\operatorname{div} w(t) = 0$ for almost all $t \in (0, T)$, and since $\frac{1}{r} + \frac{1}{r_1} + \frac{1}{2} = 1$, by the standard density argument of $C_{0,\sigma}^\infty$, we deduce $(h(t) \cdot \nabla w(t), w(t)) = 0$ for almost all $t \in (0, T)$. Therefore, (56) implies that:

$$\frac{1}{2} \|w(t)\|_{L^2(\mathbb{R}^2)}^2 + \int_0^t \|\nabla^\alpha w(\tau)\|_{L^2(\mathbb{R}^2)}^2 d\tau = \frac{1}{2} \|w_0\|_{L^2(\mathbb{R}^2)}^2 + \int_0^t (g(\tau), w(\tau))$$

for all $0 < t < T$. Now, the above energy identity directly leads to the desired estimate (i) of Lemma 3.7. Thus, Lemma 3.7 is proved.

Continuation of the proof of Lemma 3.6. We first construct a sequence $\{u_j\}_{j=1}^{\infty}$ of approximating solution to (48) by the following scheme:

$$\begin{cases} \partial_t u_j + Au_j + P(u_{j-1} \cdot \nabla u_j) = Pg & \text{in } L^2_{\sigma} \\ \text{for a.a. } t \in (0, T), \\ u_j|_{t=0} = u_0, \end{cases} \quad (57)$$

for all $j = 2, 3, \dots$, where the Stokes operator A defines $A = P(-\Delta)^{\alpha}$. Here u_1 is written as:

$$u_1(t) = e^{-tA}u_0 + \int_0^t e^{-(t-\tau)A}Pg(\tau)d\tau.$$

Since u_1 satisfies:

$$\begin{cases} \partial_t u_1 + Au_1 = Pg & \text{in } L^2_{\sigma} \text{ for a.a. } t \in (0, T), \\ u_1|_{t=0} = u_0, \end{cases} \quad (58)$$

it holds that:

$$\frac{1}{2} \frac{d}{dt} \|u_1(t)\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla^{\alpha} u_1(t)\|_{L^2(\mathbb{R}^2)}^2 = (u_1(t), g(t))$$

for $j = 2, 3, \dots$.

Since

$$\begin{aligned} |(u_1(t), g(t))| &\leq \|u_1(t)\|_{L^2(\mathbb{R}^2)} \|g(t)\|_{L^2(\mathbb{R}^2)} \\ &\leq \frac{1}{2} (1 + \|u_1(t)\|_{L^2(\mathbb{R}^2)}^2) \|g(t)\|_{L^2(\mathbb{R}^2)}, \end{aligned}$$

employing the Gronwall inequality, we find that:

$$1 + \|u_1(t)\|_{L^2(\mathbb{R}^2)}^2 \leq (1 + \|u_0\|_{L^2(\mathbb{R}^2)}^2) \exp\left(\int_0^t \|g(\tau)\|_{L^2(\mathbb{R}^2)} d\tau\right),$$

which yields

$$\begin{aligned} &\|u_1(t)\|_{L^2(\mathbb{R}^2)}^2 + 2 \int_0^t \|\nabla^{\alpha} u_1(\tau)\|_{L^2(\mathbb{R}^2)}^2 d\tau \\ &\leq \|u_0\|_{L^2(\mathbb{R}^2)}^2 + 2 \int_0^t |(u_1(\tau), g(\tau))| d\tau \\ &\leq \|u_0\|_{L^2(\mathbb{R}^2)}^2 + (1 + \|u_0\|_{L^2(\mathbb{R}^2)}^2 \exp(\int_0^T \|g(\tau)\|_{L^2(\mathbb{R}^2)} d\tau)) \\ &\quad \cdot \exp\left(\int_0^T \|g(\tau)\|_{L^2(\mathbb{R}^2)} d\tau\right) \int_0^t \|g(\tau)\|_{L^2(\mathbb{R}^2)} d\tau \end{aligned} \quad (59)$$

for all $0 < t < T$. Taking $2 < r < \infty$, $2 < \theta < \infty$ such that $\frac{2}{r} + \frac{2\alpha}{\theta} = 1$, we have $\theta(\frac{1}{\alpha} - \frac{2}{\alpha r}) = 2$. As a result, the Gagliardo-Nirenberg inequality states that:

$$\begin{aligned} &\int_0^T \|u_1(\tau)\|_{L^r(\mathbb{R}^2)}^{\theta} d\tau \\ &\leq C \int_0^T (\|u_1(\tau)\|_{L^2(\mathbb{R}^2)}^{1-\frac{1}{\alpha}+\frac{2}{\alpha r}} \|\nabla^{\alpha} u_1(\tau)\|_{L^2(\mathbb{R}^2)}^{(\frac{1}{\alpha}-\frac{2}{\alpha r})})^{\theta} d\tau \\ &\leq C \sup_{0 < t < T} \|u_1(\tau)\|_{L^2(\mathbb{R}^2)}^{(1-\frac{1}{\alpha}+\frac{2}{\alpha r})\theta} \int_0^T \|\nabla^{\alpha} u_1(\tau)\|_{L^2(\mathbb{R}^2)}^{\theta(\frac{1}{\alpha}-\frac{2}{\alpha r})} d\tau \\ &= C \sup_{0 < t < T} \|u_1(\tau)\|_{L^2(\mathbb{R}^2)}^{\theta-2} \int_0^T \|\nabla^{\alpha} u_1(\tau)\|_{L^2(\mathbb{R}^2)}^2 d\tau. \end{aligned} \quad (60)$$

Therefore, from (59), we have $u_1 \in L^\theta(0, T; L^r(\mathbb{R}^2))$ with the estimate:

$$\|u_1(\tau)\|_{L^\theta(0, T; L^r(\mathbb{R}^2))} \leq C,$$

where $C = C(r, \|u_0\|_{L^2(\mathbb{R}^2)}, \|g\|_{L^1(0, T; L^2(\mathbb{R}^2))})$. Then, it follows from the key lemma that there exists a unique weak solution u_2 of (48) in $L^\infty(0, T; L_\sigma^2(\mathbb{R}^2)) \cap L^2(0, T; H_\sigma^{\alpha, 2}(\mathbb{R}^2))$ with the estimates of Lemma 3.7 (i) and:

$$\begin{aligned} & \left\| \frac{\partial u_2}{\partial t} \right\|_{L^s(0, T; L^2(\mathbb{R}^2))} + \sum_{k=0}^2 \|\nabla^{k\alpha} u_2\|_{L^s(0, T; L^2(\mathbb{R}^2))} \\ & \leq C(\|u_0\|_{B_{2,s}^{2(1-\frac{1}{s})}(\mathbb{R}^2)} + \|g\|_{L^s(0, T; L^2(\mathbb{R}^2))}), \end{aligned}$$

the constant $C = C(s, r, T, \|u_0\|_{L^2(\mathbb{R}^2)}, \|g\|_{L^1(0, T; L^2(\mathbb{R}^2))})$. Assume that u_j is a solution of (48) with the following estimates:

$$\begin{aligned} & \|u_j(t)\|_{L^2(\mathbb{R}^2)}^2 + 2 \int_0^t \|\nabla^\alpha u_j\|_{L^2(\mathbb{R}^2)}^2 d\tau \\ & \leq \|u_0\|_{L^2(\mathbb{R}^2)}^2 + (1 + \|u_0\|_{L^2(\mathbb{R}^2)}^2) \\ & \cdot \exp\left(\int_0^T \|g(\tau)\|_{L^2(\mathbb{R}^2)} d\tau\right) \int_0^t \|g(\tau)\|_{L^2(\mathbb{R}^2)} d\tau \end{aligned} \quad (61)$$

and

$$\begin{aligned} & \left\| \frac{\partial u_j}{\partial t} \right\|_{L^s(0, T; L^2(\mathbb{R}^2))} + \sum_{k=0}^2 \|\nabla^{k\alpha} u_j\|_{L^s(0, T; L^2(\mathbb{R}^2))} \\ & \leq C(\|u_0\|_{B_{2,s}^{2(1-\frac{1}{s})}(\mathbb{R}^2)} + \|g\|_{L^s(0, T; L^2(\mathbb{R}^2))}) \end{aligned} \quad (62)$$

for all $0 < t < T$, where the constant $C = C(s, r, T, \|u_0\|_{L^2(\mathbb{R}^2)}, \|g\|_{L^1(0, T; L^2(\mathbb{R}^2))})$ is independent of j . From the key lemma and (61), with the aid of a similar estimate as in (60), it follows that there exists a unique weak solution u_{j+1} of (48) in the class of $L^\infty(0, T; L_\sigma^2(\mathbb{R}^2)) \cap$

$L^2(0, T; H_\sigma^{\alpha, 2}(\mathbb{R}^2))$ with the estimate (61) and (62) with j replaced by $j + 1$. This induction shows that the estimates (61) and (62) hold for all $j = 1, 2, 3, \dots$.

Thus, according to the compactness argument, there exists a subsequence of $\{u_j\}_{j=1}^\infty$, which we denote by $\{u_j\}_{j=1}^\infty$ itself for simplicity, and the function u :

$$\begin{aligned} u & \in L^\infty(0, T; L_\sigma^2(\mathbb{R}^2)) \cap L^2(0, T; H_\sigma^{\alpha, 2}(\mathbb{R}^2)) \cap H^{1,s}(0, T; L_\sigma^2(\mathbb{R}^2)) \\ & \cap L^s(0, T; H^{2\alpha, 2}(\mathbb{R}^2)) \end{aligned}$$

so that

$$\begin{aligned} u_j & \rightharpoonup u \text{ weakly } * \text{ in } L^\infty(0, T; L_\sigma^2(\mathbb{R}^2)); \\ u_j & \rightharpoonup u \text{ weakly } * \text{ in } L^2(0, T; H_\sigma^{\alpha, 2}(\mathbb{R}^2)); \\ \frac{\partial u_j}{\partial t} & \rightharpoonup \frac{\partial u}{\partial t} \text{ weakly } * \text{ in } L^s(0, T; L^2(\mathbb{R}^2)), \\ k & = 0, 1, 2 \text{ as } j \rightarrow \infty. \end{aligned}$$

We find that the limit u is the unique weak solution of (48) in the Leray-Hopf class having an additional regularity:

$$u \in H^{1,s}(0, T; L_\sigma^2(\mathbb{R}^2)) \cap L^s(0, T; H^{2\alpha, 2}(\mathbb{R}^2))$$

with the estimates of Lemma 3.6 (i) and (ii) by the standard argument. This proves Lemma 3.6.

4. Proof of Theorem 2.1

In this section, we shall show a local solution for large initial data and a global solution for small initial data. Concerning the fractional diffusion, we make full use of the Kato-Ponce commutator estimate.

4.1. Proof of Theorem 2.1-(i)

We first construct the mild solution in Definition 2.1 by successive approximation as follows:

$$\begin{cases} n_1(t) = K_t * n_0, \\ n_{j+1}(t) = K_t * n_0 - \int_0^t K_{t-\tau} * (u_j \cdot \nabla n_j)(\tau) d\tau - \int_0^t \nabla \cdot K_{t-\tau} * (n_j \nabla v_j)(\tau) d\tau, \end{cases} \quad (63)$$

$$v_j(t) = (-\Delta + \gamma)^{-1} n_j(t) \quad (64)$$

and

$$\begin{cases} u_1(t) = K_t * u_0, \\ u_{j+1}(t) = K_t * u_0 - \int_0^t K_{t-\tau} * P(u_j \otimes u_j + |n_j|^\beta f)(\tau) d\tau \end{cases} \quad (65)$$

for all $j = 1, 2, \dots$.

Step 1. We will first demonstrate that there are bounded sequences $\{N_j\}_{j=1}^\infty$ and $\{U_j\}_{j=1}^\infty$ such that:

$$\sup_{0 < t < T} t^{\frac{1}{\alpha}(\alpha - \frac{1}{q})} \|n_j(t)\|_{L^q(\mathbb{R}^2)} \leq N_j, \quad j = 1, 2, \dots, \quad (66)$$

$$\sup_{0 < t < T} t^{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p})} \|u_j(t)\|_{L^p(\mathbb{R}^2)} \leq U_j, \quad j = 1, 2, \dots, \quad (67)$$

Recalling on (66), since it holds that:

$$\begin{aligned} \|n_1(t)\|_{L^q(\mathbb{R}^2)} &= \|k_t * n_0\|_{L^q(\mathbb{R}^2)} \\ &\leq t^{-\frac{1}{\alpha}(\alpha-\frac{1}{q})} \sup_{0 < t < T} t^{\frac{1}{\alpha}(\alpha-\frac{1}{q})} \|n_j(t)\|_{L^q(\mathbb{R}^2)}, \quad 0 < t < T, \end{aligned}$$

we may consider:

$$N_1 := \sup_{0 < t < T} t^{\frac{1}{\alpha}(\alpha-\frac{1}{q})} \|k_t * n_0\|_{L^q(\mathbb{R}^2)}. \quad (68)$$

Suppose that (66) and (67) are true for j . By the assumption on p and q as in Theorem 2.1, we deduce:

$$1 - \frac{1}{\alpha p} - \frac{1}{2\alpha} > 0, \quad \frac{1}{\alpha} \left(\frac{1}{p} + \frac{1}{q} \right) + \frac{1}{2\alpha} - 1 > 0.$$

Then we arrive at:

$$\begin{aligned} &\left\| \int_0^t K_{t-\tau} * (u_j \cdot \nabla n_j)(\tau) d\tau \right\|_{L^q(\mathbb{R}^2)} \\ &= \left\| \int_0^t K_{t-\tau} * \nabla \cdot (u_j n_j)(\tau) d\tau \right\|_{L^q(\mathbb{R}^2)} \\ &\leq \int_0^t \|\nabla \cdot K_{t-\tau} * (u_j n_j)(\tau)\|_{L^q(\mathbb{R}^2)} d\tau \\ &\leq C \int_0^t (t-\tau)^{-\frac{1}{\alpha}(\frac{1}{q}+\frac{1}{p}-\frac{1}{q})-\frac{1}{2\alpha}} \|n_j(\tau)\|_{L^q(\mathbb{R}^2)} \|u_j(\tau)\|_{L^p(\mathbb{R}^2)} d\tau \\ &\leq CB \left(1 - \frac{1}{\alpha p} - \frac{1}{2\alpha}, \frac{1}{\alpha} \left(\frac{1}{p} + \frac{1}{q} \right) + \frac{1}{2\alpha} - 1 \right) N_j U_j t^{-\frac{1}{\alpha}(\alpha-\frac{1}{q})} \end{aligned} \quad (69)$$

for all $0 < t < T$, where $B(q, p)$ denotes the beta function and $C = C(q, p)$. Let $\frac{4\alpha}{2+\alpha} < q < 2$. Taking r such that $\frac{1}{r} = \frac{2}{q} - \frac{1}{2}$, we have $1 < r < q$. Based on Lemma 3.5, we observe:

$$\begin{aligned} &\left\| \int_0^t \nabla \cdot K_{t-\tau} * (n_j \nabla v_j)(\tau) d\tau \right\|_{L^q(\mathbb{R}^2)} \\ &\leq \int_0^t \|\nabla \cdot K_{t-\tau} * (n_j \nabla v_j)(\tau)\|_{L^q(\mathbb{R}^2)} d\tau \\ &\leq C \int_0^t (t-\tau)^{-\frac{1}{\alpha}(\frac{1}{r}-\frac{1}{q})-\frac{1}{2\alpha}} \|(n_j \nabla v_j)(\tau)\|_{L^r(\mathbb{R}^2)} d\tau \\ &\leq C \int_0^t (t-\tau)^{-\frac{1}{\alpha}(\frac{1}{q}-\frac{1}{2})-\frac{1}{2\alpha}} \|n_j(\tau)\|_{L^q(\mathbb{R}^2)}^2 d\tau \\ &\leq C \int_0^t (t-\tau)^{-\frac{1}{\alpha q}} \|n_j(\tau)\|_{L^q(\mathbb{R}^2)}^2 d\tau \\ &\leq CB \left(1 - \frac{1}{\alpha q}, \frac{2}{\alpha q} - 1 \right) N_j^2 t^{-\frac{1}{\alpha}(\alpha-\frac{1}{q})} \end{aligned} \quad (70)$$

for all $0 < t < T$, where $C = C(q)$.

Combining (63), (64) and (68)-(70), it follows that:

$$\begin{aligned} &\sup_{0 < t < T} t^{\frac{1}{\alpha}(\alpha-\frac{1}{q})} \|n_{j+1}(t)\|_{L^q(\mathbb{R}^2)} \\ &\leq N_1 + CB \left(1 - \frac{1}{\alpha p} - \frac{1}{2\alpha}, \frac{1}{\alpha} \left(\frac{1}{p} + \frac{1}{q} \right) + \frac{1}{2\alpha} - 1 \right) N_j U_j \\ &\quad + CB \left(1 - \frac{1}{\alpha q}, \frac{2}{\alpha q} - 1 \right) N_j^2, \quad j = 1, 2, \dots \end{aligned} \quad (71)$$

Next we deal with (67). Since it holds that:

$$\begin{aligned} \|u_1(t)\|_{L^p(\mathbb{R}^2)} &= \|K_t * u_0\|_{L^p(\mathbb{R}^2)} \\ &\leq t^{-\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p})} \cdot \sup_{0 < t < T} t^{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p})} \|K_t * u_0\|_{L^p(\mathbb{R}^2)} \end{aligned}$$

for all $0 < t < T$, we may consider:

$$U_1 := \sup_{0 < t < T} t^{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p})} \|K_t * u_0\|_{L^p(\mathbb{R}^2)}. \quad (72)$$

Assume that (66) and (67) are true for j , then we discover:

$$\begin{aligned} &\left\| \int_0^t \nabla \cdot K_{t-\tau} * P(u_j \otimes u_j)(\tau) d\tau \right\|_{L^p(\mathbb{R}^2)} \\ &\leq C \int_0^t (t-\tau)^{-\frac{1}{\alpha}(\frac{2}{p} - \frac{1}{p}) - \frac{1}{2\alpha}} \|P(u_j \otimes u_j)(\tau)\|_{L^{\frac{p}{2}}(\mathbb{R}^2)} d\tau \\ &\leq C \int_0^t (t-\tau)^{-\frac{1}{\alpha}(\frac{2}{p} - \frac{1}{p}) - \frac{1}{2\alpha}} \|u_j(\tau)\|_{L^p(\mathbb{R}^2)}^2 d\tau \\ &\leq CB(1 - \frac{1}{\alpha p} - \frac{1}{2\alpha}, \frac{2}{\alpha p} + \frac{1}{\alpha} - 1) U_j^2 t^{-\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p})} \end{aligned} \quad (73)$$

for all $0 < t < T$, where $C = C(p)$.

Employing (35) in Lemma 3.2 and (37) in Lemma 3.3, we deduce:

$$\begin{aligned} &\left\| \int_0^t K_{t-\tau} * P(|n_j|^\beta f)(\tau) d\tau \right\|_{L^p(\mathbb{R}^2)} \\ &\leq C \int_0^t \|K_{t-\tau} * (|n_j|^\beta f)(\tau)\|_{L^p(\mathbb{R}^2)} d\tau \\ &\leq C \int_0^t (t-\tau)^{-\frac{1}{\alpha}(\frac{1}{r} - \frac{1}{p})} \|(|n_j|^\beta f)(\tau)\|_{L_w^r(\mathbb{R}^2)} d\tau \end{aligned} \quad (74)$$

for all $1 < r < p < \infty$.

For the case $1 < \beta < \frac{3}{2}$, recalling on the assumption (13) in Theorem 2.1, we have

$$\frac{q}{\beta} > 1 \quad \text{and} \quad \frac{1}{p} < \frac{1}{r} \equiv \frac{4\alpha - 1 - 2\alpha\beta}{2} + \frac{\beta}{q} < 1,$$

then using the weak Hölder inequality, it follows that:

$$\begin{aligned} &\int_0^t (t-\tau)^{-\frac{1}{\alpha}(\frac{1}{r} - \frac{1}{p})} \|(|n_j|^\beta f)(\tau)\|_{L_w^r(\mathbb{R}^2)} d\tau \\ &\leq C \int_0^t (t-\tau)^{-\frac{1}{\alpha}(\frac{4\alpha-1-2\alpha\beta}{2} + \frac{\beta}{q} - \frac{1}{p})} \\ &\quad \cdot \|f\|_{L_w^p(\mathbb{R}^2)} \| |n_j|^\beta(\tau) \|_{L^{\frac{q}{\beta}}(\mathbb{R}^2)} d\tau \\ &= C \int_0^t (t-\tau)^{-\frac{1}{\alpha}(\frac{4\alpha-1-2\alpha\beta}{2} + \frac{\beta}{q} - \frac{1}{p})} \\ &\quad \cdot \|f\|_{L_w^p(\mathbb{R}^2)} \|n_j(\tau)\|_{L^q(\mathbb{R}^2)}^\beta d\tau, \end{aligned} \quad (75)$$

where $C = C(\beta, q)$, and ρ is given by (9).

For the case of $\beta = \frac{3}{2}$, since $\frac{3}{2} \leq q < 2$ and $\frac{2q}{3} < p < \infty$, we deduce

$$1 < p < \infty \quad \text{and} \quad \frac{1}{p} \leq \frac{1}{r} = \frac{1}{\infty} + \frac{3}{2q} < 1,$$

applying the Hölder inequality, we arrive at

$$\begin{aligned}
& \left\| \int_0^t K_{t-\tau} * P(|n_j|^{\frac{3}{2}} f)(\tau) d\tau \right\|_{L^p(\mathbb{R}^2)} \\
& \leq C \int_0^t \|K_{t-\tau} * P(|n_j|^{\frac{3}{2}} f)(\tau)\|_{L^p(\mathbb{R}^2)} d\tau \\
& \leq C \int_0^t (t-\tau)^{-\frac{1}{\alpha}(\frac{3}{2q}-\frac{1}{p})} \|f\|_{L^\infty(\mathbb{R}^2)} \|n_j(\tau)\|_{L^q(\mathbb{R}^2)}^{\frac{3}{2}} d\tau,
\end{aligned} \tag{76}$$

where $C = C(q, p)$.

Recalling on the condition (13) in Theorem 2.1, we obtain

$$1 - \frac{1}{\alpha} \left(\frac{1}{\rho} + \frac{\beta}{q} - \frac{1}{p} \right) > 0, \quad 1 - \frac{1}{\alpha} \left(\alpha - \frac{1}{q} \right) \beta > 0,$$

and therefore it holds for $1 < \beta < \frac{3}{2}$ that:

$$\begin{aligned}
& \int_0^t (t-\tau)^{-\frac{1}{\alpha}(\frac{1}{\rho}+\frac{\beta}{q}-\frac{1}{p})} \|f\|_{L_w^\rho(\mathbb{R}^2)} \|n_j(\tau)\|_{L^q(\mathbb{R}^2)}^\beta d\tau \\
& \leq \|f\|_{L_w^\rho(\mathbb{R}^2)} N_j^\beta \int_0^t (t-\tau)^{-\frac{1}{\alpha}(\frac{1}{\rho}+\frac{\beta}{q}-\frac{1}{p})} \tau^{-\frac{1}{\alpha}(\alpha-\frac{1}{q})\beta} d\tau \\
& = \|f\|_{L_w^\rho(\mathbb{R}^2)} B \left(1 - \frac{1}{\alpha} \left(\frac{1}{\rho} + \frac{\beta}{q} - \frac{1}{p} \right), 1 - \frac{1}{\alpha} \left(\alpha - \frac{1}{q} \right) \beta \right) \\
& \quad \cdot N_j^\beta t^{-\frac{1}{\alpha}(\alpha-\frac{1}{2}-\frac{1}{p})}.
\end{aligned} \tag{77}$$

Combining (75), (76) and (77), we observe

$$\begin{aligned}
& \left\| \int_0^t K_{t-\tau} * P(|n_j|^\beta f)(\tau) d\tau \right\|_{L^p(\mathbb{R}^2)} \\
& \leq C \|f\|_{L_w^\rho(\mathbb{R}^2)} B \left(1 - \frac{1}{\alpha} \left(\frac{1}{\rho} + \frac{\beta}{q} - \frac{1}{p} \right), 1 - \frac{1}{\alpha} \left(\alpha - \frac{1}{q} \right) \beta \right) \\
& \quad \cdot N_j^\beta t^{-\frac{1}{\alpha}(\alpha-\frac{1}{2}-\frac{1}{p})}
\end{aligned} \tag{78}$$

for all $0 < t < T$, where $C = C(\beta, q, p)$. As for $\beta = \frac{3}{2}$, since $\rho = \infty$, by replacing $\|f\|_{L_w^\rho(\mathbb{R}^2)}$ by $\|f\|_{L^\infty(\mathbb{R}^2)}$, the (78) holds.

Combine (65), (72), (73) and (78), we deduce

$$\begin{aligned}
& \sup_{0 < t < T} t^{\frac{1}{\alpha}(\alpha-\frac{1}{2}-\frac{1}{p})} \|u_{j+1}(t)\|_{L^p(\mathbb{R}^2)} \\
& \leq U_1 + CB \left(1 - \frac{1}{\alpha p} - \frac{1}{2\alpha}, \frac{2}{\alpha p} + \frac{1}{\alpha} - 1 \right) U_j^2 \\
& \quad + C \|f\|_{L_w^\rho(\mathbb{R}^2)} B \left(1 - \frac{1}{\alpha} \left(\frac{1}{\rho} + \frac{\beta}{q} - \frac{1}{p} \right), 1 - \frac{1}{\alpha} \left(\alpha - \frac{1}{q} \right) \beta \right) N_j^\beta,
\end{aligned} \tag{79}$$

where $C = C(\beta, q, p)$, and $\|f\|_{L_w^\rho}$ is replaced by $\|f\|_\infty$ for $\beta = \frac{3}{2}$. Then, from (71) and (79), it follows that

$$\begin{cases} N_{j+1} = N_1 + aN_1U_1 + aN_j^2, \\ U_{j+1} = U_1 + aU_j^2 + a\|f\|_Y N_j^\beta, \end{cases} \tag{80}$$

where $a = a(\beta, q, p)$ is a constant independent of j and $\|f\|_Y$ is defined by (10).

