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Research Article

New Findings in the Stability Analysis of PI-state Controlled Systems with Actuator Saturation

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Abstract

In this paper, a simple, generally valid stability proof for an anti-windup method for PI-state controlled systems is presented, with which it is possible to directly conclude the stability of the PI-state controlled system from a stable P-state controlled system with constraints in the manipulated variables, i.e. without having to perform a separate stability investigation of the anti-windup measures. The technique presented is based on the system description by means of state equations and Lyapunov's Direct Method using quadratic Lyapunov functions. Furthermore, the PI-state controller is designed in such a way that it provides the same command response as the P-state controller, for which a stability statement is already available. Both continuous-time and discrete-time systems are considered, which, apart from the saturation of the manipulated variables, show linear, time-invariant behavior. In addition, a general stability proof is given for discrete-time systems, which makes it possible to establish stable anti-windup methods for P- and PI-state controlled systems, which contain dead time elements in the path of the manipulated variables, without having to carry out separate stability investigations for them. For this purpose, the state controller design for the system with dead time elements in the manipulated variable paths is based on the principle that the same characteristics of the control behavior should be achieved as for the system without such dead time elements, but delayed by the dead time. The effectiveness of the presented methods is illustrated by an example from the field of electrical drives.

Keywords

Pi-State Controller, Actuator Saturation, Anti-Windup Methods, Continuous-Time and Discrete-Time Controllers, Controlled System with Dead Time Elements

1. Introduction

The dynamically high-quality response of a controller to the saturation of manipulated variables is an important task in controller design. Since such constraints represent non-linearities, the closed control loop is a non-linear system, even if the controlled system without actuator can be described as a linear, time-invariant system, which is assumed below. In order to avoid stability problems caused by the limitations of the manipulated variables, numerous so-called anti-windup methods are already known [1-3]. For this purpose, the Popov criterion, the circle criterion, the direct method of Lyapunov or the Kalman-Yakubovich-Popov equations [2-4] are often used for stability considerations. In more recent research studies, so-called linear matrix inequalities (LMI) are also used as an alternative for stability studies.

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However, they often only provide numerical results [5]. Other current research work is concerned with the application of the basic principles of anti-windup measures to special controllers such as PI-lead controllers [6] or with switching strategies between different anti-windup measures [7]. In addition, the use of an *Additional Dynamic Element* (ADE) is proposed, with which the stabilization in the limiting case succeeds for any controller stabilizing the unconstrained system [1]. However, even there a controller must be effective at least at one instance for which the stability in the limiting case can be proven – e.g. with the help of one of the methods listed above.

Stability analysis is especially challenging when the controller contains integral-action components to ensure steady-state accuracy. This is because the controller integral-action components are usually assigned to the controlled system during modeling, which results in an unstable or critically stable system. For this purpose, no positive definite matrices can be found for this, as required or at least aimed for in the Lyapunov theory and in the Kalman-Yakubovich-Popov equations to ensure stability. In [1], this problem is overcome by completely avoiding controller integral-action components and instead attempting to ensure steady-state accuracy with the aid of disturbance observers. But this is not always possible when the system parameters are not exactly known. The method of reference variable correction in combination with a special PI-state controller design, as explained for example in [8, 9] for discrete-time systems, provides a help in this respect. However, the stability proof described there has been greatly simplified in the meantime. Furthermore, by means of the above-mentioned procedure, it was also possible to perform the stability verification for such systems in a general way and thus greatly reduce the synthesis effort where the manipulated variables act on the system with dead time.

Due to the significant progress in stability verification for control loops with saturation of the manipulated variables, these new findings are presented in this article. The controlled system and the controller are modeled in state space. In section 2, as an introduction, the methodology of reference variable correction is first briefly explained for continuous-time systems and then the transfer to discrete-time systems is shown. Subsequently, section 3 describes special controller synthesis equations that generate a PI-state controller from an already known P-state controller in a simple way. In addition, both methods are combined in section 3 in order to establish the principles for a Lyapunov-based controller design for systems with an integral-action controller component, taking manipulated variable constraints into account. Using Lyapunov functions, the proof of stability is then provided in section 4. The measures mentioned are carried out for both continuous-time and discrete-time systems. In section 5, the method presented is extended to discrete-time systems with dead time behavior of the actuators. To illustrate the methods described, section 6 deals with an example from the field of electrical drives. A summary concludes the article.

2. Reference Variable Correction in Case of Manipulated Variable Saturation

2.1. State Equations of the Controlled System

The vectorial state differential equation of the controlled system,

$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{B}\,\underline{u}, \tag{1}$$

serves as the starting point for the following considerations [2, 10]. Here $\underline{x}$ denotes the n-dimensional state vector, $\underline{u}$ the (p,p)-dimensional manipulated variable vector, $\underline{A}$ the (n,n)-dimensional dynamics matrix and $\underline{B}$ the (n,p)-dimensional control input matrix. Disturbance variables are not considered without any generality restriction. Eq. (1) is supplemented by the output equation

$$\underline{y} = \underline{C}\,\underline{x} \tag{2}$$

using the q-dimensional vector $\underline{y}$ of the control quantities and the (q,n)-dimensional output matrix $\underline{C}$. The possibility of the manipulated variables affecting directly the control quantities is disregarded.

The differential or difference equations of the controller integrators are also included in the system description. In this respect, it is assumed below that there are as many controller integrators as controlled variables and as many reference variables as controlled variables. If the output variables of the controller integrators are summarized in the vector $\underline{v}_{\mathrm{I}}$ and the reference variables in the vector $\underline{w}$, then the vectorial differential equation of the controller integrators is as follows, provided they operate continuously in time,

$$\dot{\underline{v}}_{\mathrm{I}} = \underline{w} - \underline{y} = \underline{w} - \underline{C}\,\underline{x}. \tag{3}$$

If the controlled system is described in discrete time, the controller design is based on the controlled system state difference equation [9, 10]

$$\underline{x}_{k+1} = \underline{\Phi}\,\underline{x}_k + \underline{H}\,\underline{u}_k \tag{4}$$

instead of Eq. (1). The indices k and $k+1$ ($k \in \mathbb{N}_0$) indicate the sampling time instants kT_s respectively $(k+1)\,T_s$ of the state variables and the time instants at which the manipulated variables take effect. $\underline{\Phi}$ is the (n,n)-dimensional transition matrix for which

$$\underline{\Phi} = \mathrm{e}^{\underline{A}\,T_s} \tag{5}$$

applies [9, 10], while $\underline{H}$ is the discrete-time control input matrix with [9, 10]

$$\underline{H} = \int_0^{T_s} e^{\underline{A}\,v}\, dv\, \underline{B}. \tag{6}$$

T_s is the sampling time. The output equation is in the discrete-time case

$$\underline{y}_k = \underline{C}\,\underline{x}_k. \tag{7}$$

The vectorial difference equation of the controller integrators is as follows [9, 10]

$$\underline{v}_{I,k+1} = \underline{v}_{I,k} + \underline{w}_k - \underline{y}_k = \underline{v}_{I,k} + \underline{w}_k - \underline{C}\,\underline{x}_k. \tag{8}$$

2.2. Calculating the Corrected Reference Variables

The reference variable correction method for manipulated variable saturation [8, 9] is based on the assumption that a manipulated variable saturation becomes effective because the setpoint change is too large. If this is the case, the maximum value by which the reference variable may be changed without the manipulated variable constraints becoming effective is calculated. The vector of reference variables corrected in this way is referred to below as $\underline{w}_{corr}$. To determine the value of $\underline{w}_{corr}$ in the limiting case, the control law is specified firstly for the unlimited case and then again for the case of an active manipulated variable limitation. If the matrix of the feedback coefficients of the system state variables to the manipulated variable vector is denoted by $\underline{K}_P$, the matrix of the feedback coefficients of the output variables of the controller integral-action components to the manipulated variable vector by $\underline{K}_I$ and the matrix of the amplification factors for the reference variables, the so-called pre-filter matrix, by $\underline{M}$, then the control law in the unlimited case is as follows

$$\underline{u}_{(k)} = \underline{M}\,\underline{w}_{(k)} - \underline{K}_P\,\underline{x}_{(k)} - \underline{K}_I\,\underline{v}_{I(,k)}. \tag{9}$$

Thereby, the index k in brackets in Eq. (9) only applies to the discrete-time case.

If a manipulated variable saturation now occurs, then instead of the manipulated variable vector $\underline{u}$ requested by the controller, only a manipulated variable vector modified by the saturation can act on the system. If this is designated as $\underline{u}_{sat}$, the control law

$$\underline{u}_{sat(,k)} = \underline{M}\,\underline{w}_{corr(,k)} - \underline{K}_P\,\underline{x}_{(k)} - \underline{K}_I\,\underline{v}_{I(,k)} \tag{10}$$

is obtained, which can be derived from Eq. (9), if the possibly constrained manipulated variable vector $\underline{u}_{sat(,k)}$ is used instead of $\underline{u}_{(k)}$ and the corrected reference variable vector $\underline{w}_{corr(,k)}$ is used instead of $\underline{w}_{(k)}$. The two relationships (9) and (10) can now be interpreted in such a way that they apply at the same time. Eq. (9) generates the manipulated variable vector requested by the controller, while Eq. (10) describes with which corrected reference variable vector the realizable manipulated variable vector $\underline{u}_{sat(,k)}$ can be generated. If both equations are subtracted from each other and the resulting difference is solved for $\underline{w}_{corr(,k)}$, the result is as follows

$$\underline{w}_{corr(,k)} = \underline{w}_{(k)} - \underline{M}^{-1}\left(\underline{u}_{(k)} - \underline{u}_{sat(,k)}\right). \tag{11}$$

It specifies how $\underline{w}_{(k)}$ must be modified in order to obtain a realizable reference variable vector. The corrected reference variables are then fed to the setpoint inputs of the controller integrators. This means that instead of Eq. (3), the following applies for continuous-time control

$$\dot{\underline{v}}_I = \underline{w}_{corr} - \underline{C}\,\underline{x}, \tag{12}$$

whereas for discrete-time control, instead of Eq. (8),

$$\underline{v}_{I,k+1} = \underline{v}_{I,k} + \underline{w}_{corr,k} - \underline{C}\,\underline{x}_k \tag{13}$$

must be implemented. Thus, Eqs. (9), (11) and (12) or respectively (13) describe the equations of the controller. The corresponding block diagram is shown in Figure 1 for the case of discrete-time control, including the discrete-time modeled controlled system. Finally, it should be noted that methods that also calculate corrected reference variables and use the difference between unlimited and limited manipulated variables are sometimes referred to as reverse-correction method [11] or back-calculation (and tracking) strategy [5, 6, 12].

3. Relationship Between the Controller Coefficients of a PI-state Controller and a P-state Controller

In the study, it was shown how the coefficients of a discrete-time PI-state controller can be determined in a very simple way from an already calculated discrete-time P-state controller, provided that the same command response is to apply in both cases [9, 13]. The corresponding relationships for continuous-time state controllers were presented in [14] – and for single-input single-output systems also in [15]. With the designation $\underline{K}$ for the already known P-state controller, the following results for continuous-time controllers

$$\underline{K}_P = \underline{K} + \underline{K}_I\,\underline{C}\left(\underline{A} - \underline{B}\,\underline{K}\right)^{-1}, \tag{14}$$

$$\underline{K}_I = \underline{M}\,\underline{\Lambda}_I, \tag{15}$$

$$\underline{M} = -\left(\underline{C}\left(\underline{A} - \underline{B}\,\underline{K}\right)^{-1}\underline{B}\right)^{-1}, \tag{16}$$

and for discrete-time systems

$$\underline{K}_{\mathrm{P}} = \underline{K} + \underline{K}_{\mathrm{I}}\,\underline{C}\,\left(\underline{\Phi} - \underline{H}\,\underline{K} - \underline{I}\right)^{-1}, \tag{17}$$

$$\underline{K}_{\mathrm{I}} = \underline{M}\,\left(\underline{Z}_{\mathrm{I}} - \underline{I}\right), \tag{18}$$

$$\underline{M} = -\left(\underline{C}\,\left(\underline{\Phi} - \underline{H}\,\underline{K} - \underline{I}\right)^{-1}\underline{H}\right)^{-1}. \tag{19}$$

With continuous-time control, $\underline{\Lambda}_{\mathrm{I}}$ describes a diagonal matrix in which the main diagonal contains those control eigenvalues that have been added to the eigenvalues which result from the controlled system without controller integrators. $\underline{Z}_{\mathrm{I}}$ is the corresponding diagonal matrix for discrete-time systems. It should be emphasized in both cases that the control eigenvalues generated by $\underline{K}$ of the non-actuator-saturated P-state-controlled system are not changed by applying Eqs. (9) and (14) to (16) or (17) to (19). It should also be noted that Eqs. (14) to (16) or (17) to (19) lead to q control eigenvalues that cannot be controlled via $\underline{u}_{\mathrm{sat}}$ (see section 4). Finally, it should be noted that the calculation rules (16) and (19) for the pre-filter matrix $\underline{M}$ are the same as the corresponding calculation rules for pure P-state controllers.

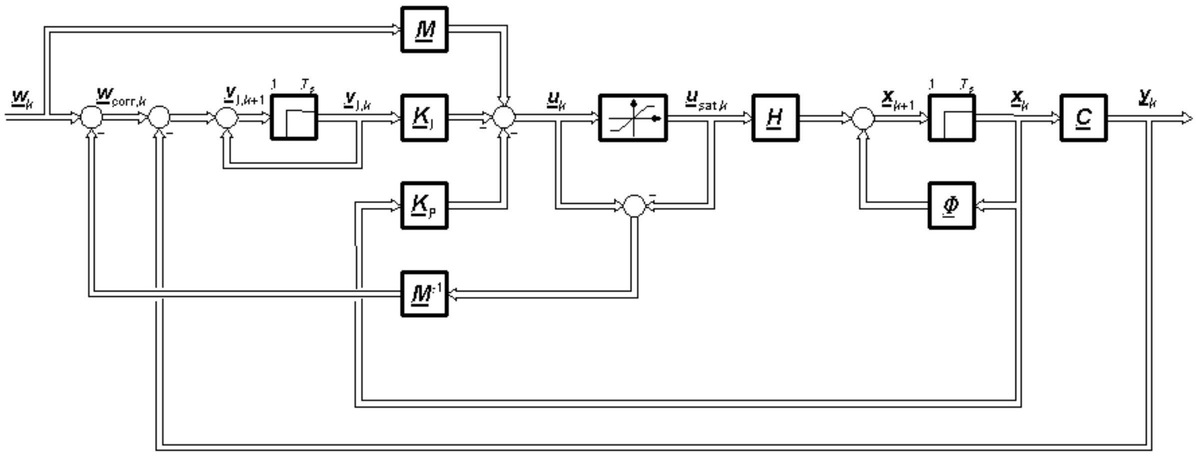

Figure 1. *Block diagram of discrete-time PI-state control with reference variable correction in the case of manipulated variable saturation.*

If we now substitute Eq. (9) into Eq. (11) and the result for continuous-time control into Eq. (12), we obtain, taking into account Eq. (15),

$$\dot{\underline{v}}_{\mathrm{I}} = \left(\underline{M}^{-1}\,\underline{K}_{\mathrm{P}} - \underline{C}\right)\underline{x} + \underline{\Lambda}_{\mathrm{I}}\,\underline{v}_{\mathrm{I}} + \underline{M}^{-1}\,\underline{u}_{\mathrm{sat}}. \tag{20}$$

If this relationship is combined with the controlled system state differential equation (1), but with $\underline{u}_{\mathrm{sat}}$ instead of $\underline{u}$, to form an overall system, the following results

$$\dot{\underline{x}}_{\mathrm{ext}} = \underline{A}_{\mathrm{ext}}\,\underline{x}_{\mathrm{ext}} + \underline{B}_{\mathrm{ext}}\,\underline{u}_{\mathrm{sat}} \tag{21}$$

with

$$\underline{x}_{\mathrm{ext}} = \begin{bmatrix} \underline{x} \\ \underline{v}_{\mathrm{I}} \end{bmatrix}, \tag{22}$$

$$\underline{A}_{\mathrm{ext}} = \begin{bmatrix} \underline{A} & \underline{0} \\ \underline{M}^{-1}\,\underline{K}_{\mathrm{P}} - \underline{C} & \underline{\Lambda}_{\mathrm{I}} \end{bmatrix}, \tag{23}$$

$$\underline{B}_{\mathrm{ext}} = \begin{bmatrix} \underline{B} \\ \underline{M}^{-1} \end{bmatrix}. \tag{24}$$

In the case of discrete-time systems, by inserting Eq. (9) into Eq. (11) and inserting the resulting intermediate result into Eq. (13) and taking Eq. (18) into account, the following is obtained

$$\underline{v}_{\mathrm{I},k+1} = \left(\underline{M}^{-1}\,\underline{K}_{\mathrm{P}} - \underline{C}\right)\underline{x}_k + \underline{Z}_{\mathrm{I}}\,\underline{v}_{\mathrm{I},k} + \underline{M}^{-1}\,\underline{u}_{\mathrm{sat},k} \tag{25}$$

The combination of this relationship with Eq. (4), again with $\underline{u}_{\mathrm{sat}}$ instead of $\underline{u}$, to form an overall system, taking into account Eq. (22), gives the result

$$\underline{x}_{\mathrm{ext},k+1} = \underline{\Phi}_{\mathrm{ext}}\,\underline{x}_{\mathrm{ext},k} + \underline{H}_{\mathrm{ext}}\,\underline{u}_{\mathrm{sat},k} \tag{26}$$

with

$$\underline{\Phi}_{\mathrm{ext}} = \begin{bmatrix} \underline{\Phi} & \underline{0} \\ \underline{M}^{-1}\,\underline{K}_{\mathrm{P}} - \underline{C} & \underline{Z}_{\mathrm{I}} \end{bmatrix}, \tag{27}$$

$$\underline{H}_{\mathrm{ext}} = \begin{bmatrix} \underline{H} \\ \underline{M}^{-1} \end{bmatrix}. \tag{28}$$

It can now be seen that both the dynamic matrix $\underline{A}_{\mathrm{ext}}$ and the transition matrix $\underline{\Phi}_{\mathrm{ext}}$ are lower block triangular matrices. Their eigenvalues are identical in their entirety with the eigenvalues of the matrix blocks on the main block diagonal. The eigenvalues of $\underline{A}_{\mathrm{ext}}$ are therefore composed of the eigenvalues of $\underline{A}$ and the control eigenvalues on the main diagonal of $\underline{\Lambda}_\mathrm{I}$ caused by the controller integrators. For discrete-time systems, the eigenvalues of the transition matrix $\underline{\Phi}_\mathrm{ext}$ are composed of the eigenvalues of $\underline{\Phi}$ and the elements on the main diagonal of $\underline{Z}_\mathrm{I}$, i.e. the control eigenvalues caused by the discrete-time controller integrators.