We next show that the limits $\lim_{j \rightarrow \infty} N_j$ and $\lim_{j \rightarrow \infty} U_j$ exist if N_1 and U_1 are small enough. To this end, we consider the following characteristic equation associated with (80):

$$\begin{cases} x = N_1 + axy + ax^2, \\ y = U_1 + ay^2 + a\|f\|_Y |x|^{\beta-1} x. \end{cases} \tag{81}$$

For the solvability of (81), we have the following proposition:

Proposition 4.1. ([24], Proposition 4.1) There exist $\delta = \delta(\beta, q, p, \|f\|_Y) > 0$, $\eta = \eta(\beta, q, p, \|f\|_Y) > 0$ and a pair of (x, y) of C^∞ -functions as $x = x(N_1, U_1)$, $y = y(N_1, U_1)$:

$$(x, y) : (N_1, U_1) \in B_\delta(\mathbf{0}) \rightarrow (x(N_1, U_1), y(N_1, U_1)) \in B_\eta(\mathbf{0})$$

with

$$x(\mathbf{0}) = y(\mathbf{0}) = 0 \quad (82)$$

so that the pair (x, y) of solution of (81) is uniquely expressed as $x = x(N_1, U_1)$, $y = y(N_1, U_1)$ for $(N_1, U_1) \in B_\delta(\mathbf{0})$. Moreover, if $(N_1, U_1) \in B_\delta(\mathbf{0})$ satisfies $N_1 > 0$ and $U_1 > 0$, then it holds that $x(N_1, U_1) > 0$ and $y(N_1, U_1) > 0$. Here $B_\delta(\mathbf{0})$ denotes the ball in \mathbb{R}^2 centered at the origin $\mathbf{0} = (0, 0)$ with the radius $\delta > 0$.

The induction and Proposition 4.1 show that for $(N_1, U_1) \in B_\delta(\mathbf{0})$ with $N_1 > 0$ and $U_1 > 0$, the sequences $\{N_j\}_{j=1}^\infty$ and $\{U_j\}_{j=1}^\infty$ satisfy:

$$\begin{cases} N_1 \leq N_2 \leq \cdots \leq N_j \leq N_{j+1} \leq \cdots \uparrow x, \\ U_1 \leq U_2 \leq \cdots \leq U_j \leq U_{j+1} \leq \cdots \uparrow y, \end{cases} \quad (83)$$

where (x, y) is the root of (81).

Step 2. Combine (66), (67) and (85), under the following assumptions:

$$(N_1, U_1) \in B_\delta(\mathbf{0}),$$

with N_1 and U_1 given by (68) and (72), respectively, we discover

$$\sup_{0 < t < T} t^{\frac{1}{\alpha}(\alpha - \frac{1}{q})} \|n_j(t)\|_{L^q(\mathbb{R}^2)} \leq x, \quad j = 1, 2, \dots, \quad (84)$$

$$\sup_{0 < t < T} t^{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p})} \|u_j(t)\|_{L^p(\mathbb{R}^2)} \leq y, \quad j = 1, 2, \dots, \quad (85)$$

where (x, y) is a solution of (81), which is given by Proposition 4.1.

Next, we consider:

$$\tilde{n}_j(t) := n_{j+1}(t) - n_j(t), \quad \tilde{u}_j(t) := u_{j+1}(t) - u_j(t), \quad j = 1, 2, \dots. \quad (86)$$

Similar to the above estimate, we derive from (84) and (85) that

$$\begin{aligned} & \sup_{0 < t < T} t^{\frac{1}{\alpha}(\alpha - \frac{1}{q})} \|\tilde{n}_{j+1}(t)\|_{L^q(\mathbb{R}^2)} \\ & \leq CB(1 - \frac{1}{\alpha p} - \frac{1}{2\alpha}, \frac{1}{\alpha}(\frac{1}{p} + \frac{1}{q}) + \frac{1}{2\alpha} - 1)x \cdot \sup_{0 < t < T} t^{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p})} \|\tilde{u}_j(t)\|_{L^p(\mathbb{R}^2)} \\ & \quad + CB(1 - \frac{1}{\alpha p} - \frac{1}{2\alpha}, \frac{1}{\alpha}(\frac{1}{p} + \frac{1}{q}) + \frac{1}{2\alpha} - 1)y \cdot \sup_{0 < t < T} t^{\frac{1}{\alpha}(\alpha - \frac{1}{q})} \|\tilde{n}_j(t)\|_{L^q(\mathbb{R}^2)} \\ & \quad + CB(1 - \frac{1}{\alpha q}, \frac{2}{\alpha q} - 1) \cdot \sup_{0 < t < T} t^{\frac{1}{\alpha}(\alpha - \frac{1}{q})} \|\tilde{n}_j(t)\|_{L^q(\mathbb{R}^2)} \end{aligned} \quad (87)$$

for all $j = 1, 2, \dots$, where $C = C(q, p)$. We also obtain

$$\begin{aligned} & \sup_{0 < t < T} t^{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p})} \|\tilde{u}_{j+1}(t)\|_{L^p(\mathbb{R}^2)} \\ & \leq CB(1 - \frac{1}{\alpha p} - \frac{1}{2\alpha}, \frac{2}{\alpha p} + \frac{1}{\alpha} - 1)y \cdot \sup_{0 < t < T} t^{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p})} \|\tilde{u}_j(t)\|_{L^p(\mathbb{R}^2)} \\ & \quad + C\|f\|_Y B(1 - \frac{1}{\alpha}(\frac{1}{\rho} + \frac{\beta}{q} - \frac{1}{p}), 1 - \frac{1}{\alpha}(\alpha - \frac{1}{q})\beta) \cdot |x|^{\beta-1} \sup_{0 < t < T} t^{\frac{1}{\alpha}(\alpha - \frac{1}{q})} \|\tilde{n}_j(t)\|_{L^q(\mathbb{R}^2)} \end{aligned} \quad (88)$$

for all $j = 1, 2, \dots$, where $C = C(\beta, q, p)$.

Now, defining $\{\tilde{N}_j\}_{j=1}^\infty$ and $\{\tilde{U}_j\}_{j=1}^\infty$ by:

$$\sup_{0 < t < T} t^{\frac{1}{\alpha}(\alpha - \frac{1}{q})} \|\tilde{n}_j(t)\|_{L^q(\mathbb{R}^2)} \leq \tilde{N}_j, \quad \sup_{0 < t < T} t^{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p})} \|\tilde{u}_j(t)\|_{L^p(\mathbb{R}^2)} \leq \tilde{U}_j, \quad j = 1, 2, \dots, \quad (89)$$

we obtain from (87), (88) and (89) that

$$\begin{cases} \tilde{N}_{j+1} = ax\tilde{U}_j + ax\tilde{N}_j + ay\tilde{N}_j, \\ \tilde{U}_{j+1} = ay\tilde{U}_j + a\|f\|_Y x^{\beta-1}\tilde{N}_j \end{cases} \quad (90)$$

for $j = 1, 2, \dots$, where a is the same constant as in (80) and $\|\cdot\|_Y$ is given by (10).

The subsequent proposition implies the convergence of (n_j, u_j) in the topology of solutions in Theorem 2.1.

Proposition 4.2. ([24], Proposition 4.2) Let the Assumption hold. Assume that (q, p) satisfies (13). Let δ be as in Proposition 4.1. There is a positive constant $\mu = \mu(\beta, q, p, \|f\|_Y) < \delta$ such that if:

$$\sqrt{\tilde{N}_1^2 + \tilde{U}_1^2} \leq \delta', \quad (91)$$

then the recurrences $\{\tilde{N}_j\}_{j=1}^\infty$ and $\{\tilde{U}_j\}_{j=1}^\infty$ defined by (90) satisfy:

$$\Sigma_{j=1}^\infty \tilde{N}_j < \infty, \quad \Sigma_{j=1}^\infty \tilde{U}_j < \infty,$$

where $\|\cdot\|_Y$ is given by (10).

Step 3. Completion of the proof of Theorem 2.1-(i).

Let δ' be obtained in Proposition 4.2. We may take $\delta_1 = \delta_1(\beta, q, p, \|f\|_Y) < \delta'$ so that:

$$a\left(\frac{\delta_1}{2}\right)^2 < \frac{\delta'}{4} \quad \text{and} \quad a\|f\|_Y \left(\frac{\delta_1}{2}\right)^\beta < \frac{\delta'}{4}. \quad (92)$$

Since C_0^∞ is dense in L^1 , there exists $n_0^* \in C_0^\infty$ so that $\|n_0 - n_0^*\|_{L^1(\mathbb{R}^2)} < \frac{\delta_1}{4} \cdot \frac{1}{G_q}$, with (11). Thus we have:

$$\begin{aligned} & \sup_{0 < t < T} t^{\frac{1}{\alpha}(\alpha - \frac{1}{q})} \|K_t * n_0\|_{L^q(\mathbb{R}^2)} \\ & \leq \sup_{0 < t < T} t^{\frac{1}{\alpha}(\alpha - \frac{1}{q})} \|K_t * (n_0 - n_0^*)\|_{L^q(\mathbb{R}^2)} + \sup_{0 < t < T} t^{\frac{1}{\alpha}(\alpha - \frac{1}{q})} \|K_t * n_0^*\|_{L^q(\mathbb{R}^2)} \\ & \leq G_q \|n_0 - n_0^*\|_{L^1(\mathbb{R}^2)} + T^{\frac{1}{\alpha}(\alpha - \frac{1}{q})} \|n_0^*\|_{L^q(\mathbb{R}^2)} \\ & \leq \frac{\delta_1}{4} + T^{\frac{1}{\alpha}(\alpha - \frac{1}{q})} \|n_0^*\|_{L^q(\mathbb{R}^2)}. \end{aligned}$$

This, together with

$$T \leq \left(\frac{\delta_1}{4\|n_0^*\|_{L^q(\mathbb{R}^2)}}\right)^{\frac{\alpha q}{\alpha q - 1}} =: T_1, \quad (93)$$

yields

$$N_1 = \sup_{0 < t < T} t^{\frac{1}{\alpha}(\alpha - \frac{1}{q})} \|K_t * n_0\|_{L^q(\mathbb{R}^2)} < \frac{\delta_1}{2}. \quad (94)$$

Similarly, we find $u_0^* \in C_0^\infty$ so that $\|u_0 - u_0^*\|_{L^1(\mathbb{R}^2)} < \frac{\delta_1}{4} \cdot \frac{1}{G_p}$, with (12), which gives that

$$\begin{aligned} & \sup_{0 < t < T} t^{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p})} \|K_t * u_0\|_{L^p(\mathbb{R}^2)} \\ & \leq \sup_{0 < t < T} t^{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p})} \|K_t * (u_0 - u_0^*)\|_{L^p(\mathbb{R}^2)} + \sup_{0 < t < T} t^{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p})} \|K_t * u_0^*\|_{L^p(\mathbb{R}^2)} \\ & \leq G_p \|u_0 - u_0^*\|_{L^2(\mathbb{R}^2)} + T^{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p})} \|u_0^*\|_{L^p(\mathbb{R}^2)} \\ & \leq \frac{\delta_1}{4} + T^{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p})} \|u_0^*\|_{L^p(\mathbb{R}^2)}. \end{aligned}$$

This, together with

$$T \leq \left(\frac{\delta_1}{4\|u_0^*\|_{L^p(\mathbb{R}^2)}}\right)^{\frac{2\alpha p}{2\alpha p - p - 2}} =: T_2, \quad (95)$$

yields

$$U_1 = \sup_{0 < t < T} t^{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p})} \|K_t * u_0\|_{L^p(\mathbb{R}^2)} < \frac{\delta_1}{2}. \quad (96)$$

Since $\delta_1 < \delta' < \delta$, we have that $(N_1, U_1) \in B_\delta(\mathbf{0})$. Hence, from Proposition 4.1, we find that there exist $\eta > 0$ and a pair (x, y) of C^∞ -functions as $x = x(N_1, U_1)$, $y = y(N_1, U_1)$:

$$(x, y) : (N_1, U_1) \in B_\delta(\mathbf{0}) \rightarrow (x(N_1, U_1), y(N_1, U_1)) \in B_\eta(\mathbf{0}) \quad (97)$$

so that a pair (x, y) of the solution of (81) is uniquely expressed as $x = x(N_1, U_1)$, $y = y(N_1, U_1)$ for $(N_1, U_1) \in B_\delta(\mathbf{0})$. Since $N_1 > 0$ and $U_1 > 0$, it follows that $x(N_1, U_1) > 0$ and $y(N_1, U_1) > 0$.

We may take \tilde{N}_1, \tilde{U}_1 as:

$$\tilde{N}_1 = aN_1U_1 + aN_1^2, \quad \tilde{U}_1 = aU_1^2 + a\|f\|_Y N_1^\beta \quad (98)$$

with the norm $\|\cdot\|_Y$ given by (10), and with (N_1, U_1) given by (68) and (72), we discover from (92), (94), (96) and (98) that

$$\sqrt{\tilde{N}_1^2 + \tilde{U}_1^2} \leq \tilde{N}_1 + \tilde{U}_1 < \mu. \quad (99)$$

Therefore, we have that (91) holds true under the hypotheses (93)-(96).

Applying (97) and (98), we can define $\{\tilde{N}_j\}_{j=1}^\infty$ and $\{\tilde{U}_j\}_{j=1}^\infty$ by (89) and (90). Let us denote T_0 by:

$$T_0 = \min\{T_1, T_2\} \quad (100)$$

with T_1 and T_2 in (93) and (95). Then T_0 depends only on β , q , p , $\|f\|_Y$, n_0 and u_0 with (11).

Since $n_j(t) = \sum_{k=1}^{j-1} \tilde{n}_k(t) + n_1(t)$ and $u_j(t) = \sum_{k=1}^{j-1} \tilde{u}_k(t) + u_1(t)$, and since (91) holds, it is evident from Proposition 4.2 that there are limits in $n \in C((0, T_0); L^q)$ of $\{n_j\}_{j=1}^\infty$ and $u \in C((0, T_0); L^p)$ of $\{u_j\}_{j=1}^\infty$ with

$$t^{\frac{1}{\alpha}(\alpha - \frac{1}{q})} n(\cdot) \in BC([0, T_0]; L^q(\mathbb{R}^2))$$

and

$$t^{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p})} u(\cdot) \in BC([0, T_0]; L^p_\sigma(\mathbb{R}^2))$$

so that

$$\lim_{j \rightarrow \infty} \left(\sup_{0 < t < T_0} t^{\frac{1}{\alpha}(\alpha - \frac{1}{q})} \|n_j(t) - n(t)\|_{L^q(\mathbb{R}^2)} \right) = 0, \quad (101)$$

$$\lim_{j \rightarrow \infty} \left(\sup_{0 < t < T_0} t^{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p})} \|u_j(t) - u(t)\|_{L^p(\mathbb{R}^2)} \right) = 0. \quad (102)$$

Then, we show that such limits n and u satisfy

$$n \in C([0, T_0]; L^1 \cap L^{\frac{1}{\alpha}}(\mathbb{R}^2)) \text{ with } \lim_{j \rightarrow \infty} \left(\sup_{0 < t < T_0} \|n_j(t) - n(t)\|_{L^{\frac{1}{\alpha}}(\mathbb{R}^2)} \right) = 0, \quad (103)$$

$$u \in C([0, T_0]; L^2 \cap L^{\frac{2}{2\alpha-1}}(\mathbb{R}^2)) \text{ with } \lim_{j \rightarrow \infty} \left(\sup_{0 < t < T_0} \|u_j(t) - u(t)\|_{L^{\frac{2}{2\alpha-1}}(\mathbb{R}^2)} \right) = 0. \quad (104)$$

For that purpose, we first show that there are bounded sequences $\{\mathcal{A}\}_{j=1}^\infty$ and $\{\mathcal{B}\}_{j=1}^\infty$ so that

$$\sup_{0 < t < T_0} \|n_j(t) - n(t)\|_{L^{\frac{1}{\alpha}}(\mathbb{R}^2)} \leq \mathcal{A}_j, \quad j = 1, 2, \dots, \quad (105)$$

$$\sup_{0 < t < T_0} \|u_j(t) - u(t)\|_{L^{\frac{2}{2\alpha-1}}(\mathbb{R}^2)} \leq \mathcal{B}_j, \quad j = 1, 2, \dots. \quad (106)$$

Since $\sup_{0 < t < \infty} \|K_t * n_0\|_{L^{\frac{1}{\alpha}}(\mathbb{R}^2)} \leq \|n_0\|_{L^{\frac{1}{\alpha}}(\mathbb{R}^2)}$, we can choose $\mathcal{A}_1 = \|n_0\|_{L^{\frac{1}{\alpha}}(\mathbb{R}^2)}$. Assume that (105) is true for j . Then it follows from (63) that

$$\begin{aligned} & \|n_{j+1}(t)\|_{L^{\frac{1}{\alpha}}(\mathbb{R}^2)} \\ & \leq \|n_0\|_{L^{\frac{1}{\alpha}}(\mathbb{R}^2)} + \int_0^t \|\nabla \cdot K_{t-\tau} * (u_j n_j)(\tau)\|_{L^{\frac{1}{\alpha}}(\mathbb{R}^2)} d\tau + \int_0^t \|\nabla \cdot K_{t-\tau} * (n_j \nabla v_j)(\tau)\|_{L^{\frac{1}{\alpha}}(\mathbb{R}^2)} d\tau \\ & \leq \|n_0\|_{L^{\frac{1}{\alpha}}(\mathbb{R}^2)} + C \int_0^t (t-\tau)^{-\frac{1}{2\alpha}} \|n_j(\tau)\|_{L^{\frac{p}{\alpha p-1}}(\mathbb{R}^2)} \\ & \quad \cdot \|u_j(\tau)\|_{L^p(\mathbb{R}^2)} d\tau + C \int_0^t (t-\tau)^{-\frac{1}{2\alpha}} \|n_j(\tau)\|_{L^{\frac{r}{\alpha r-1}}(\mathbb{R}^2)} \|\nabla v_j(\tau)\|_{L^r(\mathbb{R}^2)} d\tau \\ & \leq \|n_0\|_{L^{\frac{1}{\alpha}}(\mathbb{R}^2)} + C \int_0^t (t-\tau)^{-\frac{1}{2\alpha}} \|n_j(\tau)\|_{L^{m_3}(\mathbb{R}^2)} \\ & \quad \cdot \|n_j(\tau)\|_{L^q(\mathbb{R}^2)} d\tau + C \int_0^t (t-\tau)^{-\frac{1}{2\alpha}} \|n_j(\tau)\|_{L^{m_2}(\mathbb{R}^2)} \|u_j(\tau)\|_{L^p(\mathbb{R}^2)} d\tau \end{aligned} \quad (107)$$

with $C = C(q)$, where $\frac{1}{r} = \frac{1}{q} - \frac{1}{2}$, $\frac{1}{r} + \frac{1}{m_3} = \alpha$ and $\frac{1}{p} + \frac{1}{m_2} = \alpha$.

Since $\frac{4\alpha}{2+\alpha} < q < 2$, we have $\frac{1}{2} < \frac{1}{q} = \frac{1}{2} + \frac{1}{r} = \frac{1}{2} + \alpha - \frac{1}{m_3} < \frac{2+\alpha}{4\alpha}$, which yields $\frac{1}{\alpha} < m_3 < \frac{4\alpha}{2+\alpha}$, and it follows that

$$\frac{1}{m_3} = \alpha(1 - \theta_1) + \frac{\theta_1}{q} \text{ with } \theta_1 = \frac{2-q}{2(\alpha q - 1)} \in (0, 1), \quad (108)$$

which yields $\|n_j(\tau)\|_{L^{m_3}(\mathbb{R}^2)} \leq \|n_j(\tau)\|_{L^{\frac{1}{\alpha}}(\mathbb{R}^2)}^{1-\theta_1} \cdot \|n_j(\tau)\|_{L^q(\mathbb{R}^2)}^{\theta_1}$.

Since $p \geq \frac{q}{\alpha q - 1}$, we have $\frac{1}{\alpha} < m_2 \leq q$, and it holds that

$$\frac{1}{m_2} = \alpha(1 - \theta_2) + \frac{\theta_2}{q} \text{ with } \theta_2 = \frac{q}{p(\alpha q - 1)} \in (0, 1), \quad (109)$$

which yields $\|n_j(\tau)\|_{L^{m_2}(\mathbb{R}^2)} \leq \|n_j(\tau)\|_{L^{\frac{1}{\alpha}}(\mathbb{R}^2)}^{1-\theta_2} \cdot \|n_j(\tau)\|_{L^q(\mathbb{R}^2)}^{\theta_2}$. Thus, from (83), we find that

$$\begin{aligned} & \|n_{j+1}(t)\|_{L^{\frac{1}{\alpha}}(\mathbb{R}^2)} \leq \|n_0(t)\|_{L^{\frac{1}{\alpha}}(\mathbb{R}^2)} \\ & + C \int_0^t (t-\tau)^{-\frac{1}{2\alpha}} \|n_j(\tau)\|_{L^{\frac{1}{\alpha}}(\mathbb{R}^2)}^{1-\theta_1} \|n_j(\tau)\|_{L^q(\mathbb{R}^2)}^{1+\theta_1} d\tau \\ & + \int_0^t (t-\tau)^{-\frac{1}{2\alpha}} \|n_j(\tau)\|_{L^{\frac{1}{\alpha}}(\mathbb{R}^2)}^{1-\theta_2} \|n_j(\tau)\|_{L^q(\mathbb{R}^2)}^{\theta_2} \cdot \|u_j(\tau)\|_{L^p(\mathbb{R}^2)} d\tau \\ & \leq \mathcal{A}_1 + C(\mathcal{A}_j^{1-\theta_1} N_j^{1+\theta_1} + \mathcal{A}_j^{1-\theta_2} N_j^{\theta_2} U_j) \cdot \int_0^t (t-\tau)^{-\frac{1}{2\alpha}} \tau^{\frac{1}{2\alpha}-1} d\tau \\ & \leq \mathcal{A}_1 + CB(1 - \frac{1}{2\alpha}, \frac{1}{2\alpha})(\mathcal{A}_j^{1-\theta_1} x^{1+\theta_1} + \mathcal{A}_j^{1-\theta_2} x^{\theta_2} y), \end{aligned} \quad (110)$$

for all $0 < t < T_0$, where $C = C(q)$. Then we can choose \mathcal{A}_{j+1} as:

$$\mathcal{A}_{j+1} := \mathcal{A}_1 + b(\mathcal{A}_j^{1-\theta_1} x^{1+\theta_1} + \mathcal{A}_j^{1-\theta_2} x^{\theta_2} y), \quad (111)$$

where $b = b(q)$ is a constant independent of j .

The sequence $\{\mathcal{A}_j\}_{j=1}^\infty$ is obtained by the induction, so that (105) holds for all $j = 1, 2, \dots$. Since $\{\mathcal{A}_j\}_{j=1}^\infty$ is determined inductively by (111), we obtain

$$0 < \mathcal{A}_1 \leq \mathcal{A}_2 \leq \dots \leq \mathcal{A}_j \leq \mathcal{A}_{j+1} \leq \dots \uparrow \mathcal{A}, \quad (112)$$

where \mathcal{A} is the root of

$$\mathcal{A} = \mathcal{A}_1 + b(\mathcal{A}^{1-\theta_1} x^{1+\theta_1} + \mathcal{A}^{1-\theta_2} x^{\theta_2} y).$$

Moreover, since $\sup_{0 < t < \infty} \|K_t * u_0\|_{L^{\frac{2}{2\alpha-1}}(\mathbb{R}^2)} \leq \|u_0\|_{L^{\frac{2}{2\alpha-1}}(\mathbb{R}^2)}$, we can choose $\mathcal{B}_1 = \|u_0\|_{L^{\frac{2}{2\alpha-1}}(\mathbb{R}^2)}$. Assume that (106) is true for j . We should note that $\frac{2p}{2+(2\alpha-1)p} < \frac{p}{2}$ for all $p > \frac{2}{2\alpha-1}$. Let us denote k_1 satisfying $\frac{1}{k} = \frac{1}{k_1} + \frac{1}{p}$ as:

$$k_1 := \frac{kp}{p-k}. \quad (113)$$

If it holds that:

$$\frac{2p}{2+(2\alpha-1)p} < k < \frac{p}{2}, \quad (114)$$

then we have:

$$\frac{2}{2\alpha-1} < k_1 < p. \quad (115)$$

The cases are divided into two parts: $\frac{2}{2\alpha-1} < p < 4$ and $p > 4$. We take $k = \frac{1}{2}(\frac{2p}{2+(2\alpha-1)p} + \frac{p}{2})$ (respectively, $k = \frac{1}{2}(\frac{2p}{2+(2\alpha-1)p} + 2)$) for $\frac{2}{2\alpha-1} < p \leq 4$ (respectively, $p > 4$). Then we have $1 < k < 2$. As (114) is satisfied for both cases of $\frac{2}{2\alpha-1} < p < 4$ and $p > 4$, we derive (115) from (113).