As the eigenvalues of $\underline{\Lambda}_\mathrm{I}$ and $\underline{Z}_\mathrm{I}$ can be assumed to be stable, the dynamic matrix $\underline{A}_\mathrm{ext}$ and the transition matrix $\underline{\Phi}_\mathrm{ext}$ each describe a stable system, provided the associated controlled system is stable. This is remarkable, especially as the controller integrators included in the model – considered on their own – show critically stable behavior. The reference variable correction according to Eq. (11), in conjunction with the controller equations (14) to (16) respectively (17) to (19), has thus succeeded in forming a stable system from a critically stable system – assuming a stable controlled system but an unknown controller matrix $\underline{K}$. Exactly this is extremely advantageous for the applicability of Lyapunov's direct method in stability analysis and controller synthesis for linear systems with manipulated variable limits (see section 4).

4. Controller Synthesis and Stability Verification

The stability analysis of PI-state control with manipulated variable saturation described below is based both on controllability considerations and on Lyapunov's direct method. In combination, both methods are also suitable for controller synthesis. Initially, however, the considerations focus on the stability analysis. In a first step, the system description according to Eq. (21) respectively (26) is transformed. The extended state vector $\underline{x}_\mathrm{ext}$ is mapped to the state vector $\underline{\tilde{x}}_\mathrm{ext}$ for continuous-time control using the transformation rules

$$\underline{\tilde{x}}_\mathrm{ext} = \underline{T}\,\underline{x}_\mathrm{ext} \tag{29}$$

$$\underline{T} = \begin{bmatrix} \underline{I} & \underline{0} \\ \underline{C}\,(\underline{A}-\underline{B}\,\underline{K})^{-1} & \underline{I} \end{bmatrix}. \tag{30}$$

Deriving Eq. (29) with respect to time, replacing $\underline{\dot{x}}_\mathrm{ext}$ by the right-hand side of Eq. (21) and finally replacing $\underline{x}_\mathrm{ext}$ by Eq. (29) solved for $\underline{x}_\mathrm{ext}$ then leads to the transformed differential state equation

$$\underline{\dot{\tilde{x}}}_\mathrm{ext} = \underline{\tilde{A}}_\mathrm{ext}\,\underline{\tilde{x}}_\mathrm{ext} + \underline{\tilde{B}}_\mathrm{ext}\,\underline{u}_\mathrm{sat} \tag{31}$$

with

$$\underline{\tilde{A}}_\mathrm{ext} = \underline{T}\,\underline{A}_\mathrm{ext}\,\underline{T}^{-1}$$

$$= \begin{bmatrix} \underline{A} & \underline{0} \\ \underline{C}\,(\underline{A}-\underline{B}\,\underline{K})^{-1}\underline{A} + \underline{M}^{-1}\underline{K}_\mathrm{P} - \underline{C} - \underline{\Lambda}_\mathrm{I}\,\underline{C}\,(\underline{A}-\underline{B}\,\underline{K})^{-1} & \underline{\Lambda}_\mathrm{I} \end{bmatrix} \tag{32}$$

and

$$\underline{\tilde{B}}_\mathrm{ext} = \underline{T}\,\underline{B}_\mathrm{ext} = \begin{bmatrix} \underline{B} \\ \underline{C}\,(\underline{A}-\underline{B}\,\underline{K})^{-1}\,\underline{B} + \underline{M}^{-1} \end{bmatrix}. \tag{33}$$

If in $\underline{\tilde{A}}_\mathrm{ext}$ and $\underline{\tilde{B}}_\mathrm{ext}$ the Eqs. (14) to (16) are taken into account, then after a few reforming steps the block diagonal matrix

$$\underline{\tilde{A}}_\mathrm{ext} = \begin{bmatrix} \underline{A} & \underline{0} \\ \underline{0} & \underline{\Lambda}_\mathrm{I} \end{bmatrix} \tag{34}$$

arises and for $\underline{\tilde{B}}_\mathrm{ext}$ the result

$$\underline{\tilde{B}}_\mathrm{ext} = \begin{bmatrix} \underline{B} \\ \underline{0} \end{bmatrix} \tag{35}$$

is obtained. This shows that the lower q transformed state variables cannot be controlled from $\underline{u}_\mathrm{sat}$, neither directly because of the zero matrix in $\underline{\tilde{B}}_\mathrm{ext}$, nor indirectly via the upper $n-q$ state variables because of the zero matrix in the lower block row of $\underline{\tilde{A}}_\mathrm{ext}$. This means that the control eigenvalues contained in $\underline{\Lambda}_\mathrm{I}$ cannot be controlled via $\underline{u}_\mathrm{sat}$. As these eigenvalues can be considered to be stable, the subsystem described by the lower block row in $\underline{\tilde{A}}_\mathrm{ext}$ and $\underline{\tilde{B}}_\mathrm{ext}$ is both stable and not controllable and therefore does not need to be considered further in the stability investigations.

If the control is discrete-time, comparable statements can be made. Thus, the application of the transformation

$$\underline{\tilde{x}}_\mathrm{ext} = \underline{S}\,\underline{x}_\mathrm{ext}, \tag{36}$$

$$\underline{S} = \begin{bmatrix} \underline{I} & \underline{0} \\ \underline{C}\,(\underline{\Phi}-\underline{H}\,\underline{K}-\underline{I})^{-1} & \underline{I} \end{bmatrix} \tag{37}$$

to Eq. (26), taking into account Eqs. (17) to (19), leads to the transformed state difference equation

$$\underline{\tilde{x}}_{\mathrm{ext},k+1} = \underline{\tilde{\Phi}}_\mathrm{ext}\,\underline{\tilde{x}}_{\mathrm{ext},k} + \underline{\tilde{H}}_\mathrm{ext}\,\underline{u}_{\mathrm{sat},k} \tag{38}$$

with

$$\underline{\tilde{\Phi}}_\mathrm{ext} = \underline{S}\,\underline{\Phi}_\mathrm{ext}\,\underline{S}^{-1} = \begin{bmatrix} \underline{\Phi} & \underline{0} \\ \underline{0} & \underline{Z}_\mathrm{R,I} \end{bmatrix}, \tag{39}$$

$$\underline{\tilde{H}}_\mathrm{ext} = \underline{S}\,\underline{H}_\mathrm{ext} = \begin{bmatrix} \underline{H} \\ \underline{0} \end{bmatrix}. \tag{40}$$

This also results in a subsystem that cannot be controlled via $\underline{u}_{\mathrm{sat},k}$, which has the eigenvalues contained in $\underline{Z}_\mathrm{I}$ that can be assumed as stable. Due to its stability and non-controllability, this subsystem can also be disregarded in the further stability investigations.

The previous explanations have shown that in both continuous-time and discrete-time system description and control, only the transformed first subsystem needs to be considered further for stability analysis after the state transformation has been carried out. However, this is precisely the controlled system with its manipulated variable vector $\underline{u}_{\text{sat}}$ as the input variable. It is therefore sufficient to find a stabilizing control law for the controlled system using a P-state controller with the feedback matrix $\underline{K}$. The stability of the resulting PI-state control by means of Eqs. (14) to (16) respectively (17) to (19) is then automatically ensured on the basis of the above explanations. In order to achieve a comparable statement for discrete-time systems, a complex modal transformation of the extended controlled system was carried out in [8]. However, the above considerations have considerably simplified the proof. Comparable considerations have not yet been made for continuous-time systems.

To find a P-state controller that also stabilizes with manipulated variable limits, the following Lyapunov function is used

$$V_{(k)} = \Delta\underline{x}_{(k)}^{\mathrm{T}} \, \underline{P} \, \Delta\underline{x}_{(k)}. \tag{41}$$

In this, $\Delta\underline{x}_{(k)}$ denotes the deviation of the state vector $\underline{x}_{(k)}$ from its stationary position, which is denoted below by $\underline{x}_{\infty}$. The following therefore applies

$$\Delta\underline{x}_{(k)} = \underline{x}_{(k)} - \underline{x}_{\infty}. \tag{42}$$

Furthermore, the (n, n)-dimensional matrix $\underline{P}$ contained in Eq. (41) must be a positive definite matrix yet to be determined. This means that $V_{(k)} > 0$ holds for $\Delta\underline{x}_{(k)} \neq \underline{0}$ and $V_{(k)} = 0$ only for $\Delta\underline{x}_{(k)} = \underline{0}$. If it is now possible to ensure that $V_{(k)}$ constantly decreases for $\Delta\underline{x}_{(k)} \neq \underline{0}$ and approaches the steady state, i.e. $\Delta\underline{x}_{\infty} = \underline{0}$, for $t \to \infty$ then the stability of the system under consideration is proven [2-4].