Besides, for the case $1 < \beta < \frac{3}{2}$, since $\rho = \frac{2}{4\alpha-1-2\alpha\beta}$ and $\frac{2\beta}{2\alpha\beta-\alpha} < q < 2$, we have $1 < s < \frac{4}{3}$, satisfying $\frac{1}{s} = \frac{1}{\rho} + \frac{\beta}{q}$. Hence, by (65), it holds that:

$$\begin{aligned} & \|u_{j+1}(t)\|_{L^{\frac{2}{2\alpha-1}}(\mathbb{R}^2)} \leq \|u_0(t)\|_{L^{\frac{2}{2\alpha-1}}(\mathbb{R}^2)} \\ & + \int_0^t \|\nabla \cdot K_{t-\tau} * P(u_j \otimes u_j)(\tau)\|_{L^{\frac{2}{2\alpha-1}}(\mathbb{R}^2)} d\tau + \int_0^t \|K_{t-\tau} * P(|n_j|^\beta f)(\tau)\|_{L^{\frac{2}{2\alpha-1}}(\mathbb{R}^2)} d\tau \\ & \leq \|u_0(t)\|_{L^{\frac{2}{2\alpha-1}}(\mathbb{R}^2)} \\ & + C \int_0^t (t-\tau)^{-\frac{1}{\alpha}(\frac{1}{k}-\frac{2\alpha-1}{2})-\frac{1}{2\alpha}} \|(u_j \otimes u_j)(\tau)\|_{L^k(\mathbb{R}^2)} d\tau + C \int_0^t (t-\tau)^{-\frac{1}{\alpha}(\frac{1}{s}-\frac{2\alpha-1}{2})} \|(|n_j|^\beta f)(\tau)\|_{L_w^s(\mathbb{R}^2)} d\tau \\ & \leq \|u_0(t)\|_{L^{\frac{2}{2\alpha-1}}(\mathbb{R}^2)} + C \int_0^t (t-\tau)^{-\frac{1}{\alpha}(\frac{1}{k_1}+\frac{1}{p}-\frac{2\alpha-1}{2})-\frac{1}{2\alpha}} \|u_j(\tau)\|_{L^{k_1}(\mathbb{R}^2)} \cdot \|u_j(\tau)\|_{L^p(\mathbb{R}^2)} d\tau \\ & + C \int_0^t (t-\tau)^{-\frac{1}{\alpha}(\frac{1}{\rho}+\frac{\beta}{q}-\frac{2\alpha-1}{2})} \|f\|_{L_w^\rho(\mathbb{R}^2)} \|n_j(\tau)\|_{L^q(\mathbb{R}^2)}^\beta d\tau. \end{aligned} \quad (116)$$

for all $0 < t < T_0$, where $C = C(\beta, q, p)$.

For the case $\beta = \frac{3}{2}$, since $\frac{2\beta}{2\alpha\beta-\alpha} < q < 2$, it follows that:

$$\frac{3}{4} < \frac{1}{s} = \frac{1}{\infty} + \frac{\beta}{q} = \frac{3}{2q} \leq \alpha.$$

Taking the above into consideration, together with the Hölder inequality, (116) follows by replacing $\|f\|_{L_w^p(\mathbb{R}^2)}$ by $\|f\|_{L^\infty(\mathbb{R}^2)}$.

Based on $\frac{2}{2\alpha-1} < k_1 < p$, we discover

$$\frac{1}{k_1} = \frac{(2\alpha-1)(1-\theta_3)}{2} + \frac{\theta_3}{p} \quad \text{with} \quad \theta_3 = \frac{\frac{2\alpha-1}{2} - \frac{1}{k_1}}{\frac{2\alpha-1}{2} - \frac{1}{p}} \in (0, 1), \quad (117)$$

which yields:

$$\|u_j(\tau)\|_{L^{k_1}(\mathbb{R}^2)} \leq \|u_j(\tau)\|_{L^{\frac{2}{2\alpha-1}}(\mathbb{R}^2)}^{1-\theta_3} \|u_j(\tau)\|_{L^p(\mathbb{R}^2)}^{\theta_3}.$$

Therefore, we conclude that:

$$\begin{aligned} & \|u_{j+1}(t)\|_{L^{\frac{2}{2\alpha-1}}(\mathbb{R}^2)} \leq \|u_0(t)\|_{L^{\frac{2}{2\alpha-1}}(\mathbb{R}^2)} \\ & + C \int_0^t (t-\tau)^{-\frac{1}{\alpha}(\frac{1}{k_1} + \frac{1}{p} - \frac{2\alpha-1}{2}) - \frac{1}{2\alpha}} \cdot \|u_j(\tau)\|_{L^{\frac{2}{2\alpha-1}}(\mathbb{R}^2)}^{1-\theta_3} \|u_j(\tau)\|_{L^p(\mathbb{R}^2)}^{1+\theta_3} d\tau \\ & + C \int_0^t (t-\tau)^{-\frac{1}{\alpha}(\frac{1}{p} + \frac{\beta}{q} - \frac{2\alpha-1}{2})} \|f\|_Y \|n_j(\tau)\|_{L^q(\mathbb{R}^2)}^\beta d\tau \\ & \leq \mathcal{B}_1 + C\mathcal{B}_j^{1-\theta_3} y^{1+\theta_3} \cdot B(2 - \frac{1}{\alpha} - \frac{1}{\alpha}(\frac{1}{k_1} + \frac{1}{p}), \frac{1}{\alpha}(\frac{1}{k_1} + \frac{1}{p}) + \frac{1}{\alpha} - 1) + Cx^\beta \|f\|_Y \\ & \cdot B(1 - \frac{1}{\alpha}(\frac{1}{p} + \frac{\beta}{q} - \frac{2\alpha-1}{2}), 1 - \frac{1}{\alpha}(\alpha - \frac{1}{q})\beta), \end{aligned} \quad (118)$$

for all $0 < t < T_0$. Then we can choose \mathcal{B}_{j+1} as:

$$\mathcal{B}_{j+1} := \mathcal{B}_1 + b\mathcal{B}_j^{1-\theta_3} y^{1+\theta_3} + bx^\beta \|f\|_Y, \quad (119)$$

where $b = b(\beta, q, p)$ is a constant independent of j .

By the induction, we obtain the sequence $\{\mathcal{B}_j^\infty\}$ so that (106) holds for all $j = 1, 2, \dots$. Since $\{\mathcal{B}_j^\infty\}$ is determined inductively by (119), we arrive at

$$0 < \mathcal{B}_1 \leq \mathcal{B}_2 \leq \dots \leq \mathcal{B}_j \leq \mathcal{B}_{j+1} \leq \dots \uparrow \mathcal{B}, \quad (120)$$

where \mathcal{B} is the root of

$$\mathcal{B} = \mathcal{B}_1 + b\mathcal{B}^{1-\theta_3} y^{1+\theta_3} + bx^\beta \|f\|_Y.$$

Let \tilde{n}_j and \tilde{u}_j be the values defined by (86). Now, defining $\{\tilde{\mathcal{A}}_j\}_{j=1}^\infty$ and $\{\tilde{\mathcal{B}}_j\}_{j=1}^\infty$ by:

$$\sup_{0 < t < T_0} \|\tilde{n}_j(t)\|_{L^{\frac{1}{\alpha}}(\mathbb{R}^2)} \leq \tilde{\mathcal{A}}_j, \quad \sup_{0 < t < T_0} \|\tilde{u}_j(t)\|_{L^{\frac{2}{2\alpha-1}}(\mathbb{R}^2)} \leq \tilde{\mathcal{B}}_j$$

for $j = 1, 2, \dots$, we have

$$\begin{cases} \tilde{\mathcal{A}}_{j+1} = b\mathcal{A}^{1-\theta_1} x^{\theta_1} \tilde{N}_j + b\mathcal{A}^{1-\theta_2} x^{\theta_2} \tilde{U}_j + b\mathcal{B}^{1-\theta_4} y^{\theta_4} \tilde{N}_j, \\ \tilde{\mathcal{B}}_{j+1} = b\mathcal{B}^{1-\theta_3} y^{\theta_3} \tilde{U}_j + bx^{\beta-1} \|f\|_Y \tilde{N}_j \end{cases} \quad (121)$$

for $j = 1, 2, \dots$, where $b = b(\beta, q, p)$ is a constant independent of j . Here, θ_i ($i = 1, 2, 3$) are given by (108), (109) and (117), and $\theta_4 = \frac{\frac{1}{q} - \frac{1}{2}}{\frac{1}{2} - \frac{1}{p}}$. This means that both $\sum_{j=1}^\infty \tilde{\mathcal{A}}_j < \infty$ and $\sum_{j=1}^\infty \tilde{\mathcal{B}}_j < \infty$ since $\sum_{j=1}^\infty \tilde{N}_j < \infty$ and $\sum_{j=1}^\infty \tilde{U}_j < \infty$ by the choice of T_0 in (100).

Since $n_j(t) = \sum_{k=1}^{j-1} \tilde{n}_k(t) + n_1(t)$ and $u_j(t) = \sum_{k=1}^{j-1} \tilde{u}_k(t) + u_1(t)$, and since (91) holds, we obtain from Proposition 4.2 that there are limits in n and u with

$$n \in C([0, T_0]; L^1 \cap L^{\frac{1}{\alpha}}(\mathbb{R}^2)) \quad \text{and} \quad u \in C([0, T_0]; L^2 \cap L^{\frac{2}{2\alpha-1}}(\mathbb{R}^2))$$

so that

$$\lim_{j \rightarrow \infty} \left(\sup_{0 < t < T_0} \|n_j(t) - n(t)\|_{L^{\frac{1}{\alpha}}(\mathbb{R}^2)} \right) = 0,$$

$$\lim_{j \rightarrow \infty} \left(\sup_{0 < t < T_0} \|u_j(t) - u(t)\|_{L^{\frac{2}{2\alpha-1}}(\mathbb{R}^2)} \right) = 0.$$

Hence we get (103) and (104). Now the proof of Theorem 2.1-(i) is complete, except for (14). The proof of (14) is postponed in subsection 4.2.

4.2. Proof of Theorem 2.1-(ii)

The uniqueness will be shown under the restriction (14).

Lemma 4.1. Let the Assumption hold and let $1 < \beta < \frac{3}{2}$. Assume that (q, p) satisfies (13), then the mild solution (n, u) of (1) on $[0, T)$ in the class of $M_{q,p}(0, T)$ is unique provided

the property (14) is fulfilled.

Proof. Let $n_0 \in L^1 \cap L^{\frac{1}{\alpha}}(\mathbb{R}^2)$ and $u_0 \in L^2_{\sigma} \cap L^{\frac{2}{2\alpha-1}}(\mathbb{R}^2)$, and let (n, u) and (\hat{n}, \hat{u}) be two mild solutions of (1) on $[0, T)$ in the class of $M_{q,p}(0, T)$ for (q, p) satisfy (13) with the additional property (14).

Take $\mathcal{N}(t) = n(t) - \hat{n}(t)$ and $\mathcal{U}(t) = u(t) - \hat{u}(t)$. Recalling on Lemma 3.5 for r with $\frac{1}{r} = \frac{2}{q} - \frac{1}{2}$, in the similar manner to the proof of Theorem 2.1-(i), for $\frac{4\alpha}{2+\alpha} < q < 2$ and $\frac{q}{\alpha q - 1} \leq p < \infty$, we deduce that:

$$\begin{aligned} & \|\mathcal{N}(t)\|_{L^q(\mathbb{R}^2)} \\ & \leq C \int_0^t (t-\tau)^{-\frac{1}{\alpha}(\frac{1}{q} + \frac{1}{p} - \frac{1}{q}) - \frac{1}{2\alpha}} \|\mathcal{N}(\tau)\|_{L^q(\mathbb{R}^2)} \|u(\tau)\|_{L^p(\mathbb{R}^2)} d\tau \\ & + C \int_0^t (t-\tau)^{-\frac{1}{\alpha}(\frac{1}{q} + \frac{1}{p} - \frac{1}{q}) - \frac{1}{2\alpha}} \|\hat{n}(\tau)\|_{L^q(\mathbb{R}^2)} \|\mathcal{U}(\tau)\|_{L^p(\mathbb{R}^2)} d\tau \\ & + C \int_0^t (t-\tau)^{-\frac{1}{\alpha}(\frac{1}{r} - \frac{1}{q}) - \frac{1}{2\alpha}} \|\mathcal{N}(\tau)\|_{L^q(\mathbb{R}^2)} \|\nabla(-\Delta + \gamma)^{-1}n(\tau)\|_{L^r(\mathbb{R}^2)} d\tau \\ & + C \int_0^t (t-\tau)^{-\frac{1}{\alpha}(\frac{1}{r} - \frac{1}{q}) - \frac{1}{2\alpha}} \|\hat{n}(\tau)\|_{L^q(\mathbb{R}^2)} \|\nabla(-\Delta + \gamma)^{-1}\mathcal{N}(\tau)\|_{L^r(\mathbb{R}^2)} d\tau \\ & \leq C \int_0^t (t-\tau)^{-\frac{1}{\alpha p} - \frac{1}{2\alpha}} \|\mathcal{N}(\tau)\|_{L^p(\mathbb{R}^2)} \|u(\tau)\|_{L^p(\mathbb{R}^2)} d\tau + C \int_0^t (t-\tau)^{-\frac{1}{\alpha p} - \frac{1}{2\alpha}} \|\hat{n}(\tau)\|_{L^q(\mathbb{R}^2)} \|\mathcal{U}(\tau)\|_{L^p(\mathbb{R}^2)} d\tau \\ & + C \int_0^t (t-\tau)^{-\frac{1}{\alpha q}} \|n(\tau)\|_{L^q(\mathbb{R}^2)} \|\mathcal{N}(\tau)\|_{L^q(\mathbb{R}^2)} d\tau + C \int_0^t (t-\tau)^{-\frac{1}{\alpha q}} \|\hat{n}(\tau)\|_{L^q(\mathbb{R}^2)} \|\mathcal{N}(\tau)\|_{L^q(\mathbb{R}^2)} d\tau, \end{aligned}$$

where $C = C(q, p)$. Defining $a(t)$, $\hat{a}(t)$, $A(t)$ and $b(t)$, $\hat{b}(t)$, $B(t)$ as:

$$\begin{cases} a(t) := \sup_{0 < \tau < t} \tau^{\frac{1}{\alpha}(\alpha - \frac{1}{q})} \|n(\tau)\|_{L^q(\mathbb{R}^2)}, \\ \hat{a}(t) := \sup_{0 < \tau < t} \tau^{\frac{1}{\alpha}(\alpha - \frac{1}{q})} \|\hat{n}(\tau)\|_{L^q(\mathbb{R}^2)}, \\ A(t) := \sup_{0 < \tau < t} \tau^{\frac{1}{\alpha}(\alpha - \frac{1}{q})} \|\mathcal{N}(\tau)\|_{L^q(\mathbb{R}^2)}, \end{cases} \quad (122)$$

$$\begin{cases} b(t) := \sup_{0 < \tau < t} \tau^{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p})} \|u(\tau)\|_{L^p(\mathbb{R}^2)}, \\ \hat{b}(t) := \sup_{0 < \tau < t} \tau^{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p})} \|\hat{u}(\tau)\|_{L^p(\mathbb{R}^2)}, \\ B(t) := \sup_{0 < \tau < t} \tau^{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p})} \|\mathcal{U}(\tau)\|_{L^p(\mathbb{R}^2)}, \end{cases} \quad (123)$$

respectively, from the above estimate, we find that

$$\begin{aligned} \|\mathcal{N}(t)\|_{L^q(\mathbb{R}^2)} & \leq C(\hat{a}(t)B(t) + b(t)A(t)) \cdot \int_0^t (t-\tau)^{-\frac{1}{\alpha p} - \frac{1}{2\alpha}} \tau^{\frac{1}{\alpha}(\frac{1}{p} + \frac{1}{q}) + \frac{1}{2\alpha} - 2} d\tau \\ & + C(\hat{a}(t) + a(t))A(t) \int_0^t (t-\tau)^{-\frac{1}{\alpha q}} \tau^{\frac{2}{\alpha q} - 2} d\tau \\ & \leq C_{\mathcal{Q}1}(\hat{a}(t)B(t) + (a(t) + \hat{a}(t) + b(t))A(t)) \cdot t^{-\frac{1}{\alpha}(\alpha - \frac{1}{q})}, \quad 0 < t < T \end{aligned}$$

with $\varrho_1 = B(1 - \frac{1}{\alpha p} - \frac{1}{2\alpha}, \frac{1}{\alpha}(\frac{1}{p} + \frac{1}{q}) + \frac{1}{2\alpha} - 1) + B(1 - \frac{1}{\alpha q}, \frac{2}{\alpha q} - 1)$, which yields:

$$t^{\frac{1}{\alpha}(\alpha - \frac{1}{q})} \|\mathcal{N}(t)\|_{L^q(\mathbb{R}^2)} \leq C_{\varrho_1} (\hat{a}(t)B(t) + (a(t) + \hat{a}(t) + b(t))A(t)) \quad (124)$$

for all $0 < t < T$. Since the right-hand of (124) is a non-decreasing function of $0 < t < T$, we deduce

$$A(t) \leq C_{\varrho_1} (\hat{a}(t)B(t) + (a(t) + \hat{a}(t) + b(t))A(t)) \quad (125)$$

for all $0 < t < T$.

For \mathcal{U} , we arrive at

$$\begin{aligned} \|\mathcal{U}(t)\|_{L^p(\mathbb{R}^2)} &\leq C \int_0^t (t-\tau)^{-\frac{1}{\alpha}(\frac{2}{p}-\frac{1}{p})-\frac{1}{2\alpha}} \cdot (\|u(\tau)\|_{L^p(\mathbb{R}^2)} + \|\hat{u}(\tau)\|_{L^p(\mathbb{R}^2)}) \|\mathcal{U}(\tau)\|_{L^p(\mathbb{R}^2)} d\tau \\ &\quad + C \int_0^t (t-\tau)^{-\frac{1}{\alpha}(\frac{1}{p}+\frac{\beta}{q}-\frac{1}{p})} \|f\|_Y \|\mathcal{N}(\tau)\|_{L^q(\mathbb{R}^2)} \cdot (\|n(\tau)\|_{L^q(\mathbb{R}^2)}^{\beta-1} + \|\hat{n}(\tau)\|_{L^q(\mathbb{R}^2)}^{\beta-1}) d\tau \\ &\leq C(b(t) + \hat{b}(t))B(t) \cdot \int_0^t (t-\tau)^{-\frac{1}{\alpha}(\frac{2}{p}-\frac{1}{p})-\frac{1}{2\alpha}} \tau^{\frac{2}{\alpha p}+\frac{1}{\alpha}-2} d\tau \\ &\quad + C\|f\|_Y (a(t)^{\beta-1} + \hat{a}(t)^{\beta-1})A(t) \cdot \int_0^t (t-\tau)^{-\frac{1}{\alpha}(\frac{1}{p}+\frac{\beta}{q}-\frac{1}{p})} \tau^{-\frac{1}{\alpha}(\alpha-\frac{1}{q})\beta} d\tau. \end{aligned}$$

This implies

$$t^{\frac{1}{\alpha}(\alpha-\frac{1}{2}-\frac{1}{p})} \|\mathcal{U}(t)\|_{L^p(\mathbb{R}^2)} \leq C_{\varrho_2} ((b(t) + \hat{b}(t))B(t) + \|f\|_Y (a(t)^{\beta-1} + \hat{a}(t)^{\beta-1})A(t)),$$

where $\varrho_2 = B(1 - \frac{1}{\alpha p} - \frac{1}{2\alpha}, \frac{2}{\alpha p} + \frac{1}{\alpha} - 1) + B(1 - \frac{1}{\alpha}(\frac{1}{p} + \frac{\beta}{q} - \frac{1}{p}), 1 - \frac{1}{\alpha}(\alpha - \frac{1}{q})\beta)$, which yields that:

$$B(t) \leq C_{\varrho_2} ((b(t) + \hat{b}(t))B(t) + \|f\|_Y (a(t)^{\beta-1} + \hat{a}(t)^{\beta-1})A(t)). \quad (126)$$

Combining (125) and (126), we observe

$$\begin{aligned} A(t) + B(t) &\leq C_{\varrho_1} (\hat{a}(t)B(t) + (a(t) + \hat{a}(t) + b(t))A(t)) + C_{\varrho_2} (b(t) + \hat{b}(t))B(t) \\ &\quad + C_{\varrho_2} \|f\|_Y (a(t)^{\beta-1} + \hat{a}(t)^{\beta-1})A(t) \\ &\leq C_{\varrho_1} (\hat{a}(t) + a(t) + b(t))A(t) + C_{\varrho_2} \|f\|_Y (a(t)^{\beta-1} + \hat{a}(t)^{\beta-1})A(t) \\ &\quad + C(\varrho_1 \hat{a}(t) + \varrho_2 (b(t) + \hat{b}(t)))B(t). \end{aligned}$$

From the property (14), there is $0 < t_0 < T$ so that

$$A(t_0) + B(t_0) = 0. \quad (127)$$

This implies $n(t) = \hat{n}(t)$, $u(t) = \hat{u}(t)$ for $0 \leq t \leq t_0$.

We may show the following proposition to prove that $n(t) = \hat{n}(t)$ and $u(t) = \hat{u}(t)$ for $t_0 \leq t < T$.

Proposition 4.3. Let the Assumption hold and let $1 < \beta \leq \frac{3}{2}$. Assume that (q, p) satisfies (13). Suppose that (n, u) is a mild solution of (1) on $[0, T)$ in the class of $M_{q,p}(0, T)$. Let t_0 be the time obtained in (127). Then, there exist a constant $\xi = \xi(\beta, q, p, \|f\|_Y, \sup_{t_0 \leq \tau < T} \|n(\tau)\|_{L^q(\mathbb{R}^2)}, \sup_{t_0 \leq \tau < T} \|u(\tau)\|_{L^p(\mathbb{R}^2)})$ such that if $n(t) = \hat{n}(t)$ and $u(t) = \hat{u}(t)$ for $0 \leq t \leq s$ with some $s \in [t_0, T)$, then we obtain:

$$n(t) = \hat{n}(t) \text{ and } u(t) = \hat{u}(t) \text{ for } s \leq t \leq s + \xi.$$

Proof. Note that $\mathcal{N}(t) = \mathcal{U}(t) = 0$ for $0 \leq t \leq s$, in the same way as above, we can get:

$$\begin{aligned} \mathcal{N}(t)\|_{L^q(\mathbb{R}^2)} &\leq C \int_s^t (t-\tau)^{-\frac{1}{\alpha p}-\frac{1}{2\alpha}} (\|\hat{n}(\tau)\|_{L^q(\mathbb{R}^2)} \|\mathcal{U}(\tau)\|_{L^p(\mathbb{R}^2)} + \|\mathcal{N}(\tau)\|_{L^q(\mathbb{R}^2)} \|u(\tau)\|_{L^p(\mathbb{R}^2)}) d\tau \\ &\quad + C \int_s^t (t-\tau)^{-\frac{1}{\alpha q}} (\|n(\tau)\|_{L^q(\mathbb{R}^2)} + \|\hat{n}(\tau)\|_{L^q(\mathbb{R}^2)}) \|\mathcal{N}(\tau)\|_{L^q(\mathbb{R}^2)} d\tau \end{aligned}$$

and

$$\begin{aligned} \mathcal{U}(t) \|_{L^p(\mathbb{R}^2)} &\leq C \int_s^t (t-\tau)^{-\frac{1}{\alpha}(\frac{2}{p}-\frac{1}{p})-\frac{1}{2\alpha}} (\|\hat{u}(\tau)\|_{L^p(\mathbb{R}^2)} + \|u(\tau)\|_{L^p(\mathbb{R}^2)}) \|\mathcal{U}(\tau)\|_{L^p(\mathbb{R}^2)} d\tau \\ &\quad + C \int_s^t (t-\tau)^{-\frac{1}{\alpha}(\frac{1}{\rho}+\frac{\beta}{q}-\frac{1}{p})} \|f\|_Y \|\mathcal{N}(\tau)\|_{L^q(\mathbb{R}^2)} \cdot (\|n(\tau)\|_{L^q(\mathbb{R}^2)}^{\beta-1} + \|\hat{n}(\tau)\|_{L^q(\mathbb{R}^2)}^{\beta-1}) d\tau. \end{aligned}$$

Since $n, \hat{n} \in C([s, T]; L^q(\mathbb{R}^2))$ and $u, \hat{u} \in C([s, T]; L^p(\mathbb{R}^2))$, we deduce

$$\begin{aligned} \max\{ \sup_{s \leq \tau < T} \|n(\tau)\|_{L^q(\mathbb{R}^2)}, \sup_{s \leq \tau < T} \|\hat{n}(\tau)\|_{L^q(\mathbb{R}^2)} \} &=: a_T < \infty, \\ \max\{ \sup_{s \leq \tau < T} \|u(\tau)\|_{L^p(\mathbb{R}^2)}, \sup_{s \leq \tau < T} \|\hat{u}(\tau)\|_{L^p(\mathbb{R}^2)} \} &=: b_T < \infty, \end{aligned}$$

which yields

$$\begin{aligned} \|\mathcal{N}(t)\|_{L^q(\mathbb{R}^2)} &\leq C \int_s^t (t-\tau)^{-\frac{1}{\alpha p}-\frac{1}{2\alpha}} (a_T + b_T) \cdot (\|\mathcal{U}(\tau)\|_{L^p(\mathbb{R}^2)} + \|\mathcal{N}(\tau)\|_{L^q(\mathbb{R}^2)}) d\tau \\ &\quad + C \int_s^t (t-\tau)^{-\frac{1}{\alpha q}} \|\mathcal{N}(\tau)\|_{L^q(\mathbb{R}^2)} d\tau \end{aligned}$$

and

$$\begin{aligned} \mathcal{U}(t) \|_{L^p(\mathbb{R}^2)} &\leq C b_T \int_s^t (t-\tau)^{-\frac{1}{\alpha}(\frac{2}{p}-\frac{1}{p})-\frac{1}{2\alpha}} \|\mathcal{U}(\tau)\|_{L^p(\mathbb{R}^2)} d\tau \\ &\quad + C \|f\|_Y a_T^{\beta-1} \int_s^t (t-\tau)^{-\frac{1}{\alpha}(\frac{1}{\rho}+\frac{\beta}{q}-\frac{1}{p})} \cdot \|\mathcal{N}(\tau)\|_{L^q(\mathbb{R}^2)} d\tau. \end{aligned}$$

Defining $\hat{A}(t) := \sup_{s < \tau < t} \|\mathcal{N}(\tau)\|_{L^q(\mathbb{R}^2)}$ and $\hat{B}(t) := \sup_{s < \tau < t} \|\mathcal{U}(\tau)\|_{L^p(\mathbb{R}^2)}$, from the above estimate, we observe:

$$\|\mathcal{N}(t)\|_{L^q(\mathbb{R}^2)} \leq C(a_T + b_T)(\hat{A}(t) + \hat{B}(t))(t-s)^{-\frac{1}{\alpha p}-\frac{1}{2\alpha}+1} + C a_T \hat{A}(t)(t-s)^{1-\frac{1}{\alpha q}},$$

and

$$\mathcal{U}(t) \|_{L^p(\mathbb{R}^2)} \leq C b_T \hat{B}(t)(t-s)^{1-\frac{1}{\alpha p}-\frac{1}{2\alpha}} + C \|f\|_Y a_T^{\beta-1} \hat{A}(t)(t-s)^{1-\frac{1}{\alpha}(\frac{1}{\rho}+\frac{\beta}{q}-\frac{1}{p})}$$

for $s < t < T$. Note that $1 - \frac{1}{\alpha}(\frac{1}{\rho} + \frac{\beta}{q} - \frac{1}{p}) > 0$ by $q \geq \frac{2\beta}{2\alpha\beta+1-2\alpha}$ and $p < \infty$, the right-hand side of the above inequality is a non-decreasing function of t , we deduce:

$$\begin{aligned} \hat{A}(t) + \hat{B}(t) &\leq C(a_T + b_T)(\hat{A}(t) + \hat{B}(t))(t-s)^{1-\frac{1}{\alpha p}-\frac{1}{2\alpha}} + C a_T \hat{A}(t)(t-s)^{1-\frac{1}{\alpha q}} \\ &\quad + C b_T \hat{B}(t)(t-s)^{1-\frac{1}{\alpha p}-\frac{1}{2\alpha}} + C \|f\|_Y a_T^{\beta-1} \hat{A}(t)(t-s)^{1-\frac{1}{\alpha}(\frac{1}{\rho}+\frac{\beta}{q}-\frac{1}{p})} \\ &\leq C(a_T + b_T + a_T^{\beta-1}) \times ((t-s)^{1-\frac{1}{\alpha p}-\frac{1}{2\alpha}} + (t-s)^{1-\frac{1}{\alpha q}} \\ &\quad + \|f\|_Y (t-s)^{1-\frac{1}{\alpha}(\frac{1}{\rho}+\frac{\beta}{q}-\frac{1}{p})}) \end{aligned} \tag{128}$$

for all $s < t < T$, where $C = C(\beta, q, p)$. Now, taking ξ_1, ξ_2 with:
and ξ_3 by:

$$\xi := \min\{\xi_1, \xi_2, \xi_3\},$$

$$\xi_1 = \left(\frac{1}{4C(a_T + b_T + a_T^{\beta-1})} \right)^{\frac{2\alpha p}{(2\alpha-1)p-2}},$$

$$\xi_2 = \left(\frac{1}{4C(a_T + b_T + a_T^{\beta-1})} \right)^{\frac{\alpha q}{\alpha q-1}},$$

$$\xi_3 = \left(\frac{1}{4C(a_T + b_T + a_T^{\beta-1})} \right)^{\frac{1}{1-\frac{1}{\alpha}(\frac{1}{\rho}+\frac{\beta}{q}-\frac{1}{p})}},$$

we have from (128) that $\hat{A}(t) + \hat{B}(t) = 0$ for $s \leq t \leq s + \xi$

which yields $n(t) = \hat{n}(t)$ and $u(t) = \hat{u}(t)$ for $s \leq t \leq s + \xi$. This proves Proposition 4.3 and the proof of Lemma 4.1 is complete.

To complete the proof of Theorem 2.1-(ii), we next show that the hypothesis (14) is actually redundant for uniqueness. To this end, we need further two propositions.

Proposition 4.4. For every precompact subset K_1 of $L^1(\mathbb{R}^2) \cap L^{\frac{1}{\alpha}}(\mathbb{R}^2)$ and K_2 of $L^2(\mathbb{R}^2) \cap L^{\frac{2}{2\alpha-1}}(\mathbb{R}^2)$, and for every $1 < q < \infty$ and $\frac{2}{\alpha} < p < \infty$, there

are uniformly bounded, non-decreasing functions $\eta_q^{(1)}(t; K_1)$ with $\lim_{t \rightarrow 0^+} \eta_q^{(1)}(t; K_1) = 0$ and $\eta_p^{(2)}(t; K_2)$ with $\lim_{t \rightarrow 0^+} \eta_p^{(2)}(t; K_2) = 0$ such that

$$t^{\frac{1}{\alpha}(\alpha - \frac{1}{q})} \|K_t * n_0\|_{L^q(\mathbb{R}^2)} \leq \eta_q^{(1)}(t; K_1)$$

and

$$t^{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p})} \|K_t * u_0\|_{L^p(\mathbb{R}^2)} \leq \eta_p^{(2)}(t; K_2)$$

holds for all $n_0 \in K_1$ and $u_0 \in K_2$ and all $t > 0$.

Proof. The readers may refer to the proof of Proposition 4.5 in [24].

Proposition 4.5. For every precompact subsets K_1 of $L^1(\mathbb{R}^2) \cap L^{\frac{1}{\alpha}}(\mathbb{R}^2)$ and K_2 of $L^2_\sigma(\mathbb{R}^2) \cap L^{\frac{2}{2\alpha-1}}(\mathbb{R}^2)$ and for every (q, p) satisfying (13), there is $T_* = T_*(K_1, K_2, \alpha, \beta, q, p, \|f\|_Y)$ such that for every $n_0 \in K_1$ and $u_0 \in K_2$ there exists a mild solution (n, u) of (1) on $[0, T_*)$ in the class of $M_{q,p}(0, T_*)$ with the property:

$$\begin{aligned} & t^{\frac{1}{\alpha}(\alpha - \frac{1}{q})} \|n(t)\|_{L^q(\mathbb{R}^2)} \\ & \leq \eta_q^{(1)}(t; K_1) + o(\sqrt{(\eta_q^{(1)}(t; K_1))^2 + (\eta_p^{(2)}(t; K_2))^2}) \end{aligned}$$

and

$$\begin{aligned} & t^{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p})} \|u(t)\|_{L^p(\mathbb{R}^2)} \\ & \leq \eta_p^{(2)}(t; K_2) + o(\sqrt{(\eta_q^{(1)}(t; K_1))^2 + (\eta_p^{(2)}(t; K_2))^2}) \end{aligned}$$

for all $0 < t < T_*$. Here $\eta_q^{(1)}(t; K_1)$ and $\eta_p^{(2)}(t; K_2)$ are same functions of t given by Proposition 4.4.

Proof. One can see the proof of Proposition 4.6 in [24] for more details.

We now turn to give a complete proof of Theorem 2.1-(ii).

Completion of the proof of Theorem 2.1-(ii). Let (n, u) be a mild solution of (1) on $[0, T)$ in the class of $M_{q,p}(0, T)$ for (q, p) satisfying (13). It is shown that the uniqueness holds under the additional hypothesis (14) by Lemma 4.1. Hence, it is still necessary to prove that (n, u) satisfies the property (14).

Since $n \in C([0, T]; L^1 \cap L^{\frac{1}{\alpha}}(\mathbb{R}^2))$ and $u \in C([0, T]; L^2_\sigma \cap L^{\frac{2}{2\alpha-1}}(\mathbb{R}^2))$, we obtain that $K_1 := \{n(t); 0 < t < \frac{T}{2}\}$ and $K_2 := \{u(t); 0 < t < \frac{T}{2}\}$ are precompact subsets of $L^1 \cap L^{\frac{1}{\alpha}}$ and $L^2_\sigma \cap L^{\frac{2}{2\alpha-1}}$, respectively. By Proposition 4.6, there exist $T_* = T_*(K_1, K_2, \beta, q, p, \|f\|_Y) > 0$ such that for every $\hat{n}_0 \in K_1$ and $\hat{u}_0 \in K_2$, we find a mild solution (\hat{n}, \hat{u}) of (1) on $[0, T_*)$ in the class of $M_{q,p}(0, T_*)$ with $\hat{n}|_{t=0} = \hat{n}_0$ and $\hat{u}|_{t=0} = \hat{u}_0$ so that

$$\begin{aligned} & t^{\frac{1}{\alpha}(\alpha - \frac{1}{q})} \|\hat{n}(t)\|_{L^q(\mathbb{R}^2)} \\ & \leq \eta_q^{(1)}(t; K_1) + o(\sqrt{(\eta_q^{(1)}(t; K_1))^2 + (\eta_p^{(2)}(t; K_2))^2}) \end{aligned}$$

and

$$\begin{aligned} & t^{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p})} \|\hat{u}(t)\|_{L^p(\mathbb{R}^2)} \\ & \leq \eta_p^{(2)}(t; K_2) + o(\sqrt{(\eta_q^{(1)}(t; K_1))^2 + (\eta_p^{(2)}(t; K_2))^2}) \end{aligned}$$

for all $0 < t < T_*$. In general, we may assume that $0 < T_* \leq \frac{T}{2}$. Such a solution $\{\hat{n}(t), \hat{u}(t)\}$ can be denoted by

$$\hat{n}(t) := K_t * \hat{n}_0 \text{ and } \hat{u}(t) := K_t * \hat{u}_0.$$

Taking $0 < s < T_*$ arbitrarily, we have $n(s) \in K_1$ and $u(s) \in K_2$. Since $n \in C((0, T); L^q)$ and $u \in C((0, T); L^p)$, it holds that

$$\lim_{t \rightarrow 0^+} t^{\frac{1}{\alpha}(\alpha - \frac{1}{q})} \|n(t+s)\|_{L^q(\mathbb{R}^2)} = 0$$

and

$$\lim_{t \rightarrow 0^+} t^{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p})} \|u(t+s)\|_{L^p(\mathbb{R}^2)} = 0.$$

Therefore, by Lemma 4.1, we conclude

$$n(t+s) = K_t * n(s) \text{ and } u(t+s) = K_t * u(s) \text{ for } 0 < t < T_*.$$

Then we deduce

$$\begin{aligned} & t^{\frac{1}{\alpha}(\alpha - \frac{1}{q})} \|n(t+s)\|_{L^q(\mathbb{R}^2)} \\ & \leq \eta_q^{(1)}(t; K_1) + o(\sqrt{(\eta_q^{(1)}(t; K_1))^2 + (\eta_p^{(2)}(t; K_2))^2}) \end{aligned}$$

and

$$\begin{aligned} & t^{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p})} \|u(t+s)\|_{L^p(\mathbb{R}^2)} \\ & \leq \eta_p^{(2)}(t; K_2) + o(\sqrt{(\eta_q^{(1)}(t; K_1))^2 + (\eta_p^{(2)}(t; K_2))^2}) \end{aligned}$$

for all $0 < t < T_*$. Since $n \in C((0, T); L^q)$ and $u \in C((0, T); L^p)$, by letting $s \rightarrow 0^+$ in this estimate, we observe

$$\begin{aligned} & t^{\frac{1}{\alpha}(\alpha - \frac{1}{q})} \|n(t)\|_{L^q(\mathbb{R}^2)} \\ & \leq \eta_q^{(1)}(t; K_1) + o(\sqrt{(\eta_q^{(1)}(t; K_1))^2 + (\eta_p^{(2)}(t; K_2))^2}) \end{aligned}$$

and

$$\begin{aligned} & t^{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p})} \|u(t)\|_{L^p(\mathbb{R}^2)} \\ & \leq \eta_p^{(2)}(t; K_2) + o(\sqrt{(\eta_q^{(1)}(t; K_1))^2 + (\eta_p^{(2)}(t; K_2))^2}) \end{aligned}$$

for all $0 < t < T_*$. From Proposition 4.4, it holds $\lim_{t \rightarrow 0^+} \eta_q^{(1)}(t; K_1) = 0$ and $\lim_{t \rightarrow 0^+} \eta_p^{(2)}(t; K_2) = 0$, which yields

$$\lim_{t \rightarrow 0^+} t^{\frac{1}{\alpha}(\alpha - \frac{1}{q})} \|n(t)\|_{L^q(\mathbb{R}^2)} = 0$$

and

$$\lim_{t \rightarrow 0^+} t^{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p})} \|u(t)\|_{L^p(\mathbb{R}^2)} = 0.$$

This implies that n and u necessarily satisfy the hypothesis (14), and the proof of Theorem 2.1-(ii) is complete.

4.3. Proof of Theorem 2.1-(iii)

For the global existence, since it holds

$$\begin{aligned} N_{1,\infty} &:= \sup_{0 < t < \infty} t^{\frac{1}{\alpha}(\alpha - \frac{1}{q})} \|K_t * n_0\|_{L^q(\mathbb{R}^2)} \\ &\leq G_q \|n_0\|_{L^1 \cap L^{\frac{1}{\alpha}}(\mathbb{R}^2)} \end{aligned}$$

and

$$\begin{aligned} U_{1,\infty} &:= \sup_{0 < t < \infty} t^{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p})} \|K_t * u_0\|_{L^p(\mathbb{R}^2)} \\ &\leq G_p \|u_0\|_{L^2 \cap L^{\frac{2}{2\alpha-1}}(\mathbb{R}^2)}, \end{aligned}$$

$$N_1 = \sup_{0 < t < T} t^{\frac{1}{\alpha}(\alpha - \frac{1}{q})} \|K_t * n_0\|_{L^q(\mathbb{R}^2)} \leq \sup_{0 < t < T} t^{\frac{1}{\alpha}(\alpha - \frac{1}{q})} (Ct^{-\frac{1}{\alpha}(\frac{1}{q} - \frac{1}{q})}) \leq CT^{\frac{1}{\alpha}(\alpha - \frac{1}{q})}$$

with $C = C(q, \hat{q})$. Hence, choosing $T \leq (\frac{\delta_1}{2C\|n_0\|_{L^q(\mathbb{R}^2)}})^{\frac{\alpha\hat{q}}{\alpha\hat{q}-1}}$ with δ_1 the same as that in (92), we obtain that $N_1 \leq \frac{\delta_1}{2}$.

As for u , in the situation of $2 < \hat{p} \leq p$, since $u_0 \in L^{\hat{p}}$, we arrive at

$$U_1 = \sup_{0 < t < T} t^{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p})} \|K_t * u_0\|_{L^p(\mathbb{R}^2)} \leq \sup_{0 < t < T} t^{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p})} (Ct^{-\frac{1}{\alpha}(\frac{1}{p} - \frac{1}{p})}) \leq CT^{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p})}$$

with $C = C(p, \hat{p})$. Therefore, taking $T \leq (\frac{\delta_1}{2C\|u_0\|_{L^{\hat{p}}(\mathbb{R}^2)}})^{\frac{2\alpha\hat{p}}{(2\alpha-1)\hat{p}-2}}$ with δ_1 the same as that in (92), we find that $U_1 \leq \frac{\delta_1}{2}$.

By choosing T as:

$$T \leq \left(\frac{\delta_1}{2C\|n_0\|_{L^{\hat{q}}(\mathbb{R}^2)}}\right)^{\frac{\alpha\hat{q}}{\alpha\hat{q}-1}} \text{ for } \hat{q} \leq q, T \leq \left(\frac{\delta_1}{2C\|u_0\|_{L^{\hat{p}}(\mathbb{R}^2)}}\right)^{\frac{2\alpha\hat{p}}{(2\alpha-1)\hat{p}-2}} \text{ for } \hat{p} \leq p,$$

we observe that $N_1 + U_1 \leq \delta_1$ for some $\hat{q} > 1$ and $\hat{p} > 2$. Employing a similar argument as that in (98)-(99), we discover that (91) is fulfilled and we get the mild solution on $[0, T)$.

Next, we shall show (16)-(17) by contradiction. We consider the cases of $1 < r \leq q$ and $2 < \tilde{r} \leq p$. The constants C_*^1 and C_*^2 are chosen as $C_*^1 = C_1(\alpha, \beta, q, p, r, \|f\|_Y)$ and $C_*^2 = C_2(\alpha, \beta, q, p, \tilde{r}, \|f\|_Y)$, respectively, where C_*^1 and C_*^2 are the corresponding as C_1 and C_2 in (15). Assume that there are $0 < t_0 < T_{max}$ such that:

$$\|n(t_0)\|_{L^r(\mathbb{R}^2)} \leq C_*^1 (T_{max} - t_0)^{-\frac{1}{\alpha}(\alpha - \frac{1}{r})}$$

and

$$\|u(t_0)\|_{L^{\tilde{r}}(\mathbb{R}^2)} \leq C_*^2 (T_{max} - t_0)^{-\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{\tilde{r}})}.$$

Hence we can take $T' > T_{max}$ such that:

$$\begin{aligned} T' - t_0 = \min \left\{ \left(\frac{C_*^1}{\|n(t_0)\|_{L^r(\mathbb{R}^2)}} \right)^{\frac{\alpha r}{\alpha r - 1}}, \right. \\ \left. \left(\frac{C_*^2}{\|u(t_0)\|_{L^{\tilde{r}}(\mathbb{R}^2)}} \right)^{\frac{2\alpha\tilde{r}}{(2\alpha-1)\tilde{r}-2}} \right\}. \end{aligned}$$

$n \in C([0, \hat{T}]; L^{q_*})$ with

$$\lim_{j \rightarrow \infty} \left(\sup_{0 < t < \hat{T}} \|n_j(t) - n(t)\|_{L^{q_*}(\mathbb{R}^2)} \right) = 0$$

(130)

where G_q and G_p are given by (11) and (12), respectively. By taking $\|n_0\|_{L^1 \cap L^{\frac{1}{\alpha}}} \leq \frac{\delta_1}{2G_q}$ and $\|u_0\|_{L^2 \cap L^{\frac{2}{2\alpha-1}}} \leq \frac{\delta_1}{2G_p}$ with δ_1 the same as (92), we obtain that $N_{1,\infty} + U_{1,\infty} \leq \delta_1$. Using the same arguments from (98) to (99), we obtain that (91) in Proposition 4.2 is fulfilled and we deduce the mild solution on $[0, \infty)$. Then the proof of Theorem 2.1-(iii) is completed.

4.4. Proof of Theorem 2.1-(iv)

Assume that $n_0 \in L^1 \cap L^{\hat{q}}$ and $u_0 \in L^2 \cap L^{\hat{p}}$ for some $\hat{q} > 1$ and $\hat{p} > 2$. We will prove the above integrability of n_0 and u_0 to obtain (91).

As for n , in the situation of $\hat{q} \leq q$, since $n_0 \in L^{\hat{q}}$, we deduce

Note that (q, p) satisfies (13), and since $n(t_0) \in L^1 \cap L^q$ and $u(t_0) \in L^2 \cap L^p$, based on Theorem 2.1-(i) and (15), there exists a mild solution (\hat{n}, \hat{u}) of (1) on $[t_0, T')$ in the class $M_{q,p}(t_0, T')$ with $\hat{n}|_{t=t_0} = n(t_0)$ and $\hat{u}|_{t=t_0} = u(t_0)$. The uniqueness result of Theorem 2.1-(ii) shows:

$$n(t) = \hat{n}(t) \text{ and } u(t) = \hat{u}(t) \text{ for } t \in [t_0, T_{max}). \quad (129)$$

However, since $\hat{n} \in C((t_0, T'); L^1 \cap L^q)$ and $\hat{u} \in C((t_0, T'); L^2 \cap L^p)$, we can deduce from (129) that n can be extended as a continuous function beyond $t = T_{max}$ in the same class of $M_{q,p}(t_0, T')$. This causes a contradiction.

The proof of (18) can be easily derived similarly in [24, p.5450] and we omitted it here. This proves Theorem 2.1-(iv).

4.5. Proof of Theorem 2.1-(v)

We deal with the general cases that (q_*, p_*) satisfies (19) and conditions (1)-(4) in Theorem 2.1-(v) to prove the time continuity in L^q for n and L^p for u . In fact, we will prove that there exists a time $\hat{T} = \hat{T}(\alpha, \beta, q, p, q_*, p_*, T_0) \in (0, T)$ such that the limits of n and u are satisfied:

and

$$u \in C([0, \hat{T}); L^{p*}) \text{ with } \lim_{j \rightarrow \infty} \left(\sup_{0 < t < \hat{T}} \|u_j(t) - u(t)\|_{L^{p*}(\mathbb{R}^2)} \right) = 0. \quad (131)$$

Let $s \in (0, T_0)$ with T_0 obtained in Theorem 2.1-(i). In order to prove (130) and (131), we shall show that there are monotone increasing sequences $\{\mathcal{D}_j(s)\}_{j=1}^\infty$ and $\{\mathcal{E}_j(s)\}_{j=1}^\infty$ such that:

$$\sup_{0 < t < s} \|n_j(t)\|_{L^{q*}(\mathbb{R}^2)} \leq \mathcal{D}_j(s), \quad j = 1, 2, \dots, \quad (132)$$

and

$$\sup_{0 < t < s} \|u_j(t)\|_{L^{p*}(\mathbb{R}^2)} \leq \mathcal{E}_j(s), \quad j = 1, 2, \dots. \quad (133)$$

Similarly to (66) and (67), we define sequences $\{N_j(s)\}_{j=1}^\infty$ and $\{U_j(s)\}_{j=1}^\infty$ so that:

$$\sup_{0 < t < s} t^{\frac{1}{\alpha}(\alpha - \frac{1}{q})} \|n_j(t)\|_{L^q(\mathbb{R}^2)} \leq N_j(s), \quad j = 1, 2, \dots, \quad (134)$$

$$\sup_{0 < t < s} t^{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p})} \|u_j(t)\|_{L^p(\mathbb{R}^2)} \leq U_j(s), \quad j = 1, 2, \dots. \quad (135)$$

According to a similar argument like that in (83), we find by $(N_1(s), U_1(s)) \in B_\delta(\mathbf{0})$ with $N_1(s) > 0$ and $U_1(s) > 0$ that the sequences $\{N_j(s)\}_{j=1}^\infty$ and $\{U_j(s)\}_{j=1}^\infty$ are satisfied:

$$\begin{cases} N_1(s) \leq N_2(s) \leq \dots \leq N_j(s) \leq N_{j+1}(s) \leq \dots \uparrow x(s), \\ U_1(s) \leq U_2(s) \leq \dots \leq U_j(s) \leq U_{j+1}(s) \leq \dots \uparrow y(s), \end{cases} \quad (136)$$

where $(x(s), y(s))$ is the root of (81) with N_1 and U_1 replaced by $N_1(s)$ and $U_1(s)$, respectively. It is worth noting that Proposition 4.1 holds true for $(x(s), y(s))$ with $s \in (0, T_0)$.

At first, We proof (132) under the condition of q_* that:

$$\max \left\{ \frac{2q}{3q-2}, \frac{p}{p-1} \right\} \leq q_* \leq \infty. \quad (137)$$

We could define $\mathcal{D}_1(s)$ by $\mathcal{D}_1(s) := \sup_{0 < t < s} \|K_t * n_0\|_{L^{q*}(\mathbb{R}^2)} \leq \|n_0\|_{L^{q*}(\mathbb{R}^2)}$. Assume that (134) is true for j . For the second term in the right hand side of the first equation of (1), it follows that:

$$\begin{aligned} & \int_0^t \|\nabla K_{t-\tau} * (u_j n_j)(\tau)\|_{L^{q*}(\mathbb{R}^2)} d\tau \leq C \int_0^t (t-\tau)^{-\frac{1}{\alpha}(\frac{1}{\hat{r}_1} - \frac{1}{q_*}) - \frac{1}{2\alpha}} \|(u_j n_j)(\tau)\|_{L^{\hat{r}_1}(\mathbb{R}^2)} d\tau \\ & \leq C \int_0^t (t-\tau)^{-\frac{1}{\alpha p} - \frac{1}{2\alpha}} \|u_j(\tau)\|_{L^p(\mathbb{R}^2)} \|n_j(\tau)\|_{L^{q*}(\mathbb{R}^2)} d\tau \\ & \leq C \mathcal{D}_j(s) y(s) \int_0^t (t-\tau)^{-\frac{1}{\alpha p} - \frac{1}{2\alpha}} \tau^{-\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p})} d\tau \leq CB(1 - \frac{1}{\alpha p} - \frac{1}{2\alpha}, \frac{1}{\alpha}(\frac{1}{2} + \frac{1}{p})) \mathcal{D}_j(s) y(s), \end{aligned} \quad (138)$$

for all $0 < t < s$, where $C = C(q_*, p)$. Here we assume $\frac{1}{p} + \frac{1}{q_*} \leq 1$ and choose $1 \leq \hat{r}_1 < q_*$ such that $\frac{1}{\hat{r}_1} = \frac{1}{p} + \frac{1}{q_*}$. Hence we obtain (138) for all $p' \leq q_* \leq \infty$ with $p' = \frac{p}{p-1}$.

Next, let $q_* \geq \frac{2q}{3q-2}$. Then we have $\frac{1}{q_*} + \frac{1}{q} - \frac{1}{2} \leq 1$. For the last term in the right hand side of the first equation of (1), we arrive at

$$\begin{aligned} & \int_0^t \|\nabla K_{t-\tau} * (n_j \nabla v_j)(\tau)\|_{L^{q*}(\mathbb{R}^2)} d\tau \leq C \int_0^t (t-\tau)^{-\frac{1}{\alpha}(\frac{1}{\hat{r}_2} - \frac{1}{q_*}) - \frac{1}{2\alpha}} \|(n_j \nabla v_j)(\tau)\|_{L^{\hat{r}_2}(\mathbb{R}^2)} d\tau \\ & \leq C \int_0^t (t-\tau)^{-\frac{1}{\alpha q} - \frac{1}{2\alpha}} \|n_j(\tau)\|_{L^{q*}(\mathbb{R}^2)} \|\nabla v_j(\tau)\|_{L^{\frac{2q}{2-q}}(\mathbb{R}^2)} d\tau \leq C \int_0^t (t-\tau)^{-\frac{1}{\alpha q}} \|n_j(\tau)\|_{L^{q*}(\mathbb{R}^2)} \|n_j(\tau)\|_{L^q(\mathbb{R}^2)} d\tau \\ & \leq C \mathcal{D}_j(s) x(s) \int_0^t (t-\tau)^{-\frac{1}{\alpha q} - \frac{1}{2\alpha}} \tau^{-\frac{1}{\alpha}(\alpha - \frac{1}{q})} d\tau \leq CB(1 - \frac{1}{\alpha q}, \frac{1}{\alpha q}) \mathcal{D}_j(s) x(s) \end{aligned} \quad (139)$$

for all $0 < t < s$, where $C = C(q, q_*)$. Here we choose $1 \leq \hat{r}_2 < q_*$ so that $\frac{1}{\hat{r}_2} = \frac{1}{q} + \frac{1}{q_*} - \frac{1}{2}$.