In order to be able to evaluate whether V decreases with a continuous-time system description, V is derived with respect to time and then the sign of $\dot{V}$ is examined. This results in

$$\dot{V} = \Delta\underline{\dot{x}}^{\mathrm{T}} \, \underline{P} \, \Delta\underline{x} + \Delta\underline{x}^{\mathrm{T}} \, \underline{P} \, \Delta\underline{\dot{x}}. \tag{43}$$

$\Delta\underline{\dot{x}}$ itself is obtained from the controlled system state differential equation (1) with $\underline{u}_{\text{sat}}$ instead of $\underline{u}$. Here, $\underline{u}_{\text{sat}}$ is also divided into a stationary component $\underline{u}_{\text{sat},\infty}$ and the deviation

$$\Delta\underline{u}_{\text{sat}} = \underline{u}_{\text{sat}} - \underline{u}_{\text{sat},\infty} \tag{44}$$

of it. With further consideration of $\Delta\underline{\dot{x}}_{\infty} = \underline{0}$, the following follows from the controlled system state differential equation

$$\begin{aligned}\Delta\underline{\dot{x}} &= \underline{\dot{x}} = \underline{A} \, \underline{x} + \underline{B} \, \underline{u}_{\text{sat}} \\ &= \underline{A} \left(\Delta\underline{x} + \underline{x}_{\infty}\right) + \underline{B} \left(\Delta\underline{u}_{\text{sat}} + \underline{u}_{\text{sat},\infty}\right) \\ &= \underline{A} \, \Delta\underline{x} + \underline{B} \, \Delta\underline{u}_{\text{sat}} + \underline{A} \, \underline{x}_{\infty} + \underline{B} \, \underline{u}_{\text{sat},\infty}.\end{aligned}$$

However, since $\underline{\dot{x}}$ must be the zero vector in the steady state and $\underline{\dot{x}}$ according to Eq. (1) is identical with $\underline{A} \, \underline{x}_{\infty} + \underline{B} \, \underline{u}_{\text{sat},\infty}$ in the steady state, $\underline{A} \, \underline{x}_{\infty} + \underline{B} \, \underline{u}_{\text{sat},\infty} = \underline{0}$ holds. Overall, this gives the state differential equation

$$\Delta\underline{\dot{x}} = \underline{A} \, \Delta\underline{x} + \underline{B} \, \Delta\underline{u}_{\text{sat}} \tag{45}$$

for the deviation of the controlled system state vector from its stationary position. This used in Eq. (43) then leads to

$$\dot{V} = \Delta\underline{x}^{\mathrm{T}} \left(\underline{A}^{\mathrm{T}} \, \underline{P} + \underline{P} \, \underline{A}\right) \Delta\underline{x} + 2 \, \Delta\underline{x}^{\mathrm{T}} \, \underline{P} \, \underline{B} \, \Delta\underline{u}_{\text{sat}}. \tag{46}$$

For V to decrease for $\Delta\underline{x}_{(k)} \neq \underline{0}$, $\dot{V}$ must be negative. To ensure this even with saturated manipulated variables, the expression $\Delta\underline{x}^{\mathrm{T}} \left(\underline{A}^{\mathrm{T}} \, \underline{P} + \underline{P} \, \underline{A}\right) \Delta\underline{x}$ is considered separately from the expression $2 \, \Delta\underline{x}^{\mathrm{T}} \, \underline{P} \, \underline{B} \, \Delta\underline{u}_{\text{sat}}$. Because the former expression depends quadratically on $\Delta\underline{x}$, it is generally to be expected that $\Delta\underline{x}$ reaches values which, due to the limitation of $\Delta\underline{u}_{\text{sat}}$, lead to such a large amount of $\Delta\underline{x}^{\mathrm{T}} \left(\underline{A}^{\mathrm{T}} \, \underline{P} + \underline{P} \, \underline{A}\right) \Delta\underline{x}$ that this term determines the sign of $\dot{V}$. The matrix term $\underline{A}^{\mathrm{T}} \, \underline{P} + \underline{P} \, \underline{A}$ is therefore assigned a positive definite or at most a positive semi-definite matrix $\underline{Q}^{\mathrm{T}} \, \underline{Q}$ respectively $\underline{Q}$ by

$$\underline{A}^{\mathrm{T}} \, \underline{P} + \underline{P} \, \underline{A} = -\underline{Q}^{\mathrm{T}} \, \underline{Q} \tag{47}$$

or

$$\underline{A}^{\mathrm{T}} \, \underline{P} + \underline{P} \, \underline{A} = -\underline{Q}. \tag{48}$$

It holds that if the matrix product $\underline{Q}^{\mathrm{T}} \, \underline{Q}$ respectively $\underline{Q}$ is positive definite, the matrix $\underline{P}$ is also positive definite if the controlled system is stable, i.e. if $\underline{A}$ only has eigenvalues with a negative real part [2, 4]. Due to the special choice of the right-hand side of Eq. (47), $\underline{Q}$ can be any n-column matrix without violating the positive semi-definiteness of $\underline{Q}^{\mathrm{T}} \, \underline{Q}$. Only if $\underline{Q}^{\mathrm{T}} \, \underline{Q}$ is to be positive definite, the number of rows of $\underline{Q}$ must be at least as large as the number of columns of $\underline{Q}$ and, in addition, $\operatorname{rank}\left\{\underline{Q}\right\} = n$ must apply [16]. Furthermore, it should be noted that Eqs. (47) and (48) are Lyapunov equations. How they can be solved in principle can be read, for example, in [4].

If there is a solution for $\underline{P}$ in Eq. (47) or (48), then it can be enforced that the right-hand side of Eq. (46) does not become positive, provided the controlled system is asymptotically stable or critically stable. To do this, $\Delta\underline{u}_{\text{sat}}$ is set in the way

$$\Delta\underline{u}_{\text{sat}} = -\underline{R}^{-1} \left(\underline{B}^{\mathrm{T}} \, \underline{P} + \underline{W}^{\mathrm{T}} \, \underline{Q}\right) \Delta\underline{x} \tag{49}$$

where $\underline{R}$ is a positive definite (p, p)-dimensional diagonal matrix with otherwise arbitrary diagonal elements and $\underline{W}$ is an arbitrary matrix of suitable dimension, provided that Eq.

(47) is used as the basis for $\underline{W}$. When using Eq. (48), $\underline{W} = \underline{0}$ must be selected. The chosen approach based on Eq. (47) is oriented towards the so-called Kalman-Yakubovich-Popov equations [2-4], which generally results in more degrees of freedom than the more classical approach based on Eq. (48) with $\underline{W} = \underline{0}$. However, it is often sufficient to work with $\underline{W} = \underline{0}$ and the somewhat more simply structured formulas, using either Eq. (47) or Eq. (48) as a basis.

Eq. (49) implies the controller matrix

$$\underline{K} \;=\; \underline{R}^{-1}\left(\underline{B}^{\mathrm{T}}\,\underline{P} + \underline{W}^{\mathrm{T}}\,\underline{Q}\right). \tag{50}$$

With this and Eqs. (46), (47) and (49) we obtain

$$\begin{aligned}\dot{V} \;&=\; -\Delta\underline{x}^{\mathrm{T}}\left(\underline{Q}^{\mathrm{T}}\,\underline{Q} + 2\,\underline{P}\,\underline{B}\,\underline{R}^{-1}\left(\underline{B}^{\mathrm{T}}\,\underline{P} + \underline{W}^{\mathrm{T}}\,\underline{Q}\right)\right)\Delta\underline{x} = \\ &-\Delta\underline{x}^{\mathrm{T}}\left(\underline{Q}^{\mathrm{T}}\,\underline{Q} + 2\left(\underline{P}\,\underline{B} + \underline{Q}^{\mathrm{T}}\,\underline{W} - \underline{Q}^{\mathrm{T}}\,\underline{W}\right)\underline{R}^{-1}\,\underline{R}\,\underline{K}\right)\Delta\underline{x} \\ &= -\Delta\underline{x}^{\mathrm{T}}\left(\underline{Q}^{\mathrm{T}}\,\underline{Q} - 2\,\underline{Q}^{\mathrm{T}}\,\underline{W}\,\underline{K} + 2\,\underline{K}^{\mathrm{T}}\,\underline{R}\,\underline{K}\right)\Delta\underline{x} \\ &= -\Delta\underline{x}^{\mathrm{T}}\left(\underline{Q}^{\mathrm{T}}\,\underline{Q} - \underline{Q}^{\mathrm{T}}\,\underline{W}\,\underline{K} - \underline{K}^{\mathrm{T}}\,\underline{W}^{\mathrm{T}}\,\underline{Q} + \right. \\ &\left.\underline{K}^{\mathrm{T}}\,\underline{W}^{\mathrm{T}}\,\underline{W}\,\underline{K} - \underline{K}^{\mathrm{T}}\,\underline{W}^{\mathrm{T}}\,\underline{W}\,\underline{K} + 2\,\underline{K}^{\mathrm{T}}\,\underline{R}\,\underline{K}\right)\Delta\underline{x} \\ &= -\Delta\underline{x}^{\mathrm{T}}\left(\left(\underline{Q} - \underline{W}\,\underline{K}\right)^{\mathrm{T}}\left(\underline{Q} - \underline{W}\,\underline{K}\right) + \underline{K}^{\mathrm{T}}\left(2\,\underline{R} - \underline{W}^{\mathrm{T}}\,\underline{W}\right)\underline{K}\right)\Delta\underline{x}. \end{aligned} \tag{51}$$