Based on the condition (137), combining (83), (138) and (139), we deduce

$$\|n_{j+1}(t)\|_{L^{q_*}(\mathbb{R}^2)} \leq \mathcal{D}_1(s) + h_1 \mathcal{D}_j(s) y(s) \cdot B(1 - \frac{1}{\alpha p} - \frac{1}{2\alpha}, \frac{1}{\alpha}(\frac{1}{2} + \frac{1}{p})) + h_1 \mathcal{D}_j(s) x(s) B(1 - \frac{1}{\alpha q}, \frac{1}{\alpha q})$$

with $h_1 = h_1(q, p, q_*)$ for all $0 < t < s$. Hence we may take $\mathcal{D}_{j+1}(s)$ as:

$$\mathcal{D}_{j+1}(s) := \mathcal{D}_1(s) + h_1 \mathcal{D}_j(s)(x(s) + y(s))$$

with $h_1 = h_1(q, p, q_*)$ independent of j , where $x(s)$ and $y(s)$ are given in (136). By this induction, we can prove that there is a monotonically increasing sequence $\{\mathcal{D}_j(s)\}_{j=1}^\infty$ so that (132) holds for all $j = 1, 2, \dots$. Therefore we prove (132) under (137).

Moreover, by the induction, we shall show that there exists a time $T_1 = T_1(q, p, q_*) \leq T_0$ such that:

$$\mathcal{D}_1(T_1) \leq \mathcal{D}_2(T_1) \leq \dots \leq \mathcal{D}_j(T_1) \leq \mathcal{D}_{j+1}(T_1) \leq \dots \uparrow \mathcal{D}(T_1), \quad (140)$$

where $\mathcal{D}(T_1) > 0$ is the root of $\mathcal{D}(T_1) = \mathcal{D}_1(T_1) + h_1 \mathcal{D}(T_1)(x(T_1) + y(T_1))$, i.e.,

$$\mathcal{D}(T_1) = \frac{\mathcal{D}_1(T_1)}{1 - h_1(x(T_1) + y(T_1))}. \quad (141)$$

In fact, employing (94) and (96) to replace T with s , as we deduce from (82) in Proposition 4.1 that there exists a time $T_1 = T_1(q, p, q_*) \leq T_0$ so that:

$$0 < x(T_1) < \frac{1}{4h_1} \quad \text{and} \quad 0 < y(T_1) < \frac{1}{4h_1}. \quad (142)$$

Then $h_1(x(T_1) + y(T_1)) < h_1(\frac{1}{4h_1} + \frac{1}{4h_1}) = \frac{1}{2} < 1$ and $\mathcal{D}(T_1) = \frac{\mathcal{D}_1(T_1)}{1 - h_1(x(T_1) + y(T_1))} > \mathcal{D}_1(T_1) > 0$. Hence, by the induction, we can obtain that $\mathcal{D}_j(T_1)$ is a bounded monotone increasing sequence so that $\mathcal{D}_j(T_1) \leq \mathcal{D}(T_1)$ for all $j = 1, 2, \dots$. This implies (140) with (142).

Next, we shall show (133) under the conditions (19) and (1)-(4) of (q_*, p_*) in Theorem 2.1-(v). We may define $\mathcal{E}_1(s) := \sup_{0 < t < s} \|K_t * u_0\|_{L^{p_*}(\mathbb{R}^2)} \leq \|u_0\|_{L^{p_*}(\mathbb{R}^2)}$. Assume that (135) is true for j .

Let T_1 be as that in (140) and $s \in (0, T_1]$. Since $2 < p < \infty$, we have $p' \in (1, 2)$. We are going to estimate the Duhamel term in the second equation of (1) for the cases $1 < p_* < p'$ and $p' \leq p_* \leq \infty$.

(i) ($1 < p_* < p'$) We first choose \hat{r}_3 so that $\hat{r}_3 = \frac{2\alpha-1}{2} + \frac{2\alpha-1}{2}$. Then it holds $1 < \hat{r}_3 < q_*$. Hence it follows that

$$\begin{aligned} & \int_0^t \|\nabla K_{t-\tau} * P(u_j \otimes u_j)(\tau)\|_{L^{p_*}(\mathbb{R}^2)} d\tau \\ & \leq C \int_0^t (t-\tau)^{-\frac{1}{\alpha}(\frac{1}{\hat{r}_3} - \frac{1}{p_*}) - \frac{1}{2\alpha}} \|u_j \otimes u_j(\tau)\|_{L^{\hat{r}_3}(\mathbb{R}^2)} d\tau \\ & \leq C \int_0^t (t-\tau)^{-\frac{1}{\alpha}(2\alpha-1-\frac{1}{p_*}) - \frac{1}{2\alpha}} \|u_j(\tau)\|_{L^{\frac{2}{2\alpha-1}}(\mathbb{R}^2)}^2 d\tau \\ & \leq C \mathcal{B}^2 s^{-\frac{1}{\alpha}(\alpha-\frac{1}{2}-\frac{1}{p_*})} \end{aligned}$$

for all $0 < t < s$ with $C = C(p_*)$, where \mathcal{B} is given in (120). This implies for all (q_*, p_*) satisfying $1 < p_* < p'$ and $1 \leq q_* \leq \infty$ that

$$\int_0^t \|\nabla K_{t-\tau} * P(u_j \otimes u_j)(\tau)\|_{L^{p_*}(\mathbb{R}^2)} d\tau \leq C \mathcal{B}^2 s^{-\frac{1}{\alpha}(\alpha-\frac{1}{2}-\frac{1}{p_*})} \quad (143)$$

for all $0 < t < s$, where $C = C(p_*)$.

(ii) ($p' \leq p_* \leq \infty$) By selecting \hat{r}_4 so that $\hat{r}_4 = \frac{1}{p_*} + \frac{1}{p}$, this implies that

$$\begin{cases} (a) 1 \leq \hat{r}_4 < p_* & \text{for } p' \leq p_* < \infty, \\ (b) 1 < \hat{r}_4 < p_* & \text{for } p_* = \infty. \end{cases}$$

For the case (a), we obtain $1 < p_* < \infty$. As for case (b), we have $1 < \hat{r}_4 < \infty$. Therefore, we discover

$$\begin{aligned}
& \int_0^t \|\nabla K_{t-\tau} * P(u_j \otimes u_j)(\tau)\|_{L^{p_*}(\mathbb{R}^2)} d\tau \\
& \leq C \int_0^t (t-\tau)^{-\frac{1}{\alpha}(\frac{1}{\hat{r}_4} - \frac{1}{p_*}) - \frac{1}{2\alpha}} \|(u_j \otimes u_j)(\tau)\|_{L^{\hat{r}_4}(\mathbb{R}^2)} d\tau \\
& \leq C \int_0^t (t-\tau)^{-\frac{1}{\alpha p} - \frac{1}{2\alpha}} \|u_j(\tau)\|_{L^{p_*}(\mathbb{R}^2)} \|u_j(\tau)\|_{L^p(\mathbb{R}^2)} d\tau \\
& \leq C \mathcal{E}_j(s) y(s) \int_0^t (t-\tau)^{-\frac{1}{\alpha p} - \frac{1}{2\alpha}} \tau^{-\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p})} d\tau \\
& \leq C \mathcal{E}_j(s) y(s) B(1 - \frac{1}{\alpha p} - \frac{1}{2\alpha}, \frac{1}{\alpha}(\frac{1}{2} + \frac{1}{p}))
\end{aligned}$$

for all $0 < t < s$, where $C = C(p, p_*)$. For all (q_*, p_*) satisfying $1 \leq q_* \leq \infty$ and $p' \leq p_* \leq \infty$, we deduce

$$\int_0^t \|\nabla K_{t-\tau} * P(u_j \otimes u_j)(\tau)\|_{L^{p_*}(\mathbb{R}^2)} d\tau \leq C \mathcal{E}_j(s) y(s) \quad (144)$$

for all $0 < t < s$, where $C = C(p, p_*)$.

We will now deal with the external force term. Therefore, we divide the cases into four cases (1)-(4) under the assumption of theorem 2.1-(v).

case (1): $1 \leq q_* \leq \frac{2\beta}{2\alpha\beta+1-2\alpha} (< q)$ and $\frac{2q}{2\beta+q(4\alpha-1-2\alpha\beta)} \leq p_* \leq 2$.

Choosing \hat{r}_5 so that $\hat{r}_5 = \frac{1}{\rho} + \frac{\beta}{q}$, we find that $\hat{r}_5 \geq 1$, and in addition, $\hat{r}_5 \leq p_*$ when $\frac{2q}{2\beta+q(4\alpha-1-2\alpha\beta)} \leq p_*$. Then under the assumption $f \in L^\rho$, we observe

$$\begin{aligned}
& \int_0^t \|K_{t-\tau} * P(|n_j|^\beta f)(\tau)\|_{L^{p_*}(\mathbb{R}^2)} d\tau \leq C \int_0^t (t-\tau)^{-\frac{1}{\alpha}(\frac{1}{\hat{r}_5} - \frac{1}{p_*})} \|(|n_j|^\beta f)(\tau)\|_{L^{\hat{r}_5}(\mathbb{R}^2)} d\tau \\
& \leq C \int_0^t (t-\tau)^{-\frac{1}{\alpha}(\frac{1}{\rho} + \frac{\beta}{q} - \frac{1}{p_*})} \|f\|_{L^\rho(\mathbb{R}^2)} \|n_j(\tau)\|_{L^q(\mathbb{R}^2)}^\beta d\tau \leq C x(s)^\beta \|f\|_{L^\rho(\mathbb{R}^2)} \int_0^t (t-\tau)^{-\frac{1}{\alpha}(\frac{1}{\rho} + \frac{\beta}{q} - \frac{1}{p_*})} \tau^{-\frac{\beta}{\alpha}(\alpha - \frac{1}{q})} d\tau \\
& \leq C x(s)^\beta \|f\|_{L^\rho(\mathbb{R}^2)} B(1 - \frac{1}{\alpha}(\frac{1}{\rho} + \frac{\beta}{q} - \frac{1}{p_*}), 1 - \frac{\beta}{\alpha}(\alpha - \frac{1}{q})) \cdot s^{-\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p})}
\end{aligned}$$

provided $1 < p_* < 2$, where $C = C(\alpha, \beta, q, p_*)$. Hence, for all (q_*, p_*) satisfying $1 \leq q_* \leq \frac{2\beta}{2\alpha\beta+1-2\alpha}$ and $\frac{2q}{2\beta+q(4\alpha-1-2\alpha\beta)} \leq p_* \leq 2$ with $p_* \neq 1$, we have

$$\int_0^t \|K_{t-\tau} * P(|n_j|^\beta f)(\tau)\|_{L^{p_*}(\mathbb{R}^2)} d\tau \leq C x(s)^\beta \|f\|_{L^\rho(\mathbb{R}^2)} s^{-\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p})} \quad (145)$$

for all $0 < t < s$, where $C = C(\alpha, \beta, q, p_*)$.

case (2): $\frac{2\beta}{2\alpha\beta+1-2\alpha} < q_* \leq \infty$ and $1 < p_* < \infty$.

Choosing \hat{r}_6 so that $\hat{r}_6 = \frac{1}{\rho} + \frac{2\alpha\beta+1-2\alpha}{2}$, since $\frac{1}{\rho} + \frac{2\alpha\beta+1-2\alpha}{2} = \frac{1}{\alpha}$, we derive that $\frac{1}{\alpha} = \hat{r}_6 < p_*$. Note that $\mathcal{D}(T_1) \leq \mathcal{D}(s)$ for all $0 < s \leq T_1$ in (140), applying the assumption of $f \in L^\rho$ and (137), we deduce

$$\begin{aligned}
& \int_0^t \|K_{t-\tau} * P(|n_j|^\beta f)(\tau)\|_{L^{p_*}(\mathbb{R}^2)} d\tau \\
& \leq C \int_0^t (t-\tau)^{-\frac{1}{\alpha}(\frac{1}{\hat{r}_6} - \frac{1}{p_*})} \|(|n_j|^\beta f)(\tau)\|_{L^{\hat{r}_6}(\mathbb{R}^2)} d\tau \\
& \leq C \int_0^t (t-\tau)^{-\frac{1}{\alpha}(\frac{1}{\rho} + \frac{2\alpha\beta+1-2\alpha}{2} - \frac{1}{p_*})} \|f\|_{L^\rho(\mathbb{R}^2)} \|n_j(\tau)\|_{L^{\frac{2\beta}{2\alpha\beta+1-2\alpha}}(\mathbb{R}^2)}^\beta d\tau \\
& \leq C \mathcal{A}^{\beta(1-\theta_5)} \mathcal{D}(T_1)^{\beta\theta_5} \|f\|_{L^\rho(\mathbb{R}^2)} \int_0^t (t-\tau)^{-\frac{1}{\alpha}(\alpha - \frac{1}{p_*})} d\tau \\
& \leq C \mathcal{A}^{\beta(1-\theta_5)} \mathcal{D}(T_1)^{\beta\theta_5} \|f\|_{L^\rho(\mathbb{R}^2)} s^{\frac{1}{\alpha p_*}}
\end{aligned}$$

for all $0 < t < s$ with $C = C(\alpha, \beta, p_*)$, where \mathcal{A} is given in (112). Since $\frac{2\beta}{2\alpha\beta+1-2\alpha} < q_* \leq \infty$, $\theta_5 = \frac{\alpha - \frac{2\alpha\beta+1-2\alpha}{2\beta}}{\alpha - \frac{1}{q_*}} \in (0, 1)$. Thus recalling on $\frac{2\beta}{2\alpha\beta+1-2\alpha} < q_* \leq \infty$ and $1 < p_* < \infty$, along with (137), we arrive at

$$\int_0^t \|K_{t-\tau} * P(|n_j|^\beta f)(\tau)\|_{L^{p_*}(\mathbb{R}^2)} d\tau \leq C \mathcal{A}^{\beta(1-\theta_5)} \mathcal{D}(T_1)^{\beta\theta_5} \|f\|_{L^\rho(\mathbb{R}^2)} s^{\frac{1}{\alpha p_*}} \quad (146)$$

for all $0 < t < s$, where $C = C(\alpha, \beta, p_*)$.

case (3): $\frac{2\beta}{2\alpha\beta+1-2\alpha} < q_* < \infty$ and $p_* = \infty$.

Choosing \hat{r}_7 so that $\hat{r}_6 = \frac{1}{\rho} + \frac{\beta}{q_*}$, we obtain that $1 < \hat{r}_7 < \infty$. Then under the assumption of $f \in L^\rho$ and (137), it follows that

$$\begin{aligned} & \int_0^t \|K_{t-\tau} * P(|n_j|^\beta f)(\tau)\|_{L^{p_*}(\mathbb{R}^2)} d\tau \\ &= \int_0^t \|K_{t-\tau} * P(|n_j|^\beta f)(\tau)\|_{L^\infty(\mathbb{R}^2)} d\tau \\ &\leq C \int_0^t (t-\tau)^{-\frac{1}{\alpha\hat{r}_7}} \|(|n_j|^\beta f)(\tau)\|_{L^{\hat{r}_7}(\mathbb{R}^2)} d\tau \\ &\leq C \int_0^t (t-\tau)^{-\frac{1}{\alpha}(\frac{1}{\rho} + \frac{\beta}{q_*})} \|f\|_{L^\rho(\mathbb{R}^2)} \|n_j(\tau)\|_{L^{q_*}(\mathbb{R}^2)}^\beta d\tau \\ &\leq C \mathcal{D}(T_1)^\beta \|f\|_{L^\rho(\mathbb{R}^2)} s^{1-\frac{1}{\alpha}(\frac{1}{\rho} + \frac{\beta}{q_*})} \end{aligned}$$

for all $0 < t < s$. Since $\frac{2\beta}{2\alpha\beta+1-2\alpha} < q_* < \infty$, we have $1 - \frac{1}{\alpha}(\frac{1}{\rho} + \frac{\beta}{q_*}) > 0$. Based on $\frac{2\beta}{2\alpha\beta+1-2\alpha} < q_* < \infty$ and $p_* = \infty$, together with (137), we get

$$\int_0^t \|K_{t-\tau} * P(|n_j|^\beta f)(\tau)\|_{L^{p_*}(\mathbb{R}^2)} d\tau \leq C \mathcal{D}(T_1)^\beta \|f\|_{L^\rho(\mathbb{R}^2)} s^{1-\frac{1}{\alpha}(\frac{1}{\rho} + \frac{\beta}{q_*})} \quad (147)$$

for all $0 < t < s$, where $C = C(\alpha, \beta, q_*, p_*)$.

case (4): $q_* = \infty$ and $p_* = \infty$.

Let $1 < \beta < \frac{3}{2}$. Then from (64), we have $2 < \rho < \infty$. Hence under the assumption of $f \in L^\rho$ and (137), we discover

$$\begin{aligned} & \int_0^t \|K_{t-\tau} * P(|n_j|^\beta f)(\tau)\|_{L^{p_*}(\mathbb{R}^2)} d\tau \\ &= \int_0^t \|K_{t-\tau} * P(|n_j|^\beta f)(\tau)\|_{L^\infty(\mathbb{R}^2)} d\tau \\ &\leq C \int_0^t (t-\tau)^{-\frac{1}{\alpha\rho}} \|(|n_j|^\beta f)(\tau)\|_{L^\rho(\mathbb{R}^2)} d\tau \\ &\leq C \int_0^t (t-\tau)^{-\frac{1}{\alpha\rho}} \|f\|_{L^\rho(\mathbb{R}^2)} \|n_j(\tau)\|_{L^\infty(\mathbb{R}^2)}^\beta d\tau \\ &\leq C \mathcal{D}(T_1)^\beta \|f\|_{L^\rho(\mathbb{R}^2)} s^{1-\frac{1}{\alpha\rho}} \end{aligned}$$

for all $0 < t < s$. Therefore employing $q_* = \infty$, $p_* = \infty$ and (137), we have

$$\int_0^t \|K_{t-\tau} * P(|n_j|^\beta f)(\tau)\|_{L^{p_*}(\mathbb{R}^2)} d\tau \leq C \mathcal{D}(T_1)^\beta \|f\|_{L^\rho(\mathbb{R}^2)} s^{1-\frac{1}{\alpha\rho}} \quad (148)$$

for all $0 < t < s$, where $C = C(\alpha, \beta)$.

Then combining (143)-(147) and (148), we deduce

$$\|u_{j+1}(t)\|_{L^{p_*}(\mathbb{R}^2)} \leq \mathcal{E}_1(s) + h_2 \mathcal{E}_j(s) y(s) + h_2 (1 + \|f\|_{L^\rho(\mathbb{R}^2)}) (\mathcal{A}^\beta + \mathcal{B} + \mathcal{D}(T_1)^\beta) \cdot (1 + x(s)^\beta + y(s))(1 + s)$$

with $h_2 = h_2(\alpha, \beta, q, p, q_*, p_*)$ for all $0 < t < s \leq T_1 \leq T_0$.

Then we may take $\mathcal{E}_{j+1}(s)$ as:

$$\mathcal{E}_{j+1}(s) := \mathcal{E}_1(s) + h_2 \mathcal{E}_j(s) y(s) + h_2(1 + \|f\|_{L^\rho(\mathbb{R}^2)})(\mathcal{A}^\beta + \mathcal{B} + \mathcal{D}(T_1)^\beta) \cdot (1 + x(s)^\beta + y(s))(1 + s),$$

where $h_2 = h_2(\alpha, \beta, q, p, q_*, p_*)$ is a constant independent of j . By this induction, we can show that there exists a monotone increasing sequence $\{\mathcal{E}_j(s)\}_{j=1}^\infty$ so that (133) holds for all $j = 1, 2, \dots$. Thus we can prove (133) under (137).

Moreover, by the induction, we shall show that there exists a time

$$\hat{T} = \hat{T}(\alpha, \beta, q, p, q_*, p_*, T_0) \leq T_1$$

so that

$$0 < \mathcal{E}_1(\hat{T}) \leq \mathcal{E}_2(\hat{T}) \leq \dots \leq \mathcal{E}_j(\hat{T}) \leq \mathcal{E}_{j+1}(\hat{T}) \leq \dots \uparrow \mathcal{E}(\hat{T}), \quad (149)$$

where $\mathcal{E}(\hat{T})$ is the root of

$$\mathcal{E}(\hat{T}) := \mathcal{E}_1(\hat{T}) + h_2 \mathcal{E}(\hat{T}) y(\hat{T}) + h_2(1 + \|f\|_{L^\rho(\mathbb{R}^2)})(\mathcal{A}^\beta + \mathcal{B} + \mathcal{D}(T_1)^\beta) \cdot (1 + x(\hat{T})^\beta + y(\hat{T}))(1 + \hat{T}), \quad (150)$$

i.e., that

$$\begin{aligned} \mathcal{E}(\hat{T}) &= \frac{\mathcal{E}_1(\hat{T})}{1 - y(\hat{T})} + \frac{h_2(\mathcal{A}^\beta + \mathcal{B} + \mathcal{D}(T_1)^\beta)(1 + x(\hat{T})^\beta + y(\hat{T}))(1 + \hat{T})}{1 - y(\hat{T})} \\ &+ \frac{h_2\|f\|_{L^\rho(\mathbb{R}^2)}(\mathcal{A}^\beta + \mathcal{B} + \mathcal{D}(T_1)^\beta)(1 + x(\hat{T})^\beta + y(\hat{T}))(1 + \hat{T})}{1 - y(\hat{T})}. \end{aligned}$$

In fact, applying (94) and (96) with T replaced by s , we deduce from (82) in Proposition 4.1 that there exists a time $\hat{T} = \hat{T}(\alpha, \beta, q, p, q_*, p_*, T_0) \leq T_1$ so that

$$0 < y(\hat{T}) < \frac{1}{2h_2}. \quad (151)$$

Since $h_2 y(\hat{T}) < \frac{1}{2} < 1$ and $\mathcal{E}(\hat{T}) > \mathcal{E}_1(\hat{T}) > 0$, we can find by the induction that $\mathcal{E}_j(\hat{T})$ is a bounded monotone increasing sequence so that $\mathcal{E}_j(\hat{T}) \leq \mathcal{E}(\hat{T})$ for all $j = 1, 2, \dots$. This yields (149) with (150). Hence we prove (133) under the conditions (1)-(4) in Theorem 2.1-(v).

Let $\tilde{n}_j(t)$ and $\tilde{u}_j(t)$ be defined by (86). Now we define $\{\tilde{\mathcal{D}}_j(\hat{T})\}_{j=1}^\infty$ and $\{\tilde{\mathcal{E}}_j(\hat{T})\}_{j=1}^\infty$ by:

$$\sup_{0 < t < \hat{T}} \|\tilde{n}_j(t)\|_{L^{q_*}(\mathbb{R}^2)} \leq \tilde{\mathcal{D}}_j(\hat{T}), \quad \sup_{0 < t < \hat{T}} \|\tilde{u}_j(t)\|_{L^{p_*}(\mathbb{R}^2)} \leq \tilde{\mathcal{E}}_j(\hat{T}), \quad j = 1, 2, \dots \quad (152)$$

Let us denote $\{\tilde{N}_j(\hat{T})\}_{j=1}^\infty$ and $\{\tilde{U}_j(\hat{T})\}_{j=1}^\infty$ be the same as that in (89) with T replaced by \hat{T} . Employing (142) and (151), the same argument as in the proof of (138), (139) and (143)-(148) holds

$$\begin{aligned} \tilde{\mathcal{D}}_j(\hat{T}) &= h_1(\mathcal{D}(T_1)\tilde{U}_j + \tilde{\mathcal{D}}_j(\hat{T})y(T_1)) + h_1(\tilde{\mathcal{D}}_j(\hat{T})x(T_1) + \mathcal{D}(T_1)\tilde{N}_j) \\ &\leq h_1\mathcal{D}(T_1)(\tilde{U}_j + \tilde{N}_j) + \frac{\tilde{\mathcal{D}}_j(\hat{T})}{2} \end{aligned}$$

and

$$\begin{aligned} \tilde{\mathcal{E}}_j(\hat{T}) &= h_2 \left((\tilde{\mathcal{B}}_j U_{j+1} + \mathcal{B}_j \tilde{U}_j) \hat{T}^{-\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p})} + \tilde{\mathcal{E}}_j(\hat{T}) U_{j+1} + \mathcal{E}_j(\hat{T}) \tilde{U}_j + \|f\|_{L^\rho(\mathbb{R}^2)} (N_{j+1}^{\beta-1} \tilde{N}_j \hat{T}^{-\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p})} \right. \\ &\quad \left. + (\mathcal{A}^{\beta-1} + \mathcal{D}(T_1)^{\beta-1})(\tilde{\mathcal{A}}_j + \tilde{\mathcal{D}}_j(\hat{T})) \hat{T}^{\frac{1}{\alpha p_*}} + \mathcal{D}(T_1)^{\beta-1} \tilde{\mathcal{D}}_j(\hat{T}) \hat{T}^{1 - \frac{1}{\alpha}(\frac{1}{p} + \frac{\beta}{q_*})} \right) \\ &\leq h_2(1 + \|f\|_{L^\rho(\mathbb{R}^2)}) \left(x(\hat{T})^{\beta-1} + y(\hat{T}) + \mathcal{A}^{\beta-1} + \mathcal{B} + \mathcal{D}(T_1)^{\beta-1} + \mathcal{E}(\hat{T}) \right) (1 + \hat{T}) \\ &\quad \times \left(\tilde{U}_j + \tilde{N}_j + \tilde{\mathcal{A}}_j + \tilde{\mathcal{B}}_j + \tilde{\mathcal{D}}_j(\hat{T}) \right) + \frac{\tilde{\mathcal{E}}_j(\hat{T})}{2} \end{aligned}$$

for $j = 1, 2, \dots$, where $h_1 = h_1(q, p, q_*)$ and $h_2 = h_2(\alpha, \beta, q, p, q_*, p_*)$ are constants independent of j . This implies that both

$\sum_{j=1}^{\infty} \tilde{D}_j(\hat{T}) < \infty$ and $\sum_{j=1}^{\infty} \tilde{E}_j(\hat{T}) < \infty$ since $\sum_{j=1}^{\infty} \tilde{N}_j < \infty$, $\sum_{j=1}^{\infty} \tilde{U}_j < \infty$, $\sum_{j=1}^{\infty} \tilde{A}_j < \infty$ and $\sum_{j=1}^{\infty} \tilde{B}_j < \infty$ by the choice of T_0 in (100) and the identities for \tilde{A}_j and \tilde{B}_j in (121).

Note that $n_j(t) = \sum_{k=1}^{j-1} \tilde{n}_k(t) + n_1(t)$ and $u_j(t) = \sum_{k=1}^{j-1} \tilde{u}_k(t) + u_1(t)$, and since (91) holds, we obtain from Proposition 4.2 that there are limits in n and u with

$$n \in C([0, \hat{T}); L^{q_*}(\mathbb{R}^2)) \text{ and } u \in C([0, \hat{T}); L^{p_*}(\mathbb{R}^2))$$

so that

$$\lim_{j \rightarrow \infty} \left(\sup_{0 < t < \hat{T}} \|n_j(t) - n(t)\|_{L^{q_*}(\mathbb{R}^2)} \right) = 0$$

and

$$\lim_{j \rightarrow \infty} \left(\sup_{0 < t < \hat{T}} \|u_j(t) - u(t)\|_{L^{p_*}(\mathbb{R}^2)} \right) = 0.$$

Hence we have (130) and (131) and complete the proof of Theorem 2.1-(v).

4.6. Proof of Theorem 2.1-(vi)

We deal with the case of (q_*, p_*) satisfying $\frac{2\beta}{2\alpha\beta+1-2\alpha} \leq q_* < 2$ and $\frac{q_*}{q_*-1} \leq p_* < \infty$ to prove the time continuity in $\dot{H}^{k,q_*}(\mathbb{R}^2)$ for n and in $\dot{H}^{k,p_*}(\mathbb{R}^2)$ for u with $0 < k < 2(\frac{\beta}{\alpha}(\alpha - \frac{1}{q_*}) - (\alpha - \frac{1}{2} - \frac{1}{p_*}))$.

We are going to show that there exists a time $\tilde{T} = \tilde{T}(\alpha, \beta, q, p, q_*, p_*, k, \|f\|_Y, \hat{T}) \in (0, T_0)$ such that the limits n and u satisfy

$$t^{\frac{k}{2\alpha}} n \in BC([0, \tilde{T}); \dot{H}^{k,q_*}) \text{ with } \lim_{j \rightarrow \infty} \left(\sup_{0 < t < \tilde{T}} t^{\frac{k}{2\alpha}} \|(-\Delta)^{\frac{k}{2}}(n_j(t) - n(t))\|_{L^{q_*}(\mathbb{R}^2)} \right) = 0 \quad (153)$$

and

$$t^{\frac{k}{2\alpha}} u \in BC([0, \tilde{T}); \dot{H}^{k,p_*}) \text{ with } \lim_{j \rightarrow \infty} \left(\sup_{0 < t < \tilde{T}} t^{\frac{k}{2\alpha}} \|(-\Delta)^{\frac{k}{2}}(u_j(t) - u(t))\|_{L^{p_*}(\mathbb{R}^2)} \right) = 0. \quad (154)$$

Let $s \in (0, \tilde{T}]$ obtained in (130) and (131). To prove (153) and (154), we shall show that there are monotone increasing sequences $\{\mathcal{F}_j(s)\}_{j=1}^{\infty}$ and $\{\mathcal{L}_j(s)\}_{j=1}^{\infty}$ so that

$$\sup_{0 < t < s} t^{\frac{k}{2\alpha}} \|(-\Delta)^{\frac{k}{2}} n_j(t)\|_{L^{q_*}(\mathbb{R}^2)} \leq \mathcal{F}_j(s) \quad (155)$$

and

$$\sup_{0 < t < s} t^{\frac{k}{2\alpha}} \|(-\Delta)^{\frac{k}{2}} u_j(t)\|_{L^{p_*}(\mathbb{R}^2)} \leq \mathcal{L}_j(s) \quad (156)$$

for $j = 1, 2, \dots$.

At first, we construct $\{\mathcal{L}_j(s)\}_{j=1}^{\infty}$ satisfying (156), under the condition on (q_*, p_*) and k such that

$$\frac{2\beta}{2\alpha\beta+1-2\alpha} \leq q_* < 2, \frac{q_*}{q_*-1} \leq p_* < \infty \text{ and } 0 < k < 2\left(\frac{\beta}{\alpha}\left(\alpha - \frac{1}{q_*}\right) - \left(\alpha - \frac{1}{2} - \frac{1}{p_*}\right)\right) \quad (157)$$

We may define $\mathcal{L}_1(s) := \sup_{0 < t < s} t^{\frac{k}{2\alpha}} \|(-\Delta)^{\frac{k}{2}} K_t * u_0\|_{L^{p_*}(\mathbb{R}^2)} \leq C \|u_0\|_{L^{p_*}(\mathbb{R}^2)}$ with $C = C(q_*, k)$. Assume that (156) is true for j . Then, from (65), it follows that

$$\begin{aligned} \|(-\Delta)^{\frac{k}{2}} u_{j+1}(t)\|_{L^{p_*}(\mathbb{R}^2)} &\leq t^{-\frac{k}{2\alpha}} \sup_{0 < t < s} t^{\frac{k}{2\alpha}} \|(-\Delta)^{\frac{k}{2}} K_t * u_0\|_{L^{p_*}(\mathbb{R}^2)} \\ &+ \int_0^t \|(-\Delta)^{\frac{k}{2}} \nabla K_{t-\tau} * P(u_j \otimes u_j)(\tau)\|_{L^{p_*}(\mathbb{R}^2)} d\tau \\ &+ \int_0^t \|(-\Delta)^{\frac{k}{2}} K_{t-\tau} * P(|n_j|^{\beta} f)(\tau)\|_{L^{p_*}(\mathbb{R}^2)} d\tau := t^{-\frac{k}{2\alpha}} \mathcal{L}_1(s) + I_1(t) + I_2(t) \end{aligned} \quad (158)$$

for all $0 < t < s$.

For the term $I_1(t)$, since $2 < \frac{q_*}{q_*-1} \leq q_* < \infty$, for $0 < k < 2\alpha$, employing Lemma 3.4, we arrive at

$$\begin{aligned}
& I_1(t) \\
& \leq C \int_0^t (t-\tau)^{-\frac{1}{\alpha}(\frac{2}{p_*}-\frac{1}{p_*})-\frac{1}{2\alpha}} \\
& \quad \cdot \|(-\Delta)^{\frac{k}{2}}(u_j \otimes u_j)(\tau)\|_{L^{\frac{p_*}{2}}(\mathbb{R}^2)} d\tau \\
& \leq C \int_0^t (t-\tau)^{-\frac{1}{\alpha p_*}-\frac{1}{2\alpha}} \|(-\Delta)^{\frac{k}{2}} u_j(\tau)\|_{L^{p_*}(\mathbb{R}^2)} \\
& \quad \cdot \|u_j(\tau)\|_{L^{p_*}(\mathbb{R}^2)} d\tau \\
& \leq C\mathcal{E}(\hat{T})\mathcal{L}_j(s) \int_0^t (t-\tau)^{-\frac{1}{\alpha p_*}-\frac{1}{2\alpha}} \tau^{-\frac{k}{2\alpha}} d\tau \\
& \leq C\mathcal{E}(\hat{T})\mathcal{L}_j(s)B(1 - \frac{1}{\alpha p_*} - \frac{1}{2\alpha}, 1 - \frac{k}{2\alpha})s^{\frac{1}{\alpha}(\alpha-\frac{1}{2}-\frac{1}{p_*})}t^{-\frac{k}{2\alpha}}
\end{aligned} \tag{159}$$

for all $0 < t < s \leq \hat{T}$, where $C = C(p_*, k)$.

As for $I_2(t)$, recalling on $\frac{2\beta}{2\alpha\beta+1-2\alpha} \leq q_* < 2$ and $\frac{q_*}{q_*-1} \leq p_* < \infty$, choosing \hat{r}_8 so that $\frac{1}{\hat{r}_8} = \frac{1}{\rho} + \frac{\beta}{q_*}$, we find that $1 \leq \hat{r}_8 < p_*$. Then, for $0 < k < 2(\frac{\beta}{\alpha}(\alpha - \frac{1}{q_*}) - (\alpha - \frac{1}{2} - \frac{1}{p_*}))$, we deduce

$$\begin{aligned}
& I_2(t) \\
& \leq C \int_0^t (t-\tau)^{-\frac{1}{\alpha}(\frac{1}{\hat{r}_8}-\frac{1}{p_*})-\frac{k}{2\alpha}} \|(|n_j|^\beta f)(\tau)\|_{L^{\hat{r}_8}(\mathbb{R}^2)} d\tau \\
& \leq C \int_0^t (t-\tau)^{-\frac{1}{\alpha}(\frac{1}{\rho}+\frac{\beta}{q_*}-\frac{1}{p_*})-\frac{k}{2\alpha}} \|n_j(\tau)\|_{L^{p_*}(\mathbb{R}^2)} \|f\|_{L^\rho(\mathbb{R}^2)} d\tau \\
& \leq C\mathcal{D}(\hat{T})^\beta \|f\|_{L^\rho(\mathbb{R}^2)} \int_0^t (t-\tau)^{-\frac{1}{\alpha}(\frac{1}{\rho}+\frac{\beta}{q_*}-\frac{1}{p_*})-\frac{k}{2\alpha}} d\tau \\
& \leq C\mathcal{D}(\hat{T})^\beta \|f\|_{L^\rho(\mathbb{R}^2)} s^{1-\frac{1}{\alpha}(\frac{1}{\rho}+\frac{\beta}{q_*}-\frac{1}{p_*})}t^{-\frac{k}{2\alpha}}
\end{aligned} \tag{160}$$

for all $0 < t < s \leq \hat{T}$, where $C = C(\alpha, \beta, q_*, p_*, k)$.

Using (157), plugging (159) and (160) into (158), we obtain

$$\begin{aligned}
& \sup_{0 < t < s} t^{\frac{k}{2\alpha}} \|(-\Delta)^{\frac{k}{2}} u_{j+1}(t)\|_{L^{p_*}(\mathbb{R}^2)} \\
& \leq \mathcal{L}_1(s) + h_3\mathcal{D}(\hat{T})^\beta \|f\|_{L^\rho(\mathbb{R}^2)} s^{1-\frac{1}{\alpha}(\frac{1}{\rho}+\frac{\beta}{q_*}-\frac{1}{p_*})} \\
& \quad + h_3\mathcal{E}(\hat{T})\mathcal{L}_j(s)B(1 - \frac{1}{\alpha p_*} - \frac{1}{2\alpha}, 1 - \frac{k}{2\alpha})s^{\frac{1}{\alpha}(\alpha-\frac{1}{2}-\frac{1}{p_*})}
\end{aligned}$$

with $h_3 = h_3(\alpha, \beta, q_*, p_*, k)$. Hence we may take $\mathcal{L}_{j+1}(s)$ as

$$\mathcal{L}_{j+1}(s) := \mathcal{L}_1(s) + h_3\mathcal{D}(\hat{T})^\beta \|f\|_{L^\rho(\mathbb{R}^2)} s^{1-\frac{1}{\alpha}(\frac{1}{\rho}+\frac{\beta}{q_*}-\frac{1}{p_*})} + h_3\mathcal{E}(\hat{T})\mathcal{L}_j(s)s^{\frac{1}{\alpha}(\alpha-\frac{1}{2}-\frac{1}{p_*})}$$

for all $s \in (0, \hat{T}]$, where $h_3 = h_3(\alpha, \beta, q_*, p_*, k)$ is a constant independent of t . By this induction, we can show that there is a monotone increasing sequence $\{\mathcal{L}_j(s)\}_{j=1}^\infty$ so that (156) holds for all $j = 1, 2, \dots$. Thus we prove (156) under the condition (157).

Moreover, by the induction, we shall prove that there is a time

$$T_2 = T_2(\alpha, \beta, q, p, q_*, p_*, k, \|f\|_{L^\rho(\mathbb{R}^2)}) \leq \hat{T}$$

such that

$$0 < \mathcal{L}_1(T_2) \leq \mathcal{L}_2(T_2) \leq \dots \leq \mathcal{L}_j(T_2) \leq \mathcal{L}_{j+1}(T_2) \leq \dots \uparrow \mathcal{L}(T_2), \tag{161}$$

where $\mathcal{L}(T_2) > 0$ is the root of

$$\mathcal{L}(T_2) := \mathcal{L}_1(T_2) + h_3 \mathcal{D}(\hat{T})^\beta \|f\|_{L^\rho(\mathbb{R}^2)} T_2^{1-\frac{1}{\alpha}(\frac{1}{\rho}+\frac{\beta}{q_*}-\frac{1}{p_*})} + h_3 \mathcal{E}(\hat{T}) \mathcal{L}(T_2) T_2^{\frac{1}{\alpha}(\alpha-\frac{1}{2}-\frac{1}{p_*})}$$

i.e., that

$$\mathcal{L}(T_2) = \frac{\mathcal{L}_1(T_2) + h_3 \mathcal{D}(\hat{T})^\beta \|f\|_{L^\rho(\mathbb{R}^2)} T_2^{1-\frac{1}{\alpha}(\frac{1}{\rho}+\frac{\beta}{q_*}-\frac{1}{p_*})}}{1 - h_3 \mathcal{E}(\hat{T}) T_2^{\frac{1}{\alpha}(\alpha-\frac{1}{2}-\frac{1}{p_*})}}. \quad (162)$$

Choosing a time $T_2 = T_2(\alpha, \beta, q, p, q_*, p_*, k, \|f\|_{L^\rho(\mathbb{R}^2)}) \leq \hat{T}$ so that

$$0 < T_2 \leq \left(\frac{1}{2h_3 \mathcal{E}(\hat{T})} \right)^{\frac{2\alpha p_*}{(2\alpha-1)p_*-2}}, \quad (163)$$

we have $h_3 \mathcal{E}(\hat{T}) T_2^{\frac{1}{\alpha}(\alpha-\frac{1}{2}-\frac{1}{p_*})} \leq \frac{1}{2} < 1$. By the induction, we can find that $\mathcal{L}_j(T_2)$ is a bounded monotone increasing sequence such that $\mathcal{L}_j(T_2) \leq \mathcal{L}(T_2)$ for all $j = 1, 2, \dots$. This implies (161) with (162).

Let $s \in (0, T_2]$ and T_2 be as that in (163). We construct $\{\mathcal{F}_j(s)\}_{j=1}^\infty$ satisfying (155), under the condition of (q_*, p_*) and k so that

$$\frac{4}{3} \leq q_* < 2, \quad \frac{q_*}{q_*-1} \leq p_* < \infty \quad \text{and} \quad 0 < k < 2\alpha. \quad (164)$$

We may define $\mathcal{F}_1(s)$ by

$$\mathcal{F}_1(s) := \sup_{0 < t < s} t^{\frac{k}{2\alpha}} \|(-\Delta)^{\frac{k}{2}} K_t * n_0\|_{L^{q_*}(\mathbb{R}^2)} \leq C \|n_0\|_{L^{q_*}(\mathbb{R}^2)}$$

with $C = C(q_*, k)$. Suppose that (155) is true for j . Then, from (63), we discover

$$\begin{aligned} \|(-\Delta)^{\frac{k}{2}} n_{j+1}(t)\|_{L^{q_*}(\mathbb{R}^2)} &\leq t^{-\frac{k}{2\alpha}} \sup_{0 < t < s} t^{\frac{k}{2\alpha}} \|(-\Delta)^{\frac{k}{2}} K_t * n_0\|_{L^{q_*}(\mathbb{R}^2)} \\ &+ \int_0^t \|(-\Delta)^{\frac{k}{2}} \nabla K_{t-\tau} * (u_j n_j)(\tau)\|_{L^{q_*}(\mathbb{R}^2)} d\tau + \int_0^t \|(-\Delta)^{\frac{k}{2}} \nabla K_{t-\tau} * (n_j \nabla v_j)(\tau)\|_{L^{q_*}(\mathbb{R}^2)} d\tau \\ &=: t^{-\frac{k}{2\alpha}} \mathcal{F}_1(s) + I_3(t) + I_4(t) \end{aligned} \quad (165)$$

for all $0 < t < s$.

We first estimate the term $I_3(t)$. Taking \hat{r}_9 so that $\frac{1}{\hat{r}_9} = \frac{1}{p_*} + \frac{1}{q_*}$, since $\frac{4}{3} \leq q_* < 2$ and $\frac{q_*}{q_*-1} \leq p_* < \infty$, we have $1 \leq \hat{r}_9 < q_*$. Then, for $0 < k < 2\alpha$, employing Lemma 3.4, we deduce

$$\begin{aligned} I_3(t) &\leq C \int_0^t (t-\tau)^{-\frac{1}{\alpha}(\frac{1}{\hat{r}_9}-\frac{1}{q_*})-\frac{1}{2\alpha}} \|(-\Delta)^{\frac{k}{2}} (u_j n_j)(\tau)\|_{L^{\hat{r}_9}(\mathbb{R}^2)} d\tau \\ &\leq C \int_0^t (t-\tau)^{-\frac{1}{\alpha p_*}-\frac{1}{2\alpha}} \|(-\Delta)^{\frac{k}{2}} u_j(\tau)\|_{L^{p_*}(\mathbb{R}^2)} \cdot \|n_j(\tau)\|_{L^{q_*}(\mathbb{R}^2)} d\tau \\ &\quad + C \int_0^t (t-\tau)^{-\frac{1}{\alpha p_*}-\frac{1}{2\alpha}} \|u_j(\tau)\|_{L^{p_*}(\mathbb{R}^2)} \cdot \|(-\Delta)^{\frac{k}{2}} n_j(\tau)\|_{L^{q_*}(\mathbb{R}^2)} d\tau \\ &\leq C \left(\mathcal{D}(\hat{T}) \mathcal{L}(T_2) + \mathcal{E}(\hat{T}) \mathcal{F}_j(s) \right) \cdot \int_0^t (t-\tau)^{-\frac{1}{\alpha p_*}-\frac{1}{2\alpha}} \tau^{-\frac{k}{2\alpha}} d\tau \\ &\leq C \left(\mathcal{D}(\hat{T}) \mathcal{L}(T_2) + \mathcal{E}(\hat{T}) \mathcal{F}_j(s) \right) \cdot B \left(1 - \frac{1}{\alpha p_*} - \frac{1}{2\alpha}, 1 - \frac{k}{2\alpha} \right) s^{\frac{1}{\alpha}(\alpha-\frac{1}{2}-\frac{1}{p_*})} t^{-\frac{k}{2\alpha}} \end{aligned} \quad (166)$$

for all $0 < t < s \leq T_2$, where $C = C(q_*, p_*, k)$.

Next we are going to deal with the term $I_4(t)$. Choosing \hat{r}_{10} so that $\frac{1}{\hat{r}_{10}} = \frac{1}{q_*} - \frac{1}{2}$, together with $\frac{4}{3} \leq q_* < 2$, we find that $4 \leq \hat{r}_{10} \leq \infty$. Taking r so that $\frac{1}{r} = \frac{1}{\hat{r}_{10}} + \frac{1}{q_*}$, we have $1 < r \leq q_*$. Then, for $0 < k < 2\alpha$, applying Lemma 3.4, we observe

$$\begin{aligned}
I_4(t) &\leq C \int_0^t (t-\tau)^{-\frac{1}{\alpha}(\frac{1}{r}-\frac{1}{q_*})-\frac{1}{2\alpha}} \|(-\Delta)^{\frac{k}{2}}(n_j \nabla v_j)(\tau)\|_{L^r(\mathbb{R}^2)} d\tau \\
&\leq C \int_0^t (t-\tau)^{-\frac{1}{\alpha r_{10}}-\frac{1}{2\alpha}} \|(-\Delta)^{\frac{k}{2}} n_j(\tau)\|_{L^{q_*}(\mathbb{R}^2)} \times \|\nabla v_j(\tau)\|_{L^{\hat{r}_{10}}(\mathbb{R}^2)} d\tau \\
&\quad + C \int_0^t (t-\tau)^{-\frac{1}{\alpha r_{10}}-\frac{1}{2\alpha}} \|n_j(\tau)\|_{L^{q_*}(\mathbb{R}^2)} \times \|(-\Delta)^{\frac{k}{2}} \nabla v_j(\tau)\|_{L^{\hat{r}_{10}}(\mathbb{R}^2)} d\tau \\
&\leq C \int_0^t (t-\tau)^{-\frac{1}{\alpha q_*}} \|(-\Delta)^{\frac{k}{2}} n_j(\tau)\|_{L^{q_*}(\mathbb{R}^2)} \|n_j(\tau)\|_{L^{q_*}(\mathbb{R}^2)} d\tau \\
&\leq C \mathcal{D}(\hat{T}) \mathcal{F}_j(s) \int_0^t (t-\tau)^{-\frac{1}{\alpha q_*}} \tau^{-\frac{k}{2\alpha}} d\tau \\
&\leq C \mathcal{D}(\hat{T}) \mathcal{F}_j(s) B(1 - \frac{1}{\alpha q_*}, 1 - \frac{k}{2\alpha}) s^{1-\frac{1}{\alpha q_*}} t^{-\frac{k}{2\alpha}}
\end{aligned} \tag{167}$$

for all $0 < t < s \leq T_2$, where $C = C(q_*, k)$.

Substituting (166) and (167) into (165), along with (164), we arrive at

$$\begin{aligned}
t^{-\frac{k}{2\alpha}} \|(-\Delta)^{\frac{k}{2}} n_{j+1}(t)\|_{L^{q_*}(\mathbb{R}^2)} &\leq \mathcal{F}_1(s) + h_4 \mathcal{D}(\hat{T}) \mathcal{F}_j(s) B(1 - \frac{1}{\alpha q_*}, 1 - \frac{k}{2\alpha}) s^{1-\frac{1}{\alpha q_*}} \\
&\quad + h_4 \left(\mathcal{D}(\hat{T}) \mathcal{L}(T_2) + \mathcal{E}(\hat{T}) \mathcal{F}_j(s) \right) B(1 - \frac{1}{\alpha p_*} - \frac{1}{2\alpha}, 1 - \frac{k}{2\alpha}) s^{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p_*})}
\end{aligned}$$

with $h_4 = h_4(q_*, p_*, k)$. Therefore, we can take $\mathcal{F}_{j+1}(s)$ as

$$\mathcal{F}_{j+1}(s) := \mathcal{F}_1(s) + h_4 \mathcal{D}(\hat{T}) \mathcal{L}(T_2) s^{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p_*})} + h_4 \mathcal{F}_j(s) \left(\mathcal{D}(\hat{T}) s^{1-\frac{1}{\alpha q_*}} + \mathcal{E}(\hat{T}) s^{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p_*})} \right),$$

where the constant h_4 is independent of j . By this induction, we can prove that there is a time $\tilde{T} = \tilde{T}(\alpha, \beta, q, p, q_*, p_*, k, \|f\|_{L^\rho(\mathbb{R}^2)}, \hat{T}) \leq T_2$ such that

$$0 < \mathcal{F}_1(\tilde{T}) \leq \mathcal{F}_2(\tilde{T}) \leq \dots \leq \mathcal{F}_j(\tilde{T}) \leq \mathcal{F}_{j+1}(\tilde{T}) \leq \dots \uparrow \mathcal{F}(\tilde{T}), \tag{168}$$

where $\mathcal{F}(\tilde{T}) > 0$ is the root of

$$\mathcal{F}(\tilde{T}) = \mathcal{F}_1(\tilde{T}) + h_4 \mathcal{D}(\hat{T}) \mathcal{L}(T_2) \tilde{T}^{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p_*})} + h_4 \mathcal{F}(\tilde{T}) \left(\mathcal{D}(\hat{T}) \tilde{T}^{1-\frac{1}{\alpha q_*}} + \mathcal{E}(\hat{T}) \tilde{T}^{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p_*})} \right)$$

i.e., that

$$\mathcal{F}(\tilde{T}) = \frac{\mathcal{F}_1(\tilde{T}) + h_4 \mathcal{D}(\hat{T}) \mathcal{L}(T_2) \tilde{T}^{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p_*})}}{1 - h_4 \left(\mathcal{D}(\hat{T}) \tilde{T}^{1-\frac{1}{\alpha q_*}} + \mathcal{E}(\hat{T}) \tilde{T}^{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p_*})} \right)}. \tag{169}$$

Choosing $\tilde{T} = \tilde{T}(\alpha, \beta, q, p, q_*, p_*, k, \|f\|_{L^\rho(\mathbb{R}^2)}, \hat{T}) \leq T_2$ such that

$$0 < \tilde{T} \leq \min \left\{ \left(\frac{1}{4h_3 \mathcal{D}(\hat{T})} \right)^{\frac{\alpha q_*}{\alpha q_* - 1}}, \left(\frac{1}{4h_3 \mathcal{E}(\hat{T})} \right)^{\frac{2\alpha p_*}{(2\alpha - 1)p_* - 2}} \right\}, \tag{170}$$

we deduce $h_4 \left(\mathcal{D}(\hat{T}) \tilde{T}^{1-\frac{1}{\alpha q_*}} + \mathcal{E}(\hat{T}) \tilde{T}^{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p_*})} \right) \leq \frac{1}{2} < 1$. By this induction, we can obtain that $\mathcal{F}_j(\tilde{T})$ is a bounded monotone increasing sequence so that $\mathcal{F}_j(\tilde{T}) \leq \mathcal{F}(\tilde{T})$ for all $j = 1, 2, \dots$. This implies (168) with (169).

Let $\tilde{n}_j(t)$ and $\tilde{u}_j(t)$ be ones that are defined by (86). Now we define $\{\tilde{\mathcal{F}}_j(\tilde{T})\}_{j=1}^\infty$ and $\{\tilde{\mathcal{L}}_j(\tilde{T})\}_{j=1}^\infty$ by

$$\sup_{0 < t < s} t^{\frac{k}{2\alpha}} \|(-\Delta)^{\frac{k}{2}} \tilde{n}_j(t)\|_{L^{q_*}(\mathbb{R}^2)} \leq \tilde{\mathcal{F}}_j(s)$$

and

$$\sup_{0 < t < s} t^{\frac{k}{2\alpha}} \|(-\Delta)^{\frac{k}{2}} \tilde{u}_j(t)\|_{L^{p_*}(\mathbb{R}^2)} \leq \tilde{\mathcal{L}}_j(s)$$

for $j = 1, 2, \dots$.