If $\underline{R}$ is now chosen so that $2\,\underline{R} - \underline{W}^{\mathrm{T}}\,\underline{W}$ is positive definite – for which the diagonal elements of $\underline{R}$ only have to be chosen sufficiently large – then $\underline{K}^{\mathrm{T}}\left(2\,\underline{R} - \underline{W}^{\mathrm{T}}\,\underline{W}\right)\underline{K}$ is a positive semi-definite matrix term [16] due to $p < n$. Furthermore, $\left(\underline{Q} - \underline{W}\,\underline{K}\right)^{\mathrm{T}}\left(\underline{Q} - \underline{W}\,\underline{K}\right)$ is also at least positive semi-definite. This ensures that $\dot{V}$ does not become positive. With $\underline{W} = \underline{0}$, $\underline{Q}$ quadratic and positive definite and $\underline{R}$ positive definite in any case, $\dot{V}$ is then always negative for $\Delta\underline{x} \neq \underline{0}$ and zero for $\Delta\underline{x} = \underline{0}$ itself. $\dot{V}$ is negative definite in this case, while V is positive definite. Since these properties apply to the entire state space, the stability condition of Lyapunov's direct method is fulfilled and thus the system under consideration is stable [2-4]. The same applies if the considerations are based on Eq. (48), which with $\underline{W} = \underline{0}$ results in

$$\dot{V} \;=\; -\Delta\underline{x}^{\mathrm{T}}\left(\underline{Q} \;+\; 2\,\underline{K}^{\mathrm{T}}\,\underline{R}\,\underline{K}\right)\Delta\underline{x}. \tag{52}$$

Both, Eqs. (51) and (52), also apply in particular when manipulated variable constraints take effect. In this case, the limitation causes the amount of a certain element of $\Delta\underline{u}_{\mathrm{sat}}$ to be smaller than specified by the controller. However, this can be modeled according to Eq. (49) by a corresponding increase in the relevant diagonal element of $\underline{R}$. However, since the diagonal elements of $\underline{R}$ can be chosen to be arbitrarily large as long as they are only positive and fulfill the condition $2\,\underline{R} - \underline{W}^{\mathrm{T}}\,\underline{W} \succcurlyeq 0$, this has no negative influence on the definiteness of $\dot{V}$ and thus on the stability conclusion.

It is particularly worth mentioning in this context that for the proof of stability described above, both matrices $\underline{P}$ and $\underline{Q}$ must be positive definite if V is positive definite and $\dot{V}$ is negative definite. This is possible if the controlled system is stable. However, if the controller integrators had been included in the system model without splitting off the uncontrollable, stable subsystem as described above, then the simultaneous specification of $\underline{P}$ and $\underline{Q}$ as positive definite matrices would not have been possible due to the then critically stable system.

If the description of the controlled system is available in discrete-time form, the difference

$$\Delta V_k \;=\; V_{k+1} \;-\; V_k \tag{53}$$

is formed. If it is negative definite, V_k decreases at each sampling step as long as $\Delta\underline{x}_k$ is not a zero vector. The aim is therefore to ensure that ΔV_k is negative definite in the entire state space. For this purpose, V_{k+1} and V_k are replaced in Eq. (53) according to the right-hand side of Eq. (41), from which

$$\Delta V_k \;=\; \Delta\underline{x}_{k+1}^{\mathrm{T}}\,\underline{P}\,\Delta\underline{x}_{k+1} \;-\; \Delta\underline{x}_k^{\mathrm{T}}\,\underline{P}\,\Delta\underline{x}_k \tag{54}$$

follows. Furthermore, for $\Delta\underline{x}_{k+1}$ according to Eqs. (4) and (42), but with $\underline{u}_{\mathrm{sat},k}$ instead of $\underline{u}_k$, we obtain

$$\begin{aligned}\Delta\underline{x}_{k+1} \;&=\; \underline{x}_{k+1} \;-\; \underline{x}_\infty \;=\; \underline{\Phi}\,\underline{x}_k \;+\; \underline{H}\,\underline{u}_{\mathrm{sat},k} \;-\; \underline{x}_\infty \;= \\ &\underline{\Phi}\left(\Delta\underline{x}_k + \underline{x}_\infty\right) \;+\; \underline{H}\left(\Delta\underline{u}_{\mathrm{sat},k} + \underline{u}_\infty\right) \;-\; \underline{x}_\infty \;=\; \underline{\Phi}\,\Delta\underline{x}_k \;+ \\ &\underline{H}\,\Delta\underline{u}_{\mathrm{sat},k} \;+\; \underline{\Phi}\,\underline{x}_\infty \;+\; \underline{H}\,\underline{u}_\infty \;-\; \underline{x}_\infty\end{aligned} \tag{55}$$

In addition, Eq. (4) leads to the following relationship for the steady state due to $\lim_{k\to\infty}\underline{x}_k = \lim_{k\to\infty}\underline{x}_{k+1} = \underline{x}_\infty$ and $\lim_{k\to\infty}\underline{u}_{\mathrm{sat},k} = \underline{u}_\infty$:

$$\underline{x}_\infty \;=\; \underline{\Phi}\,\underline{x}_\infty \;+\; \underline{H}\,\underline{u}_\infty$$

From Eq. (55) therefore arises

$$\Delta\underline{x}_{k+1} \;=\; \underline{\Phi}\,\Delta\underline{x}_k \;+\; \underline{H}\,\Delta\underline{u}_{\mathrm{sat},k}. \tag{56}$$

If we now insert Eq. (56) into Eq. (54), we finally receive

$$\begin{aligned}\Delta V_k \;&=\; \Delta\underline{x}_k^{\mathrm{T}}\left(\underline{\Phi}^{\mathrm{T}}\,\underline{P}\,\underline{\Phi} - \underline{P}\right)\Delta\underline{x}_k + 2\,\Delta\underline{x}_k^{\mathrm{T}}\,\underline{\Phi}^{\mathrm{T}}\,\underline{P}\,\underline{H}\,\Delta\underline{u}_{\mathrm{sat},k} \;+ \\ &\Delta\underline{u}_{\mathrm{sat},k}^{\mathrm{T}}\,\underline{\mathrm{H}}^{\mathrm{T}}\,\underline{P}\,\underline{H}\,\Delta\underline{u}_{\mathrm{sat},k}\end{aligned} \tag{57}$$

If we proceed according to Lyapunov's direct method, we must first ensure that the first summand $\Delta\underline{x}_k^{\mathrm{T}}\left(\underline{\Phi}^{\mathrm{T}}\,\underline{P}\,\underline{\Phi} - \underline{P}\right)\Delta\underline{x}_k$ is negative definite or at least negative semi-definite. This is ensured by

$$\underline{\Phi}^{\mathrm{T}}\,\underline{P}\,\underline{\Phi} \;-\; \underline{P} \;=\; -\underline{Q}^{\mathrm{T}}\,\underline{Q} \tag{58}$$

respectively

$$\underline{\Phi}^{\mathrm{T}}\,\underline{P}\,\underline{\Phi} - \underline{P} = -\underline{Q} \tag{59}$$

and by setting $\underline{Q}$ as a matrix with at least as many rows as columns at maximum rank when applying Eq. (58) and aiming for positive definiteness of $\Delta\underline{x}_k^{\mathrm{T}}\,\underline{Q}^{\mathrm{T}}\,\underline{Q}\,\Delta\underline{x}_k$ as in the continuous-time system description. If $\underline{\Phi}$ only has stable eigenvalues, then the positive definiteness of $\underline{P}$ follows from the positive definiteness of $\underline{Q}^{\mathrm{T}}\,\underline{Q}$ respectively $\underline{Q}$ [2, 4] and thus also the positive definiteness of V according to Eq. (41). If, on the other hand, $\underline{\Phi}$ has stable and/or critically stable eigenvalues, $\underline{P}$ can be chosen to be positive definite, but $\underline{Q}^{\mathrm{T}}\,\underline{Q}$ respectively $\underline{Q}$ can then at best be positive semi-definite (see example from section 6).

In order for ΔV_k to be negative (semi-)definite, the sum of all three summands must be negative (semi-)definite in addition to the first summand of Eq. (57), which can now be represented in the form $-\Delta\underline{x}_k^{\mathrm{T}}\,\underline{Q}^{\mathrm{T}}\,\underline{Q}\,\Delta\underline{x}_k$ respectively $-\Delta\underline{x}_k^{\mathrm{T}}\,\underline{Q}\,\Delta\underline{x}_k$. To achieve this, starting from the second summand, the control law

$$\Delta\underline{u}_{\mathrm{beg},k} = -\underline{R}^{-1}\left(\underline{H}^{\mathrm{T}}\,\underline{P}\,\underline{\Phi} + \underline{W}^{\mathrm{T}}\,\underline{Q}\right)\Delta\underline{x}_k \tag{60}$$

or the associated controller matrix

$$\underline{K} = \underline{R}^{-1}\left(\underline{H}^{\mathrm{T}}\,\underline{P}\,\underline{\Phi} + \underline{W}^{\mathrm{T}}\,\underline{Q}\right) \tag{61}$$

is applied – if applicable with $\underline{W} = \underline{0}$ in the case of Eq. (59). Eq. (60), after insertion into Eq. (57) and taking into account Eqs. (58) and (61) yields the result