Let us denote $\{\tilde{\mathcal{D}}(\hat{T})\}_{j=1}^{\infty}$ and $\{\tilde{\mathcal{E}}(\hat{T})\}_{j=1}^{\infty}$ be the same ones in (152). The same argument as in the proof of (159), (160), (166) and (167) holds

$$\begin{aligned}\tilde{\mathcal{F}}_{j+1}(\tilde{T}) &= h_4 \left(\tilde{\mathcal{L}}_j(\tilde{T})\mathcal{D}(\hat{T}) + \mathcal{L}(T_2)\tilde{\mathcal{D}}_j(\hat{T}) + \tilde{\mathcal{E}}_j(\hat{T})\mathcal{F}(\tilde{T}) + \mathcal{E}(\hat{T})\tilde{\mathcal{F}}_j(\tilde{T}) \right) \tilde{T}^{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p_*})} \\ &\quad + h_4 \left(\tilde{\mathcal{F}}_j(\tilde{T})\mathcal{D}(\hat{T}) + \tilde{\mathcal{D}}_j(\hat{T})\mathcal{F}(\tilde{T}) \right) \tilde{T}^{\frac{1}{\alpha}(\alpha - \frac{1}{q_*})} \\ &\leq \frac{1}{2}\tilde{\mathcal{F}}_j(\tilde{T}) + h_4 \left(\mathcal{D}(\hat{T}) + \mathcal{F}(\tilde{T}) + \mathcal{L}(T_2) \right) \times (1 + \tilde{T})^{\max\{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p_*}), \frac{1}{\alpha}(\alpha - \frac{1}{q_*})\}} \\ &\quad \times \left(\tilde{\mathcal{D}}_j(\hat{T}) + \tilde{\mathcal{E}}_j(\hat{T}) + \tilde{\mathcal{L}}_j(\tilde{T}) \right)\end{aligned}$$

and

$$\begin{aligned}\tilde{\mathcal{L}}_{j+1}(\tilde{T}) &= h_3 \left(\mathcal{E}(\hat{T})\tilde{\mathcal{L}}_j(\tilde{T}) + \mathcal{L}(T_2)\tilde{\mathcal{E}}_j(\hat{T}) \right) \tilde{T}^{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p_*})} + h_3 \|f\|_{L^p(\mathbb{R}^2)} \mathcal{D}(\hat{T})^\beta \tilde{\mathcal{D}}_j(\hat{T}) \tilde{T}^{1 - \frac{1}{\alpha}(\frac{1}{\rho} + \frac{\beta}{q_*} - \frac{1}{p_*})} \\ &\leq \frac{1}{2}\tilde{\mathcal{L}}_j(\tilde{T}) + h_3(1 + \|f\|_{L^p(\mathbb{R}^2)}) \left(\mathcal{D}(\hat{T})^{\beta-1} + \mathcal{L}(T_2) \right) \times (1 + \tilde{T})^{\max\{\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p_*}), 1 - \frac{1}{\alpha}(\frac{1}{\rho} + \frac{\beta}{q_*} - \frac{1}{p_*})\}} \\ &\quad \times \left(\tilde{\mathcal{D}}_j(\hat{T}) + \tilde{\mathcal{E}}_j(\hat{T}) \right)\end{aligned}$$

for all $j = 1, 2, \dots$, where the constants h_3 and h_4 are independent of j . This implies that both $\sum_{j=1}^{\infty} \tilde{\mathcal{F}}_j(\tilde{T}) < \infty$ and $\sum_{j=1}^{\infty} \tilde{\mathcal{L}}_j(\tilde{T}) < \infty$ since $\sum_{j=1}^{\infty} \tilde{\mathcal{D}}(\hat{T}) < \infty$ and $\sum_{j=1}^{\infty} \tilde{\mathcal{E}}(\hat{T}) < \infty$.

Since $n_j(t) = \sum_{k=1}^{j-1} \tilde{n}_k(t) + n_1(t)$ and $u_j(t) = \sum_{k=1}^{j-1} \tilde{u}_k(t) + u_1(t)$, we find that there are limits in n and u with

$$t^{\frac{k}{2\alpha}} n \in BC([0, \tilde{T}]; \dot{H}^{k, q_*}) \quad \text{and} \quad t^{\frac{k}{2\alpha}} u \in BC([0, \tilde{T}]; \dot{H}^{k, p_*})$$

so that

$$\lim_{j \rightarrow \infty} \left(\sup_{0 < t < \tilde{T}} t^{\frac{k}{2\alpha}} \|(-\Delta)^{\frac{k}{2}}(n_j(t) - n(t))\|_{L^{q_*}(\mathbb{R}^2)} \right) = 0$$

and

$$\lim_{j \rightarrow \infty} \left(\sup_{0 < t < \tilde{T}} t^{\frac{k}{2\alpha}} \|(-\Delta)^{\frac{k}{2}}(u_j(t) - u(t))\|_{L^{p_*}(\mathbb{R}^2)} \right) = 0.$$

Hence we derive (153) and (154), which implies (22). Then we completed the proof of Theorem 2.1-(vi).

$\eta \in (1, 2\alpha)$, and $\varphi(x)$ be defined by (27). Then

$$(-\Delta)^\alpha \varphi \in L^\infty(\mathbb{R}^2). \quad (172)$$

5. Proof of Theorem 2.2

The main role in our proof of the blow up of solutions to (1) is played by the smooth weight function $\varphi(x)$ on \mathbb{R}^2 with $\eta \in (1, 2\alpha]$. Since $(1 + |x|^2)^\eta \leq (1 + |x|^\eta)^2$, we have for each $\varepsilon > 0$, suitably chosen $C(\varepsilon) > 0$, and for every $x \in \mathbb{R}^2$

$$\varphi(x) \leq |x|^\eta \leq \varepsilon + C(\varepsilon)\varphi(x). \quad (171)$$

Now, let us state two auxiliary results concerning the weight function $\varphi(x)$ which will be found in [3] and we omit the proof here.

Lemma 5.1. ([3], Lemma 4.1) Assume that $\alpha \in (0, 1)$,

Lemma 5.2. ([3], Lemma 4.3) For every $\eta \in (1, 2\alpha)$, the function $\varphi(x)$ defined in (27) is locally uniformly convex on \mathbb{R}^2 . Furthermore, there exists $Q = Q(\eta)$ such that the following inequality

$$(\nabla \varphi(x) - \nabla \varphi(y)) \cdot (x - y) \geq \frac{Q|x - y|^2}{1 + |x|^{2-\eta} + |y|^{2-\eta}} \quad (173)$$

holds true for all $x, y \in \mathbb{R}^2$.

Proof of Theorem 2.2. We consider the function

$$w = w(t) \equiv \int_{\mathbb{R}^2} n(x, t) \varphi(x) dx \quad (174)$$

where $\varphi(x)$ is defined in ((27) and $1 < \eta < 2\alpha$. Moreover, it satisfies the relation

$$\begin{aligned}
\frac{d}{dt}w(t) &= - \int_{\mathbb{R}^2} (-\Delta)^\alpha n(x, t) \varphi(x) dx + \int_{\mathbb{R}^2} u(x, t) n(x, t) \cdot \nabla \varphi(x) dx \\
&\quad + \int_{\mathbb{R}^2} n(x, t) \nabla (-\Delta)^{-1} n(x, t) \cdot \nabla \varphi(x) dx \\
&= - \int_{\mathbb{R}^2} (-\Delta)^\alpha n(x, t) \varphi(x) dx + \int_{\mathbb{R}^2} u(x, t) n(x, t) \cdot \nabla \varphi(x) dx \\
&\quad - \frac{1}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (\nabla \varphi(x) - \nabla \varphi(y)) \cdot (x - y) \frac{n(x, t) n(y, t)}{|x - y|^2} dx dy
\end{aligned} \tag{175}$$

after using the symmetrization of the double integral. For any fixed $M > 0$, we are going to use the simple identity as follows

$$\begin{aligned}
M^2 &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} n(x, t) n(y, t) dx dy \\
&= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} n(x, t) n(y, t) \frac{(1 + |x|^{2-\eta} + |y|^{2-\eta})^\delta}{(1 + |x|^{2-\eta} + |y|^{2-\eta})^\delta} dx dy
\end{aligned} \tag{176}$$

with some $\delta > 0$. Applying the Hölder inequality, choosing the exponents $q > 1$ and $q' = \frac{q}{q-1}$ such that

$$\delta q = 1, \quad (2 - \eta) \delta q' = \eta. \tag{177}$$

As a consequence, we obtain

$$M^2 \leq J(t)^{\frac{1}{q}} \left(\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} n(x, t) n(y, t) \cdot (1 + |x|^{2-\eta} + |y|^{2-\eta})^{\delta q'} dx dy \right)^{\frac{1}{q'}}, \tag{178}$$

where the integral $J(t)$ satisfies

$$\begin{aligned}
J(t) &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{n(x, t) n(y, t)}{1 + |x|^{2-\eta} + |y|^{2-\eta}} dx dy \\
&\leq \frac{1}{Q} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (\nabla \varphi(x) - \nabla \varphi(y)) \cdot (x - y) \cdot \frac{n(x, t) n(y, t)}{|x - y|^2} dx dy
\end{aligned} \tag{179}$$

by Lemma 5.2.

Recalling (171) and (177), there exists a constant C_1 such that

$$(1 + |x|^{2-\eta} + |y|^{2-\eta})^{\delta q'} \leq C_1 (1 + \varphi(x) + \varphi(y)). \tag{180}$$

Therefore, (178) can be written as

$$M^2 \leq C_1^{\frac{1}{q'}} J(t)^{\frac{1}{q}} (M^2 + 2Mw(t))^{\frac{1}{q'}}. \tag{181}$$

Going back to (175), employing Lemma 5.1 and (179)-(181), we arrive at

$$\frac{d}{dt}w(t) \leq C_2 M - C_3 \frac{M^{2q}}{(M^2 + 2Mw(t))^{\frac{q}{q'}}} + \int_{\mathbb{R}^2} u(x, t) n(x, t) \cdot \nabla \varphi(x) dx \tag{182}$$

with $C_2 = \|(-\Delta)^\alpha \varphi\|_{L^\infty(\mathbb{R}^2)}$. Then integrating it from 0 to t , we observe

$$w(t) - w(0) \leq \int_0^t C_2 M - C_3 \frac{M^{2q}}{(M^2 + 2Mw(\tau))^{\frac{q}{q'}}} d\tau + \int_0^t \int_{\mathbb{R}^2} u(x, \tau) n(x, \tau) \cdot \nabla \varphi(x) dx d\tau. \tag{183}$$

To complete the proof, we will estimate the term $\|u\|_{L^s(0,t;L^\infty(\mathbb{R}^2))}$. Let $\hat{r} = \frac{\beta}{\alpha\beta+1-2\alpha}$. Since $\frac{1}{2} = \frac{1}{\rho} + \frac{1}{\hat{r}}$, then we deduce

$$\begin{aligned} \| |n|^\beta f \|_{L^s(0,t;L^2(\mathbb{R}^2))} &\leq \|f\|_{L^\rho(\mathbb{R}^2)} \|n\|_{L^{\beta s}(0,t;L^{\beta\hat{r}}(\mathbb{R}^2))}^\beta \\ &= \|f\|_{L^\rho(\mathbb{R}^2)} \|n\|_{L^{\beta s}(0,t;L^{\frac{\beta}{\alpha\beta+1-2\alpha}}(\mathbb{R}^2))}^\beta. \end{aligned} \quad (184)$$

If we assume that $\frac{\beta}{\alpha\beta+1-2\alpha} \leq r \leq \infty$, then we have

$$\begin{aligned} \| |n|^\beta f \|_{L^s(0,t;L^2(\mathbb{R}^2))} &\leq \|f\|_{L^\rho(\mathbb{R}^2)} \|n\|_{L^{\beta s}(0,t;L^1(\mathbb{R}^2))}^{\frac{r(\alpha\beta+1-2\alpha)-\beta}{r-1}} \|n\|_{L^{\beta s}(0,t;L^r(\mathbb{R}^2))}^{\frac{r(2\alpha-1+\beta-\alpha\beta)}{r-1}} \\ &\leq \|f\|_{L^\rho(\mathbb{R}^2)} \|n\|_{L^\infty(0,t;L^1(\mathbb{R}^2))}^{\frac{r(\alpha\beta+1-2\alpha)-\beta}{r-1}} \|n\|_{L^\infty(0,t;L^r(\mathbb{R}^2))}^{\frac{r(2\alpha-1+\beta-\alpha\beta)}{r-1}} t^{\frac{1}{s}}. \end{aligned} \quad (185)$$

Hence, applying (184), (185) and Lemma 3.6, we obtain that $u \in L^s(0,t;L^\infty(\mathbb{R}^2))$ and

$$\|u\|_{L^s(0,t;L^\infty(\mathbb{R}^2))} \leq K_{s,r,t} \quad (186)$$

for all $0 < t < T$, where $K_{s,r,t} = K_{s,r,t}(\alpha, \beta, s, t, \|u_0\|_{L^2(\mathbb{R}^2)}, \|u_0\|_{B_{2,s}^{2(1-\frac{1}{s})}(\mathbb{R}^2)}, \|f\|_{L^\rho(\mathbb{R}^2)},$

$\|n\|_{L^\infty(0,t;L^1(\mathbb{R}^2))}, \|n\|_{L^\infty(0,t;L^r(\mathbb{R}^2))}$).

For the term $\int_0^t \int_{\mathbb{R}^2} u(x, \tau) n(x, \tau) \cdot \nabla \varphi(x) dx d\tau$, based on (186), applying Hölder inequality, we discover

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}^2} u(x, \tau) n(x, \tau) \cdot \nabla \varphi(x) dx d\tau \\ &= \eta \int_0^t \int_{\mathbb{R}^2} u(x, \tau) n(x, \tau) (1 + |x|^2)^{\frac{\eta}{2}} \cdot \frac{|x|}{1 + |x|^2} dx d\tau \\ &\leq \frac{\eta}{2} \int_0^t \int_{\mathbb{R}^2} u(x, \tau) n(x, \tau) [(1 + |x|^2)^{\frac{\eta}{2}} - 1 + 1] dx d\tau \\ &= \frac{\eta}{2} \int_0^t \int_{\mathbb{R}^2} u(x, \tau) n(x, \tau) \varphi(x) dx d\tau + \frac{\eta}{2} \int_0^t \int_{\mathbb{R}^2} u(x, \tau) n(x, \tau) dx d\tau \\ &= \frac{\eta}{2} \int_0^t \|u\|_{L^\infty(\mathbb{R}^2)} w(\tau) d\tau + \frac{\eta}{2} \int_0^t \|u\|_{L^\infty(\mathbb{R}^2)} \|n_0\|_{L^1(\mathbb{R}^2)} d\tau \\ &\leq C_4 \|u\|_{L^2(0,t;L^\infty(\mathbb{R}^2))}^2 + \varepsilon \int_0^t (w^2(\tau) + \|n_0\|_{L^1(\mathbb{R}^2)}^2) d\tau. \end{aligned} \quad (187)$$

Plugging (187) into (183), we deduce

$$w(t) \leq w(0) + C_4 \|u\|_{L^2(0,t;L^\infty(\mathbb{R}^2))}^2 + \int_0^t \left(C_2 M - C_3 \frac{M^{2q}}{(M^2 + 2Mw(\tau))^{\frac{q}{q'}}} + \varepsilon w^2(\tau) + \varepsilon \|n_0\|_{L^1(\mathbb{R}^2)}^2 \right) d\tau. \quad (188)$$

Next, for a suitable fixed constant $M = M_0 > w(0) + C_4 \|u\|_{L^2(0,T_{max};L^\infty(\mathbb{R}^2))}$, define

$$X_t := \{t \mid \forall 0 \leq \tau < t, w(\tau) < M_0\} \quad (189)$$

and

$$T_* = \sup X_t, \quad (190)$$

where $0 < t < T_{max}$.

Assume that $0 \leq T_* < T_{max}$, and choosing C_3 suitably so that

$$\begin{aligned}
 w(T_*) &\leq w(0) + C_4 \|u\|_{L^2(0, T_{max}; L^\infty(\mathbb{R}^2))} + \lim_{\epsilon \rightarrow 0} \int_0^{T_* - \epsilon} \left(C_2 M - C_3 \frac{M^{2q}}{(M^2 + 2Mw(\tau))^{\frac{q}{q'}}} \right. \\
 &\quad \left. + \varepsilon w^2(\tau) + \varepsilon \|n_0\|_{L^1(\mathbb{R}^2)}^2 \right) d\tau \\
 &< M_0 + \lim_{\epsilon \rightarrow 0} \int_0^{T_* - \epsilon} \left(C_2 M_0 - C_3 \frac{M_0^{2q}}{(M_0^2 + 2M_0^2)^{\frac{q}{q'}}} + \varepsilon M_0^2 + \varepsilon \|n_0\|_{L^1(\mathbb{R}^2)}^2 \right) d\tau \\
 &< M_0 + \left(C_2 M_0 - C_3 \frac{M_0^{2q}}{(M_0^2 + 2M_0^2)^{\frac{q}{q'}}} + \varepsilon M_0^2 + \varepsilon \|n_0\|_{L^1(\mathbb{R}^2)}^2 \right) T_* \\
 &< M_0.
 \end{aligned} \tag{191}$$

This causes a contradiction and we conclude that $T_* = T_{max}$.

Furthermore, we are going to deduce that $T_{max} < \frac{M_0}{C_5 M_0^2 - C_2 M_0 - \varepsilon \|n_0\|_{L^1(\mathbb{R}^2)}^2}$. Let us first assume $T_{max} \geq \frac{M_0}{C_5 M_0^2 - C_2 M_0 - \varepsilon \|n_0\|_{L^1(\mathbb{R}^2)}^2}$. Recalling (191), we discover

$$\begin{aligned}
 w(t) &< M_0 + \left(C_2 M_0 - C_3 \frac{M_0^{2q}}{(M_0^2 + 2M_0^2)^{\frac{q}{q'}}} + \varepsilon M_0^2 + \varepsilon \|n_0\|_{L^1(\mathbb{R}^2)}^2 \right) T_{max} \\
 &= M_0 + \left(C_2 M_0 - \frac{C_3}{3^{\frac{q}{q'}}} M_0^2 + \varepsilon M_0^2 + \varepsilon \|n_0\|_{L^1(\mathbb{R}^2)}^2 \right) T_{max} \\
 &< M_0 + \left(C_2 M_0 - C_5 M_0^2 + \varepsilon \|n_0\|_{L^1(\mathbb{R}^2)}^2 \right) T_{max} < 0.
 \end{aligned} \tag{192}$$

This implies $T_{max} < \frac{M_0}{C_5 M_0^2 - C_2 M_0 - \varepsilon \|n_0\|_{L^1(\mathbb{R}^2)}^2}$.

We next prove (26) by contradiction. To this end, suppose that there exists a constant \tilde{C} such that

$$\limsup_{t \rightarrow T_{max}^-} (\|n(t)\|_{L^{q_*}(\mathbb{R}^2)} + \|u(t)\|_{L^p(\mathbb{R}^2)} + \|u(t)\|_{L^{p_*}(\mathbb{R}^2)}) \leq \tilde{C}. \tag{193}$$

Since $1 < q < 2 < \frac{\alpha}{\alpha-1} \leq q_*$, applying Gagliardo-Nirenberg inequality, we find that

$$\limsup_{t \rightarrow T_{max}^-} (\|n(t)\|_{L^{q_*}(\mathbb{R}^2)} + \|n(t)\|_{L^q(\mathbb{R}^2)} + \|u(t)\|_{L^p(\mathbb{R}^2)} + \|u(t)\|_{L^{p_*}(\mathbb{R}^2)}) \leq 2\tilde{C} + \|n_0\|_{L^1(\mathbb{R}^2)}. \tag{194}$$

Then there exists a time $t_* \in (0, T_{max})$ such that

$$\sup_{t_* < \tau < T_{max}} (\|n(t)\|_{L^{q_*}(\mathbb{R}^2)} + \|n(t)\|_{L^q(\mathbb{R}^2)} + \|u(t)\|_{L^p(\mathbb{R}^2)} + \|u(t)\|_{L^{p_*}(\mathbb{R}^2)}) \leq 2\tilde{C} + \|n_0\|_{L^1(\mathbb{R}^2)} + 1. \tag{195}$$

We construct a time sequence $\{t_l\}_{l=1}^\infty$ with $t_l \rightarrow T_{max}$ and functions $n_* \in L^1 \cap L^{q_*}(\mathbb{R}^2)$ and $u_* \in L^{p_*} \cap H_\sigma^{1,2}(\mathbb{R}^2)$ so that

$$\lim_{l \rightarrow \infty} n(x, t_l) = n_*(x) \text{ for a.a. } x \in \mathbb{R}^2 \text{ and } n_*(x) \geq 0 \text{ for a.a. } x \in \mathbb{R}^2 \tag{196}$$

and

$$\lim_{l \rightarrow \infty} u(x, t_l) = u_*(x) \text{ for a.a. } x \in \mathbb{R}^2. \tag{197}$$

For that purpose, assume that k satisfying the following condition

$$0 < k < 2 \min \left\{ \alpha - \frac{1}{q}, \alpha - \frac{1}{2} - \frac{1}{p}, \frac{1}{p_*} p, \beta \left(1 - \frac{1}{\alpha q_*} \right) - 1 + \frac{1}{2\alpha} \right\}. \tag{198}$$

Since $\frac{p}{p-1} \leq q_*$ and $\frac{2q}{3q-2} \leq q_*$ by (19), for $\frac{t_*+T_{max}}{2} < t < T_{max}$, we have

$$\begin{aligned}
 & \|(-\Delta)^{\frac{k}{2}} n(t)\|_{L^{q_*}(\mathbb{R}^2)} \\
 & \leq \|(-\Delta)^{\frac{k}{2}} K_{t-t_*} * n(t_*)\|_{L^{q_*}(\mathbb{R}^2)} + \int_{t_*}^t \|(-\Delta)^{\frac{k}{2}} \nabla \cdot K_{t-\tau} * (un + n \nabla (-\Delta)^{-1} n(\tau))\|_{L^{q_*}(\mathbb{R}^2)} d\tau \\
 & \leq C(t-t_*)^{-\frac{k}{2\alpha}} \|n(t_*)\|_{L^{q_*}(\mathbb{R}^2)} \\
 & \quad + C \int_{t_*}^t (t-\tau)^{-\frac{1}{\alpha p} - \frac{k+1}{2\alpha}} \|(un)(\tau)\|_{L^{\frac{q_* p}{q_*+p}}(\mathbb{R}^2)} d\tau + C \int_{t_*}^t (t-\tau)^{-\frac{1}{\alpha}(\frac{1}{q}-\frac{1}{2}) - \frac{k+1}{2\alpha}} \|(n \nabla (-\Delta)^{-1} n)(\tau)\|_{L^{\frac{2q_* q}{2q+q_*(2-q)}}(\mathbb{R}^2)} d\tau \\
 & \leq C \left(\frac{2}{T_{max}-t_*} \right)^{\frac{k}{2\alpha}} \|n(t_*)\|_{L^{q_*}(\mathbb{R}^2)} \\
 & \quad + C \int_{t_*}^t (t-\tau)^{-\frac{1}{\alpha p} - \frac{k+1}{2\alpha}} \|u(\tau)\|_{L^p(\mathbb{R}^2)} \|n(\tau)\|_{L^{q_*}(\mathbb{R}^2)} d\tau + C \int_{t_*}^t (t-\tau)^{-\frac{1}{\alpha}(\frac{1}{q}-\frac{1}{2}) - \frac{k+1}{2\alpha}} \|n(\tau)\|_{L^{q_*}(\mathbb{R}^2)} \|n(\tau)\|_{L^q(\mathbb{R}^2)} d\tau,
 \end{aligned}$$

which yields

$$\begin{aligned}
 & \|(-\Delta)^{\frac{k}{2}} n(t)\|_{L^{q_*}(\mathbb{R}^2)} \\
 & \leq C \left(\frac{2}{T_{max}-t_*} \right)^{\frac{k}{2\alpha}} \|n(t_*)\|_{L^{q_*}(\mathbb{R}^2)} \\
 & \quad + C \sup_{t_* < \tau < t} \|u(\tau)\|_{L^p(\mathbb{R}^2)} \sup_{t_* < \tau < t} \|n(\tau)\|_{L^{q_*}(\mathbb{R}^2)} \cdot \int_{t_*}^t (t-\tau)^{-\frac{1}{\alpha p} - \frac{k+1}{2\alpha}} d\tau \\
 & \quad + C \sup_{t_* < \tau < t} \|n(\tau)\|_{L^{q_*}(\mathbb{R}^2)} \sup_{t_* < \tau < t} \|n(\tau)\|_{L^q(\mathbb{R}^2)} \cdot \int_{t_*}^t (t-\tau)^{-\frac{1}{\alpha}(\frac{1}{q}-\frac{1}{2}) - \frac{k+1}{2\alpha}} d\tau,
 \end{aligned} \tag{199}$$

where $C = C(q, p, q_*, p_*, k)$. Employing (195) and (198), we deduce

$$\begin{aligned}
 & \|(-\Delta)^{\frac{k}{2}} n(t)\|_{L^{q_*}(\mathbb{R}^2)} \\
 & \leq C \left(\frac{2}{T_{max}-t_*} \right)^{\frac{k}{2\alpha}} \|n(t_*)\|_{L^{q_*}(\mathbb{R}^2)} + C(2\tilde{C} + \|n_0\|_{L^1(\mathbb{R}^2)} + 1)^2 \int_{t_*}^t (t-\tau)^{-\frac{1}{\alpha p} - \frac{k+1}{2\alpha}} d\tau \\
 & \quad + C(2\tilde{C} + \|n_0\|_{L^1(\mathbb{R}^2)} + 1)^2 \int_{t_*}^t (t-\tau)^{-\frac{1}{\alpha}(\frac{1}{q}-\frac{1}{2}) - \frac{k+1}{2\alpha}} d\tau \\
 & \leq C \left(\frac{2}{T_{max}-t_*} \right)^{\frac{k}{2\alpha}} \|n(t_*)\|_{L^{q_*}(\mathbb{R}^2)} + C(2\tilde{C} + \|n_0\|_{L^1(\mathbb{R}^2)} + 1)^2 T_{max}^{1-\frac{1}{\alpha p} - \frac{k+1}{2\alpha}} \\
 & \quad + C(2\tilde{C} + \|n_0\|_{L^1(\mathbb{R}^2)} + 1)^2 T_{max}^{1-\frac{1}{\alpha}(\frac{1}{q}-\frac{1}{2}) - \frac{k+1}{2\alpha}}
 \end{aligned} \tag{200}$$

for all $\frac{t_*+T_{max}}{2} < t < T_{max}$. Let us now introduce the well-known property: For every $1 \leq r \leq \infty$ and $0 < k < 2$, there exists a positive constant C such that if $f \in L^r(\mathbb{R}^2)$ and $(-\Delta)^{\frac{k}{2}} f \in L^r(\mathbb{R}^2)$, then

$$\|K_s * f - f\|_{L^r(\mathbb{R}^2)} \leq C s^{\frac{k}{2\alpha}} \|(-\Delta)^{\frac{k}{2}} f\|_{L^r(\mathbb{R}^2)} \tag{201}$$

for all $0 < s < \infty$. Hence, recalling (200), we arrive at

$$\begin{aligned}
 & \|K_{t-\tau} * n(\tau) - n(\tau)\|_{L^{q_*}(\mathbb{R}^2)} \\
 & \leq C(t-\tau)^{\frac{k}{2\alpha}} \sup_{\frac{t_*+T_{max}}{2} < \tau < T_{max}} \|(-\Delta)^{\frac{k}{2}} n(\tau)\|_{L^{q_*}(\mathbb{R}^2)} \rightarrow 0 \quad \text{as } t, \tau \rightarrow T_{max}.
 \end{aligned} \tag{202}$$

Assume that $\frac{t_* + T_{max}}{2} < \tau < t < T_{max}$. Similarly to (200), using (24) and (202), we conclude

$$\begin{aligned} & \|n(t) - n(\tau)\|_{L^{q_*}(\mathbb{R}^2)} \\ & \leq \|K_{t-\tau} * n(\tau) - n(\tau)\|_{L^{q_*}(\mathbb{R}^2)} + \int_{\tau}^t \|\nabla \cdot K_{t-s} * (un + n(-\Delta)^{-1}n)(s)\|_{L^{q_*}(\mathbb{R}^2)} ds \\ & \leq \|K_{t-\tau} * n(\tau) - n(\tau)\|_{L^{q_*}(\mathbb{R}^2)} + C(2\tilde{C} + \|n_0\|_{L^1(\mathbb{R}^2)} + 1)^2(t - \tau)^{1 - \frac{1}{\alpha p} - \frac{1}{2\alpha}} \\ & \quad + C(2\tilde{C} + \|n_0\|_{L^1(\mathbb{R}^2)} + 1)^2(t - \tau)^{1 - \frac{1}{\alpha q}} \rightarrow 0 \quad \text{as } t, \tau \rightarrow T_{max}, \end{aligned} \quad (203)$$

where $C = C(q, p, q_*, p_*)$. Since $L^{q_*}(\mathbb{R}^2)$ is a complete metric space, we obtain that there exists a function $n_* \in L^{q_*}(\mathbb{R}^2)$ such that

$$\|n_* - n(\tau)\|_{L^{q_*}(\mathbb{R}^2)} \rightarrow 0 \quad \text{as } \tau \rightarrow T_{max}. \quad (204)$$

Hence, there exists a time sequence $\{t_l\}_{l=1}^{\infty}$ with $t_l \rightarrow T_{max}$ such that

$$\lim_{l \rightarrow \infty} n(x, t_l) = n_*(x) \quad \text{for a.a. } x \in \mathbb{R}^2. \quad (205)$$

Since $n(x, t_l) \geq 0$, we deduce $n_*(x) \geq 0$ for almost all $x \in \mathbb{R}^2$.