$$\begin{aligned}
\Delta V_k &= -\Delta\underline{x}_k^{\mathrm{T}}\left(\underline{Q}^{\mathrm{T}}\,\underline{Q} + 2\,\underline{\Phi}^{\mathrm{T}}\,\underline{P}\,\underline{H}\,\underline{K} - \underline{K}^{\mathrm{T}}\underline{\mathrm{H}}^{\mathrm{T}}\,\underline{P}\,\underline{H}\,\underline{K}\right)\Delta\underline{x}_k \\
&= -\Delta\underline{x}_k^{\mathrm{T}}\left(\underline{Q}^{\mathrm{T}}\,\underline{Q} + +2\left(\underline{\Phi}^{\mathrm{T}}\,\underline{P}\,\underline{H} + \underline{Q}^{\mathrm{T}}\,\underline{W} - \underline{Q}^{\mathrm{T}}\,\underline{W}\right)\underline{R}^{-1}\underline{R}\,\underline{K} \right.\\
&\qquad \left. - \underline{K}^{\mathrm{T}}\underline{\mathrm{H}}^{\mathrm{T}}\,\underline{P}\,\underline{H}\,\underline{K}\right)\Delta\underline{x}_k \\
&= -\Delta\underline{x}_k^{\mathrm{T}}\left(\underline{Q}^{\mathrm{T}}\,\underline{Q} - 2\,\underline{Q}^{\mathrm{T}}\,\underline{W}\,\underline{K} + 2\,\underline{K}^{\mathrm{T}}\,\underline{R}\,\underline{K} \right.\\
&\qquad \left. - \underline{K}^{\mathrm{T}}\underline{\mathrm{H}}^{\mathrm{T}}\,\underline{P}\,\underline{H}\,\underline{K}\right)\Delta\underline{x}_k \\
&= -\Delta\underline{x}_k^{\mathrm{T}}\left(\underline{Q}^{\mathrm{T}}\,\underline{Q} - \underline{Q}^{\mathrm{T}}\,\underline{W}\,\underline{K} - \underline{K}^{\mathrm{T}}\,\underline{W}^{\mathrm{T}}\,\underline{Q} + \right.\\
&\qquad +\underline{K}^{\mathrm{T}}\underline{W}^{\mathrm{T}}\underline{W}\,\underline{K} - \underline{K}^{\mathrm{T}}\underline{W}^{\mathrm{T}}\underline{W}\,\underline{K} + 2\,\underline{K}^{\mathrm{T}}\underline{R}\,\underline{K} \\
&\qquad \left. - \underline{K}^{\mathrm{T}}\underline{\mathrm{H}}^{\mathrm{T}}\underline{P}\,\underline{H}\,\underline{K}\right)\Delta\underline{x}_k \\
&= -\Delta\underline{x}_k^{\mathrm{T}}\left(\left(\underline{Q} - \underline{W}\,\underline{K}\right)^{\mathrm{T}}\left(\underline{Q} - \underline{W}\,\underline{K}\right) + \right.\\
&\qquad \left. + \underline{K}^{\mathrm{T}}\left(2\,\underline{R} - \underline{\mathrm{H}}^{\mathrm{T}}\underline{P}\,\underline{H} - \underline{W}^{\mathrm{T}}\underline{W}\right)\underline{K}\right)\Delta\underline{x}_k.
\end{aligned} \tag{62}$$

Using Eq. (59) as a basis, $\underline{W}$ is then simply specified as the zero matrix and $\underline{Q}^{\mathrm{T}}\,\underline{Q}$ is replaced by $\underline{Q}$. If $\underline{R}$ is now chosen so that the bracket expression $2\,\underline{R} - \underline{H}^{\mathrm{T}}\,\underline{P}\,\underline{H} - \underline{W}^{\mathrm{T}}\,\underline{W}$ is positive definite, then the entire matrix term in the last row of Eq. (62) is positive semi-definite due to $n > p$. Again, this can always be achieved with sufficiently large amounts r_{ii} $(i = 1, \dots, p)$ of $\underline{R}$. Furthermore, because the matrix term in the second last row of Eq. (62) is at least positive semi-definite, ΔV_k cannot become positive. With $\underline{W} = \underline{0}$, quadratic, positive definite specification of $\underline{Q}$ and $2\,\underline{R} - \underline{H}^{\mathrm{T}}\,\underline{P}\,\underline{H}$, a stable discrete-time controlled system according to Eq. (4) can therefore always be stabilized using the control law (60) via a PI-state controller. This also applies in particular when manipulated variable constraints occur, because when limiting $u_{\mathrm{sat},i,k}$ $(i = 1, \dots, p)$, only the element r_{ii} of $\underline{R}$ needs to be increased in thought until $u_{\mathrm{sat},i,k}$ corresponds to the relevant limiting value using Eq. (60).

5. Discrete-time Systems with Dead Time for Manipulated Variable Determination

With discrete-time systems, it is often the case that the manipulated variables do not act on the system – not even approximately – from the instant at which the state variables from which the respective manipulated variables were determined are sampled. A dead time between the calculation of the manipulated variables and their becoming effective must therefore be taken into account when creating the model. In order to have defined and at the same time easily manageable correlations, a dead time is usually introduced that incorporates exactly one sampling interval [9, 13]. The calculated, if necessary limited manipulated variable vector $\underline{u}_{\mathrm{sat},k}$, which was determined on the basis of the state vector $\underline{x}_k$, is then set to a newly introduced vector $\underline{v}_{\mathrm{T}}$, using the difference equation

$$\underline{v}_{\mathrm{T},k+1} = \underline{u}_{\mathrm{sat},k}. \tag{63}$$

If $\underline{v}_{\mathrm{T},k}$ is now used instead of $\underline{u}_{\mathrm{sat},k}$ as the manipulated variable vector acting on the controlled system, the dead time is taken into account in the model. If $\underline{x}$ and $\underline{v}_{\mathrm{T}}$ are then combined to form the overall state vector

$$\underline{x}_{\mathrm{ext}} = \begin{bmatrix} \underline{x} \\ \underline{v}_{\mathrm{T}} \end{bmatrix} \tag{64}$$

this results in the vectorial controlled system state difference equation

$$\underline{x}_{\mathrm{ext},k+1} = \underline{\Phi}_{\mathrm{ext}}\,\underline{x}_{\mathrm{ext},k} + \underline{H}_{\mathrm{ext}}\,\underline{u}_{\mathrm{sat},k}, \tag{65}$$

using the extended transition matrix

$$\underline{\Phi}_{\mathrm{ext}} = \begin{bmatrix} \underline{\Phi} & \underline{H} \\ \underline{0} & \underline{0} \end{bmatrix} \tag{66}$$

and the extended discrete-time control input matrix

$$\underline{H}_{\mathrm{ext}} = \begin{bmatrix} \underline{0} \\ \underline{I} \end{bmatrix}. \tag{67}$$

For the output equation of the system with dead time, it is correspondingly obtained

$$\underline{y}_k = \underline{C}_{\text{ext}} \, \underline{x}_{\text{ext},k} \quad (68)$$

with

$$\underline{C}_{\text{ext}} = [\, \underline{C} \quad \underline{0} \,]. \quad (69)$$

If it is now successful to establish a generally valid correlation between the stability behavior of the system extended by dead times and the dead-time-free system with and without manipulated variable constraints, then the effort for stability analysis and controller synthesis can be significantly reduced for systems with manipulated variable saturation and dead times in the manipulated variable paths.

For the difference equation (65) of the controlled system with dead time, the same stability considerations can now be made as those based on the system state difference equation (4). Analogous to Eq. (57), the relationship

$$\begin{aligned} \Delta V_k = {} & \Delta \underline{x}_{\text{ext},k}^{\mathrm{T}} \left(\underline{\Phi}_{\text{ext}}^{\mathrm{T}} \underline{P}_{\text{ext}} \underline{\Phi}_{\text{ext}} - \underline{P}_{\text{ext}} \right) \Delta \underline{x}_{\text{ext},k} + \\ & + 2 \, \Delta \underline{x}_{\text{ext},k}^{\mathrm{T}} \underline{\Phi}_{\text{ext}}^{\mathrm{T}} \underline{P}_{\text{ext}} \underline{H}_{\text{ext}} \Delta \underline{u}_{\text{sat},k} + \\ & + \Delta \underline{u}_{\text{sat},k}^{\mathrm{T}} \underline{H}_{\text{ext}}^{\mathrm{T}} \underline{P}_{\text{ext}} \underline{H}_{\text{ext}} \Delta \underline{u}_{\text{sat},k} \end{aligned} \quad (70)$$

then occurs, where first for $\Delta \underline{x}_{\text{ext},k}^{\mathrm{T}} \left(\underline{\Phi}_{\text{ext}}^{\mathrm{T}} \underline{P}_{\text{ext}} \underline{\Phi}_{\text{ext}} - \underline{P}_{\text{ext}} \right) \Delta \underline{x}_{\text{ext},k}$ and then for the remaining terms negative definiteness or negative semi-definiteness must be ensured. With the approaches

$$\underline{\Phi}_{\text{ext}}^{\mathrm{T}} \underline{P}_{\text{ext}} \underline{\Phi}_{\text{ext}} - \underline{P}_{\text{ext}} = -\underline{Q}_{\text{ext}}^{\mathrm{T}} \underline{Q}_{\text{ext}}, \quad (71)$$

$$\Delta \underline{u}_{\text{sat},k} = -\underline{R}^{-1} \left(\underline{H}_{\text{ext}}^{\mathrm{T}} \underline{P}_{\text{ext}} \underline{\Phi}_{\text{ext}} + \underline{W}_{\text{ext}}^{\mathrm{T}} \underline{Q}_{\text{ext}} \right) \Delta \underline{x}_k \quad (72)$$

respectively $\underline{Q}_{\text{ext}}$ instead of $\underline{Q}_{\text{ext}}^{\mathrm{T}} \underline{Q}_{\text{ext}}$ and at the same time $\underline{W}_{\text{ext}} = \underline{0}$, this is achieved in the same way as previously described in section 4 for discrete-time systems without dead times in the manipulated variable paths. If you now specify $\underline{P}_{\text{ext}}$ as a symmetrical matrix in block matrix notation

$$\underline{P}_{\text{ext}} = \begin{bmatrix} \underline{P}_{11} & \underline{P}_{21}^{\mathrm{T}} \\ \underline{P}_{21} & \underline{P}_{22} \end{bmatrix}, \quad (73)$$