Furthermore, applying Fatou lemma, we observe

$$\int_{\mathbb{R}^2} n_*(x) dx \leq \liminf_{l \rightarrow \infty} \int_{\mathbb{R}^2} n(x, t_l) dx = \|n_0\|_{L^1(\mathbb{R}^2)}. \quad (206)$$

Therefore, there exists a function $n_* \in L^1 \cap L^{q_*}(\mathbb{R}^2)$ such that $n_*(x) \geq 0$ for almost all $x \in \mathbb{R}^2$ and (196) holds.

Next, we are going to prove (197). Employing (195) and (198), the similar argument as in the proof of (143)-(144) holds

$$\int_{t_*}^t \|(-\Delta)^{\frac{k}{2}} \nabla \cdot K_{t-\tau} * P(u \otimes u)(\tau)\|_{L^{p_*}(\mathbb{R}^2)} d\tau \leq C(2\tilde{C} + \|n_0\|_{L^1(\mathbb{R}^2)} + 1)^2 \left(T_{max}^{1 - \frac{1}{\alpha p} - \frac{k+1}{2\alpha}} + T_{max}^{-\frac{1}{\alpha}(\alpha - 1 - \frac{1}{p_*}) - \frac{k+1}{2\alpha}} \right) \quad (207)$$

for all $\frac{t_* + T_{max}}{2} < t < T_{max}$, where $C = C(p, p_*, k)$.

On the other hand, applying the similar argument of (145)-(148), we observe

$$\begin{aligned} & \int_{t_*}^t \|(-\Delta)^{\frac{k}{2}} K_{t-\tau} * P(|n|^{\beta} f)(\tau)\|_{L^{p_*}(\mathbb{R}^2)} d\tau \\ & \leq C(2\tilde{C} + \|n_0\|_{L^1(\mathbb{R}^2)} + 1)^{\beta} \|f\|_{L^{\rho}(\mathbb{R}^2)} \\ & \quad \times \left(T_{max}^{1 - \frac{1}{\alpha}(\frac{1}{\rho} + \frac{\beta}{q} - \frac{1}{p_*}) - \frac{k}{2\alpha}} + T_{max}^{1 - \frac{1}{\alpha}(\frac{1}{\rho} + \frac{2\alpha\beta + 1 - 2\alpha}{2} - \frac{1}{p_*}) - \frac{k}{2\alpha}} + T_{max}^{1 - \frac{1}{\alpha}(\frac{1}{\rho} + \frac{\beta}{q_*}) - \frac{k}{2\alpha}} + T_{max}^{1 - \frac{1}{\alpha\rho} - \frac{k}{2\alpha}} \right) \end{aligned} \quad (208)$$

for all $\frac{t_* + T_{max}}{2} < t < T_{max}$, where $C = C(\alpha, \beta, q, q_*, p_*, k)$. Combining (207) and (208), we have

$$\begin{aligned} & \|(-\Delta)^{\frac{k}{2}} u(t)\|_{L^{p_*}(\mathbb{R}^2)} \\ & \leq \|(-\Delta)^{\frac{k}{2}} K_{t-t_*} * u(t_*)\|_{L^{p_*}(\mathbb{R}^2)} + \int_{t_*}^t \|(-\Delta)^{\frac{k}{2}} \nabla \cdot K_{t-\tau} * P(u \otimes u)(\tau)\|_{L^{p_*}(\mathbb{R}^2)} d\tau \\ & \quad + \int_{t_*}^t \|(-\Delta)^{\frac{k}{2}} K_{t-\tau} * P(|n|^{\beta} f)(\tau)\|_{L^{p_*}(\mathbb{R}^2)} d\tau \\ & \leq C(2\tilde{C} + \|n_0\|_{L^1(\mathbb{R}^2)} + 1)^2 \left(T_{max}^{1 - \frac{1}{\alpha p} - \frac{k+1}{2\alpha}} + T_{max}^{-\frac{1}{\alpha}(\alpha - 1 - \frac{1}{p_*}) - \frac{k+1}{2\alpha}} \right) \\ & \quad + C(2\tilde{C} + \|n_0\|_{L^1(\mathbb{R}^2)} + 1)^{\beta} \|f\|_{L^{\rho}(\mathbb{R}^2)} \\ & \quad \times \left(T_{max}^{1 - \frac{1}{\alpha}(\frac{1}{\rho} + \frac{\beta}{q} - \frac{1}{p_*}) - \frac{k}{2\alpha}} + T_{max}^{1 - \frac{1}{\alpha}(\frac{1}{\rho} + \frac{2\alpha\beta + 1 - 2\alpha}{2} - \frac{1}{p_*}) - \frac{k}{2\alpha}} + T_{max}^{1 - \frac{1}{\alpha}(\frac{1}{\rho} + \frac{\beta}{q_*}) - \frac{k}{2\alpha}} + T_{max}^{1 - \frac{1}{\alpha\rho} - \frac{k}{2\alpha}} \right), \end{aligned} \quad (209)$$

where $C = C(\alpha, \beta, q, p, q_*, p_*, k)$. This yields that

$$\begin{aligned} & \|K_{t-\tau} * u(\tau) - u(\tau)\|_{L^{p_*}(\mathbb{R}^2)} \\ & \leq C(t-\tau)^{\frac{k}{2\alpha}} \sup_{\frac{t_*+T_{max}}{2} < \tau < T_{max}} \|(-\Delta)^{\frac{k}{2}} u(\tau)\|_{L^{p_*}(\mathbb{R}^2)} \rightarrow 0 \quad \text{as } t, \tau \rightarrow T_{max}. \end{aligned} \quad (210)$$

Let $\frac{t_*+T_{max}}{2} < \tau < t < T_{max}$. From (198) and (210), we conclude

$$\begin{aligned} & \|u(t) - u(\tau)\|_{L^{p_*}(\mathbb{R}^2)} \\ & \leq \|K_{t-\tau} * u(\tau) - u(\tau)\|_{L^{p_*}(\mathbb{R}^2)} \\ & \quad + \int_{\tau}^t \|\nabla \cdot K_{t-s} * P(u \otimes u)(s)\|_{L^{p_*}(\mathbb{R}^2)} ds + \int_{\tau}^t \|K_{t-s} * P(|n|^{\beta} f)(s)\|_{L^{p_*}(\mathbb{R}^2)} ds \\ & \leq \|K_{t-\tau} * u(\tau) - u(\tau)\|_{L^{p_*}(\mathbb{R}^2)} + C(2\tilde{C} + \|n_0\|_{L^1(\mathbb{R}^2)} + 1)^2(t-\tau)^{1-\frac{1}{\alpha p} - \frac{1}{2\alpha}} \\ & \quad + C(2\tilde{C} + \|n_0\|_{L^1(\mathbb{R}^2)} + 1)^2(t-\tau)^{-\frac{1}{\alpha}(\alpha - \frac{1}{2} - \frac{1}{p_*})} + C(2\tilde{C} + \|n_0\|_{L^1(\mathbb{R}^2)} + 1)^{\beta} \|f\|_{L^{\rho}(\mathbb{R}^2)} \\ & \quad \times \left((t-\tau)^{1-\frac{1}{\alpha}(\frac{1}{\rho} + \frac{\beta}{q} - \frac{1}{p_*})} + (t-\tau)^{1-\frac{1}{\alpha}(\frac{1}{\rho} + \frac{2\alpha\beta+1-2\alpha}{2} - \frac{1}{p_*})} \right. \\ & \quad \left. + (t-\tau)^{1-\frac{1}{\alpha}(\frac{1}{\rho} + \frac{\beta}{q_*})} + (t-\tau)^{1-\frac{1}{\alpha\rho}} \right) \rightarrow 0 \quad \text{as } t, \tau \rightarrow T_{max}, \end{aligned} \quad (211)$$

where $C = C(\alpha, \beta, q, p, q_*, p_*)$. Since $L^{p_*}(\mathbb{R}^2)$ is a complete metric space, we find that there exists a function $u_* \in L^{p_*}(\mathbb{R}^2)$ such that

$$\|u_* - u(\tau)\|_{L^{p_*}(\mathbb{R}^2)} \rightarrow 0 \quad \text{as } \tau \rightarrow T_{max}. \quad (212)$$

Therefore, there exists a time sequence $\{t_l\}_{l=1}^{\infty}$ with $t_l \rightarrow T_{max}$ such that

$$\lim_{l \rightarrow \infty} u(x, t_l) = u_*(x) \quad \text{for a.a. } x \in \mathbb{R}^2. \quad (213)$$

In addition, recalling Fatou lemma, together with Lemma 3.6, we observe: there exists $t_{**} \in (0, T_{max})$ such that

$$\begin{aligned} \int_{\mathbb{R}^2} |u_*(x)|^2 dx & \leq \liminf_{l \rightarrow \infty} \int_{\mathbb{R}^2} |u(x, t_l)|^2 dx \\ & \leq \sup_{t_{**} < t < T_{max}} \|u(t)\|_{L^2(\mathbb{R}^2)}^2 + 1 \leq \|u_0\|_{L^2(\mathbb{R}^2)}^2 + C(1 + \|u_0\|_{L^2(\mathbb{R}^2)}^2) + 1, \end{aligned} \quad (214)$$

where $C = C(\alpha, \beta, q_*, \|n_0\|_{L^1(\mathbb{R}^2)}, \|f\|_{L^{\rho}(\mathbb{R}^2)}, T_{max}, \tilde{C})$. Hence, there exists a function $u_* \in L^2 \cap L^{p_*}(\mathbb{R}^2)$ with (197).

We next show that $u_* \in H^{1,2}(\mathbb{R}^2)$ and $\nabla \cdot u_*(x) = 0$ for almost all $x \in \mathbb{R}^2$. Applying Lemma 3.6 with $s = 2$, we deduce

$$u \in H^{1,2}(0, T_{max}; L^2_{\sigma}(\mathbb{R}^2)) \cap L^2(0, T_{max}; H^{2\alpha,2}(\mathbb{R}^2)) \quad (215)$$

with the estimates as follows

$$\left\| \frac{\partial u}{\partial t} \right\|_{L^2(0, T_{max}; L^2(\mathbb{R}^2))} + \sum_{k=0}^2 \|\partial^{k\alpha} u\|_{L^2(0, T_{max}; L^2(\mathbb{R}^2))} \leq C. \quad (216)$$

Then, based on the standard embedding theorem, we arrive at

$$u \in BUC([0, T_{max}]; H^{1,2}(\mathbb{R}^2)), \quad (217)$$

where BUC implies the bounded uniformly continuous space. Thus, we obtain that there exists a function $\chi_* \in L^2(\mathbb{R}^2)$ such that

$$\|\chi_* - \nabla u(\tau)\|_{L^2(\mathbb{R}^2)} \rightarrow 0 \quad \text{as } \tau \rightarrow T_{max}. \quad (218)$$

Hence, we observe $\chi_* = \nabla u(x)$ for almost all $x \in L^2(\mathbb{R}^2)$

and there exists a time sequence $\{t_l\}_{l=1}^{\infty}$ with $t_l \rightarrow T_{max}$ so that

$$\lim_{l \rightarrow \infty} \nabla u(x, t_l) = \nabla u_*(x) \quad \text{for a.a. } x \in \mathbb{R}^2. \quad (219)$$

This yields that $u_* \in H^{1,2}(\mathbb{R}^2)$ and $\nabla \cdot u(x, t_l) \rightarrow \nabla \cdot u_*(x) = 0$ for almost all $x \in L^2(\mathbb{R}^2)$ as $t_l \rightarrow T_{max}$. Therefore we establish (197).

By virtue of (196) and (197), using Theorem 2.1-(i), (ii), (iv), we could extend the mild solution beyond T_{max} . This

contradicts that T_{max} in (23) is the maximal existence time. This complete the proof of Theorem 2.2.

Acknowledgments

The work is partially supported by National Natural Science Foundation of China (11771380 and 12401654).

Conflicts of Interest

The authors declared that they have no conflicts of interest to this work.

References

- [1] P. Biler, G. Karch and J. Zienkiewicz, "Large global-in-time solutions to a nonlocal model of chemotaxis," *Advances in Mathematics*. 2018, Vol.330, 834-875. <https://doi.org/10.1016/j.aim.2018.03.036>
- [2] P. Biler, T. Cieřlak, G. Karch and J. Zienkiewicz, "Local criteria for blowup in two-dimensional chemotaxis models," *Discrete and Continuous Dynamical Systems*. 2017, Vol. 37, no. 4, 1841-1856. <https://doi.org/10.3934/dcds.2017077>
- [3] P. Biler and G. Karch, "Blowup of solutions to generalized Keller-Segel model," *Journal of Evolution Equations*. 2010, Vol. 10, no. 2, 247-262. <https://doi.org/10.1007/s00028-009-0048-0>
- [4] L. Brandolese and G. Karch, "Far field asymptotics of solutions to convection equation with anomalous diffusion," *Journal of Evolution Equations*. 2008, Vol. 8, no. 2, 307-326. <https://doi.org/10.1007/s00028-008-0356-9>
- [5] H. Brezis and T. Cazenave, "A nonlinear heat equation with singular initial data," *Journal d'Analyse Mathématique*. 1996, Vol. 68, 277-304.
- [6] J. Burczak and R. Granero-Belinchón, "Boundedness and homogeneous asymptotics for a fractional logistic Keller-Segel equations," *Discrete and Continuous Dynamical Systems Series S*. 2020, Vol. 13, no. 2, 139-164. <https://doi.org/10.3934/dcdss.2020008>
- [7] J. Burczak and R. Granero-Belinchón, "Suppression of blow up by a logistic source in 2D Keller-Segel system with fractional dissipation," *Journal of Differential Equations*. 2017, Vol. 263, no. 9, 6115-6142. <https://doi.org/10.1016/j.jde.2017.07.007>
- [8] J. Burczak and R. Granero-Belinchón, "On a generalized doubly parabolic Keller-Segel system in one spatial dimension," *Mathematical Models and Methods in Applied Sciences*. 2016, Vol. 26, no. 1, 111-160. <https://doi.org/10.1142/S0218202516500044>
- [9] M. Chae, K. Kang, J. Lee and K. Lee, "A regularity condition and temporal asymptotics for chemotaxis-fluid equations," *Nonlinearity*. 2018, Vol. 31, no. 2, 351-387. <https://doi.org/10.1088/1361-6544/aa92ec>
- [10] L. Caffarelli and Y. Sire, "On some pointwise inequalities involving nonlocal operators," *Applied and Numerical Harmonic Analysis*. 2017, Birkhäuser/Springer, Cham.
- [11] X. Cao, S. Kurima and M. Mizukami, "Global existence and asymptotic behavior of classical solutions for a 3D two species chemotaxis-Stokes system with competitive kinetics," *Mathematical Methods in the Applied Sciences*. 2018, Vol. 41, no. 8, 3138-3154. <https://doi.org/10.1002/mma.4807>
- [12] M. D'Elia, Q. Du, C. Glusa, M. Gunzburger, X.C. Tian and Z. Zhou, "Numerical methods for nonlocal and fractional models," *Acta Numerica*. 2020, Vol.29, 1-124. <https://doi.org/10.1017/S096249292000001X>
- [13] Q. Du, X.C. Tian and Z. Zhou, "Nonlocal diffusion models with consistent local and fractional limits," *A³N²M: approximation, applications, and analysis of nonlocal, nonlinear models*. 2023, Vol. 165, 175-213.
- [14] R. Duan, A. Lorz and P. Markowich, "Global solutions to the coupled chemotaxis-fluid equations," *Communications in Partial Differential Equations*. 2010, Vol. 35, no. 9, 1635-1673. <https://doi.org/10.1080/03605302.2010.497199>
- [15] C. Escudero, "The fractional Keller-Segel model," *Nonlinearity*. 2006, Vol. 19, no. 12, 2909-2918. <https://doi.org/10.1088/0951-7715/19/12/010>
- [16] A. Garfinkel, Y. Tintut, D. Petrasek, K. Boström and L. L. Demer, "Pattern formation by vascular mesenchymal cells," *Proceedings of the National Academy of Sciences*. 2004, Vol. 101, 9247-9250. <https://doi.org/10.1073/pnas.0308436101>
- [17] Y. Giga and H. Sohr, "Abstract L^p estimates for the Cauchy problem with applications to the Navier-Stokes equations in exterior domains," *Journal of Functional Analysis*. 1991, Vol. 102, no. 1, 72-94. [https://doi.org/10.1016/0022-1236\(91\)90136-S](https://doi.org/10.1016/0022-1236(91)90136-S)
- [18] R. Granero-Belinchón, "Global solutions for a hyperbolic-parabolic system of chemotaxis," *Journal of Mathematical Analysis and Applications*. 2017, Vol. 449, no. 1, 872-883. <https://doi.org/10.1016/j.jmaa.2016.12.050>
- [19] B.L. Guo, X.K. Pu and F.H. Huang, "Fractional Partial differential Equations and their Numerical Solutions," 2011, *Science Press*.
- [20] Y.S. Guo and S.M. He, "On the 8π -critical mass threshold of a Patlak-Keller-Segel-Navier-Stokes system," *SIAM Journal on Mathematical Analysis*. 2021, Vol. 53, no. 3, 2925-2956. <https://doi.org/10.1137/20M1340629>

- [21] T. Kato and G. Poince, "Commutator estimates and the Euler and Navier-Stokes equations," *Communications on Pure and Applied Mathematics*. 1988, Vol. 41, no. 7, 891-907. <https://doi.org/10.1002/cpa.3160410704>
- [22] M. Hirata, S. Kurima, M. Mizukami and T. Yokota, "Boundedness and stabilization in a two-dimensional two-species chemotaxis-Navier-Stokes system with competitive kinetics," *Journal of Differential Equations*. 2017, Vol. 263, no. 1, 470-490. <https://doi.org/10.1016/j.jde.2017.02.045>
- [23] H. Huang and J.G. Liu, "Well-posedness for Keller-Segel equation with fractional laplacian and the theory of propagation of chaos," *Kinetic and Related Models*. 2016, Vol. 9, no. 4, 715-748. <https://doi.org/10.3934/krm.2016013>
- [24] H. Kozono, M. Miura and Y. Sugiyama, "Time global existence and finite time blow-up criterion for solutions to the Keller-Segel system coupled with the Navier-Stokes fluid," *Journal of Differential Equations*. 2019, Vol. 267, no. 9, 5410-5492. <https://doi.org/10.1016/j.jde.2019.05.035>
- [25] H. Kozono, M. Miura and Y. Sugiyama, "Existence and uniqueness theorem on mild solutions to the Keller-Segel system coupled with the Navier-Stokes fluid," *Journal of Functional Analysis*. 2016, Vol. 270, no. 5, 1663-1683. <https://doi.org/10.1016/j.jfa.2015.10.016>
- [26] A. Lorz, "A coupled Keller-Segel-Stokes model: global existence for small initial data and blow-up delay," *Communications in Mathematical Sciences*. 2012, Vol. 10, no. 2, 555-574. <https://dx.doi.org/10.4310/CMS.2012.v10.n2.a7>
- [27] A. Lorz, "Coupled chemotaxis fluid model," *Mathematical Models and Methods in Applied Sciences*. 2010, Vol. 20, no. 6, 987-1004. <https://doi.org/10.1142/S0218202510004507>
- [28] Y. Li and Y. Li, "Global boundedness of solutions for the chemotaxis-Navier-Stokes system in \mathbb{R}^2 ," *Journal of Differential Equations*. 2016, Vol. 261, no. 11, 6570-6631. <https://doi.org/10.1016/j.jde.2016.08.045>
- [29] I. Tuval, L. Cisneros, C. Dombrowski, C.W. Wolgemuth, J.O. Kessler and R.E. Goldstein, "Bacterial swimming and oxygen transport near contact lines," *Proceedings of the National Academy of Sciences*. 2005, Vol. 102, 2277-2282. <https://doi.org/10.1073/pnas.0406724102>
- [30] Y. Tao and M. Winkler, "Blow-up prevention by quadratic degradation in a two-dimensional Keller-Segel-Navier-Stokes system," *Zeitschrift für angewandte Mathematik und Physik*. 2016, Vol. 67, no. 6, Art. 138, 23pp. <https://doi.org/10.1007/s00033-016-0732-1>
- [31] Y. Tao and M. Winkler, "Boundedness and decay enforced by quadratic degradation in a three-dimensional chemotaxis-fluid system," *Zeitschrift für angewandte Mathematik und Physik*. 2015, Vol. 66, no. 5, 2555-2573. <https://doi.org/10.1007/s00033-015-0541-y>
- [32] Z. Tan and X. Zhang, "Decay estimates of the coupled chemotaxis-fluid equations in \mathbb{R}^3 ," *Journal of Mathematical Analysis and Applications*. 2014, Vol. 410, no. 1, 27-38. <https://doi.org/10.1016/j.jmaa.2013.08.008>
- [33] M. Winkler, "Reaction-driven relaxation in three-dimensional Keller-Segel-Navier-Stokes interaction," *Communications in Mathematical Physics*. 2022, Vol. 389, no. 1, 439-489. <https://doi.org/10.1007/s00220-021-04272-y>
- [34] M. Winkler, "Small-mass solutions in the two-dimensional Keller-Segel system coupled to the Navier-Stokes equations," *SIAM Journal on Mathematical Analysis*. 2020, Vol. 52, no. 2, 2041-2080. <https://doi.org/10.1137/19M1264199>
- [35] M. Winkler, "A three-dimensional Keller-Segel-Navier-Stokes system with logistic source: global weak solutions and asymptotic stabilization," *Journal of Functional Analysis*. 2019, Vol. 276, no. 5, 1339-1401. <https://doi.org/10.1016/j.jfa.2018.12.009>
- [36] M. Winkler, "How far do chemotaxis-driven forces influence regularity in the Navier-Stokes system?" *Transactions of the American Mathematical Society*. 2017, Vol. 369, no. 5, 3067-3125. <http://dx.doi.org/10.1090/tran/6733>
- [37] X. Wang, Z. Liu and L. Zhou, "Asymptotic decay for the classical solution of the chemotaxis system with fractional Laplacian in high dimensions," *Discrete and Continuous Dynamical Systems-B*. 2018, Vol. 23, no. 9, 4003-4020. <https://doi.org/10.3934/dcdsb.2018121>
- [38] G. Wu and X. Zheng, "On the well-posedness for Keller-Segel system with fractional diffusion," *Mathematical Methods in the Applied Sciences*. 2011, Vol. 34, no. 14, 1739-1750. <https://doi.org/10.1002/mma.1480>