$\underline{Q}_{\text{ext}}$ as block diagonal matrix

$$\underline{Q}_{\text{ext}} = \begin{bmatrix} \underline{Q}_{11} & \underline{0} \\ \underline{0} & \underline{Q}_{22} \end{bmatrix} \quad (74)$$

and $\underline{W}_{\text{ext}}$ in block matrix notation

$$\underline{W}_{\text{ext}} = \begin{bmatrix} \underline{W}_{11} \\ \underline{W}_{21} \end{bmatrix}, \quad (75)$$

then it follows from Eq. (71) by block-by-block writing, taking into account Eq. (66)

$$\underline{\Phi}^{\mathrm{T}} \underline{P}_{11} \underline{\Phi} - \underline{P}_{11} = -\underline{Q}_{11}^{\mathrm{T}} \underline{Q}_{11}, \quad (76)$$

$$\underline{H}^{\mathrm{T}} \underline{P}_{11} \underline{\Phi} - \underline{P}_{21} = \underline{0}, \quad (77)$$

$$\underline{H}^{\mathrm{T}} \underline{P}_{11} \underline{H} - \underline{P}_{22} = -\underline{Q}_{22}^{\mathrm{T}} \underline{Q}_{22}. \quad (78)$$

While Eq. (76) represents the discrete-time Lyapunov equation of the dead time-free controlled system, Eqs. (77) and (78) directly yield the solutions

$$\underline{P}_{21} = \underline{H}^{\mathrm{T}} \underline{P}_{11} \underline{\Phi}, \quad (79)$$

$$\underline{P}_{22} = \underline{H}^{\mathrm{T}} \underline{P}_{11} \underline{H} + \underline{Q}_{22}^{\mathrm{T}} \underline{Q}_{22} \quad (80)$$

for the matrix blocks $\underline{P}_{21}$ and $\underline{P}_{22}$ of $\underline{P}_{\text{ext}}$. The specification of $\underline{Q}_{\text{ext}}$ therefore determines $\underline{P}_{\text{ext}}$. Because $\underline{P}_{\text{ext}}$ is positive definite if $\underline{Q}_{\text{ext}}^{\mathrm{T}} \underline{Q}_{\text{ext}}$ respectively $\underline{Q}_{\text{ext}}$ with stable transition matrix $\underline{\Phi}_{\text{ext}}$ is positive definite [2, 4] and $\underline{\Phi}_{\text{ext}}$ as upper block triangular matrix with stable matrix $\underline{\Phi}$ fulfills this condition, the positive definiteness of $\underline{P}_{\text{ext}}$ no longer needs to be proven separately. If we also evaluate Eq. (61) for $\underline{P}_{\text{ext}}$ according to Eq. (73) instead of $\underline{P}$, $\underline{\Phi}_{\text{ext}}$ according to Eq. (66) instead of $\underline{\Phi}$, $\underline{H}_{\text{ext}}$ according to Eq. (67) instead of $\underline{H}$, $\underline{Q}_{\text{ext}}$ according to Eq. (74) instead of $\underline{Q}$ and $\underline{W}_{\text{ext}}$ according to Eq. (75) instead of $\underline{W}$, then the following results with $\underline{K}_{\text{ext}}$ instead of $\underline{K}$, taking into account Eq. (79):

$$\begin{aligned} \underline{K}_{\text{ext}} = {} & \underline{R}^{-1} \left([\, \underline{0} \; \underline{I} \,] \begin{bmatrix} \underline{P}_{11} & \underline{P}_{21} \\ \underline{P}_{21} & \underline{P}_{22} \end{bmatrix} \begin{bmatrix} \underline{\Phi} & \underline{H} \\ \underline{0} & \underline{0} \end{bmatrix} \right. \\ & \left. + [\, \underline{W}_{11}^{\mathrm{T}} \; \underline{W}_{21}^{\mathrm{T}} \,] \begin{bmatrix} \underline{Q}_{11} & \underline{0} \\ \underline{0} & \underline{Q}_{22} \end{bmatrix} \right) \\ = {} & \underline{R}^{-1} \left(\underline{P}_{21} [\, \underline{\Phi} \; \underline{H} \,] + [\underline{W}_{11}^{\mathrm{T}} \underline{Q}_{11} \quad \underline{W}_{21}^{\mathrm{T}} \underline{Q}_{22}] \right) \\ = {} & \underline{R}^{-1} \left(\underline{H}^{\mathrm{T}} \underline{P}_{11} \underline{\Phi} [\, \underline{\Phi} \; \underline{H} \,] + [\underline{W}_{11}^{\mathrm{T}} \underline{Q}_{11} \quad \underline{W}_{21}^{\mathrm{T}} \underline{Q}_{22}] \right). \end{aligned} \quad (81)$$

A comparison with Eq. (61) shows that for $\underline{W}_{\text{ext}} = \underline{0}$ the controller matrix $\underline{K}_{\text{ext}}$ of the system with dead time is associated with the controller matrix $\underline{K}$ of the system without dead time via the relationship

$$\underline{K}_{\text{ext}} = \underline{K} \, [\, \underline{\Phi} \; \underline{H} \,]. \quad (82)$$

In [13], Eq. (82) was already derived – without taking manipulated variable limits into account – for the problem that a controller matrix (here $\underline{K}_{\text{ext}}$) is requested for the system with dead time, which produces the same control behavior as the associated dead time-free system, only delayed by one sampling interval. Interestingly, this controller setting also fulfills the requirement for stability in the case of manipulated variable constraints, provided that the dead time-free system (with the controller matrix $\underline{K}$) fulfills this requirement. Furthermore, when applying Eq. (82), the pre-filter matrix ensuring stationary accuracy in the command behavior, which is denoted below as $\underline{M}_{\text{ext}}$ to distinguish it from $\underline{M}$ from Eq. (19), is identical to $\underline{M}$, i.e. it holds

$$\underline{M}_{\text{ext}} = \underline{M}. \tag{83}$$

To recognize this, the determination equation for $\underline{M}_{\text{ext}}$ is first written according to Eq. (19) with $\underline{C}_{\text{ext}}$ instead of $\underline{C}$, $\underline{\Phi}_{\text{ext}}$ instead of $\underline{\Phi}$, $\underline{K}_{\text{ext}}$ instead of $\underline{K}$ and $\underline{H}_{\text{ext}}$ instead of $\underline{H}$. It reads

$$\underline{M}_{\text{ext}} = -\left(\underline{C}_{\text{ext}}\left(\underline{\Phi}_{\text{ext}} - \underline{H}_{\text{ext}}\,\underline{K}_{\text{ext}} - \underline{I}\right)^{-1}\underline{H}_{\text{ext}}\right)^{-1}. \tag{84}$$

In this context, taking into account Eqs. (66) and (67), it holds

$$\underline{\Phi}_{\text{ext}} - \underline{H}_{\text{ext}}\,\underline{K}_{\text{ext}} - \underline{I} = \begin{bmatrix} \underline{\Phi} - \underline{I} & \underline{H} \\ -\underline{K}\,\underline{\Phi} & -\left(\underline{I} + \underline{K}\,\underline{H}\right) \end{bmatrix} \tag{85}$$

as well as

$$\left(\underline{\Phi}_{\text{ext}} - \underline{H}_{\text{ext}}\,\underline{K}_{\text{ext}} - \underline{I}\right)^{-1} = \begin{bmatrix} \left(\underline{\Phi} - \underline{H}\,\underline{K} - \underline{I}\right)^{-1}\underline{\Phi} - \underline{I} & \left(\underline{\Phi} - \underline{H}\,\underline{K} - \underline{I}\right)^{-1}\underline{H} \\ -\underline{K}\left(\underline{\Phi} - \underline{H}\,\underline{K} - \underline{I}\right)^{-1}\underline{\Phi} & -\underline{K}\left(\underline{\Phi} - \underline{H}\,\underline{K} - \underline{I}\right)^{-1}\underline{H} - \underline{I} \end{bmatrix}. \tag{86}$$

The easiest way to verify the above relationship is to multiply $\left(\underline{\Phi}_{\text{ext}} - \underline{H}_{\text{ext}}\,\underline{K}_{\text{ext}} - \underline{I}\right)^{-1}$ by $\underline{\Phi}_{\text{ext}} - \underline{H}_{\text{ext}}\,\underline{K}_{\text{ext}} - \underline{I}$ to obtain the unit matrix. If $\left(\underline{\Phi}_{\text{ext}} - \underline{H}_{\text{ext}}\,\underline{K}_{\text{ext}} - \underline{I}\right)^{-1}$ is then multiplied from the left by $\underline{C}_{\text{ext}}$ according to Eq. (69) and from the right by $\underline{H}_{\text{ext}}$ according to Eq. (67), the following results

$$\underline{C}_{\text{ext}}\left(\underline{\Phi}_{\text{ext}} - \underline{H}_{\text{ext}}\,\underline{K}_{\text{ext}} - \underline{I}\right)^{-1}\underline{H}_{\text{ext}} = \underline{C}\left(\underline{\Phi} - \underline{H}\,\underline{K} - \underline{I}\right)^{-1}\underline{H},$$

which just equals $-\underline{M}^{-1}$. The renewed inversion and negation according to Eq. (84) finally leads to the statement of Eq. (83).

Finally, it should be noted that the stability statement made in this section also applies if the controlled system includes a dead time in the maniplated variable paths and is to be controlled with a PI-state controller. In this case, as described above, a P-state controller is first designed for the controlled system with dead time using Eqs. (82) and (83) (without controller integral-action component) and then $\underline{K}_{\text{ext}}$ from Eq. (82) is inserted as the controller matrix $\underline{K}$ in Eq. (17). The extended system matrices $\underline{\Phi}_{\text{ext}}$, $\underline{H}_{\text{ext}}$ and $\underline{C}_{\text{ext}}$ from Eqs. (66), (67) respectively (69) can be used. The pre-filter matrix $\underline{M}$ can remain unchanged due to Eq. (83) and the statements in section 3.

6. Example

The following example from the field of electrical drives is intended to illustrate the methodology described above. It concerns the speed control system of a three-phase drive to be controlled. The associated model consists of the series connection of a dead time element and a P-T$_1$ element to simulate the subordinate closed current control loop as well as an integrator to model the mechanics (single-mass oscillator). Figure 2 shows the continuous-time structure of the model. To concentrate on the essentials, the setpoint $T_{\text{e,ref}}$ respectively the actual value T_{e} of the electric torque are used directly as input and output variables of the subordinate current control loop instead of the corresponding torque-forming current (setpoint) components. The time constant of the closed current control loop is τ_{CCL}. The dead time element comes from the modeling of the computing time of the signal processor used for control. The difference between the electric torque T_{e} and the load torque T_{L} results in the acceleration torque, which leads to the speed (angular velocity) ω when integrated via the moment of inertia J.

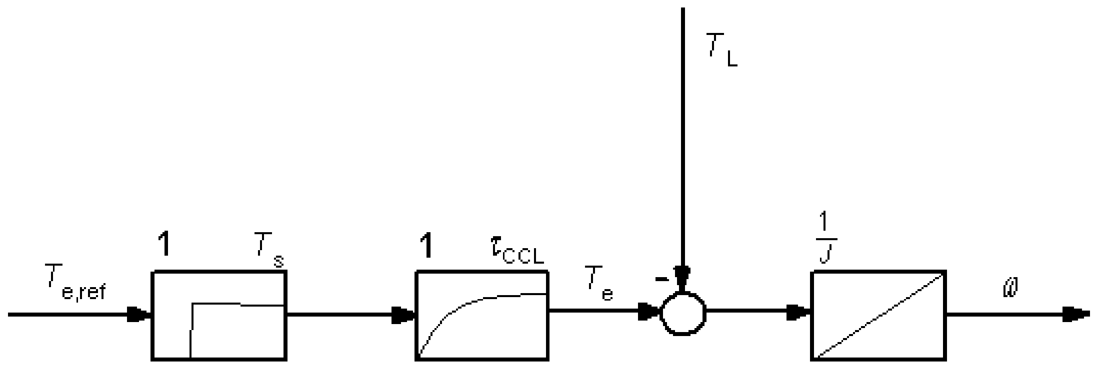

Figure 2. *Continuous-time block diagram of the exemplary controlled system.*

A discrete-time PI-state controller is to be used as the speed controller. For this purpose, the system model must be discretized beforehand. Using the sampling time T_s and neglecting the load torque, the discrete-time state equations of the dead-time-free system are approximately as follows [8, 17]

$$x_{1,k+1} = z_{\mathrm{CCL}}\, x_{1,k} + (1 - z_{\mathrm{CCL}})\, u_k,$$

$$x_{2,k+1} = \frac{T_s}{J}\,\frac{1 + z_{\mathrm{CCL}}}{2}\, x_{1,k} + x_{2,k} + \frac{T_s}{J}\,\frac{1 - z_{\mathrm{CCL}}}{2}\, u_k,$$

$$y_{\mathrm{R},k} = x_{2,k},$$

if the state variable x_1 is understood to be the actual torque value and the state variable x_2, which is also the controlled variable y, is understood to be the actual speed value. In contrast, the output variable of the dead time element from Figure 2 serves as the manipulated variable u in the dead time-free system. For z_{CCL}, it holds $z_{\mathrm{CCL}} = \mathrm{e}^{-\frac{T_s}{\tau_{\mathrm{CCL}}}}$. For the relevant system matrices, this results in the values

$$\underline{\Phi} = \begin{bmatrix} z_{\mathrm{CCL}} & 0 \\ \frac{T_s}{J}\,\frac{1 + z_{\mathrm{CCL}}}{2} & 1 \end{bmatrix},$$

$$\underline{H} = \begin{bmatrix} 1 - z_{\mathrm{CCL}} \\ \frac{T_s}{J}\,\frac{1 - z_{\mathrm{CCL}}}{2} \end{bmatrix},$$

$$\underline{C} = [\,0 \quad 1\,].$$

As can be seen, the transition matrix $\underline{\Phi}$ has an eigenvalue at $z = 1$, which is why the controlled system is not asymptotically stable. It can therefore be assumed that $\underline{Q}^{\mathrm{T}}\,\underline{Q}$ respectively $\underline{Q}$ will not be a positive definite matrix term. The calculation of $\underline{\Phi}^{\mathrm{T}}\,\underline{P}\,\underline{\Phi} - \underline{P}$ according to Eq. (58) or (59) shows this immediately. Because with the symmetrical matrix

$$\underline{P} = \begin{bmatrix} p_{11} & p_{21} \\ p_{21} & p_{22} \end{bmatrix}$$

and the abbreviation $c = \frac{T_s}{J}\,\frac{1 + z_{\mathrm{CCL}}}{2}$ you get

$$\underline{\Phi}^{\mathrm{T}}\,\underline{P}\,\underline{\Phi} - \underline{P} = \begin{bmatrix} (z_{\mathrm{CCL}}^2 - 1)\, p_{11} + 2\, c\, z_{\mathrm{CCL}}\, p_{21} + c^2 p_{22} & (z_{\mathrm{CCL}} - 1)\, p_{21} + c\, p_{22} \\ (z_{\mathrm{CCL}} - 1)\, p_{21} + c\, p_{22} & 0 \end{bmatrix}.$$

Because of the zero element on the main diagonal, $\underline{Q}^{\mathrm{T}}\,\underline{Q}$ respectively $\underline{Q}$ can at best be positive semi-definite, and only if the secondary diagonal elements are zero. From this follows directly

$$p_{22} = \frac{1 - z_{\mathrm{CCL}}}{c}\, p_{21}.$$

Since it is sufficient for the example to carry out the controller calculation with $\underline{W} = \underline{0}$ and thus use Eq. (48) as a basis, it applies that

$$\underline{Q} = \begin{bmatrix} (1 - z_{\mathrm{CCL}}^2)\, p_{11} - c\,(1 + z_{\mathrm{CCL}})\, p_{21} & 0 \\ 0 & 0 \end{bmatrix}.$$

Finally, the condition

$$p_{21} \le \frac{1 - z_{\mathrm{CCL}}}{c}\, p_{11}$$

must be fulfilled so that $\underline{Q}$ can be positive semi-definite. For the matrix $\underline{P}$, it follows under the above conditions

$$\underline{P} = \begin{bmatrix} p_{11} & p_{21} \\ p_{21} & \frac{1 - z_{\mathrm{CCL}}}{c}\, p_{21} \end{bmatrix}.$$

It has positive definiteness for $p_{11} > 0$ and $0 < p_{21} < \frac{1 - z_{\mathrm{CCL}}}{c}\, p_{11}$. Furthermore, the positive semi-definiteness of $2\,\underline{R} - \underline{H}^{\mathrm{T}}\,\underline{P}\,\underline{H}$ must also be fulfilled, which leads in the example to the condition

$$r \ge \frac{(1 - z_{\mathrm{CCL}})^2}{2} \left(p_{11} + \frac{c\,(3 + z_{\mathrm{CCL}})}{(1 + z_{\mathrm{CCL}})^2}\, p_{21} \right)$$

using $\underline{R} = r$. In the next step, the P-state controller matrix $\underline{K}$ is calculated by means of Eq. (61). Because $\underline{W} = \underline{0}$ was selected, the result is

$$\underline{K} = \frac{1 - z_{\mathrm{CCL}}}{r} \left[z_{\mathrm{CCL}}\, p_{11} + \frac{c\,(2 + z_{\mathrm{CCL}})}{1 + z_{\mathrm{CCL}}}\, p_{21} \quad \frac{2}{1 + z_{\mathrm{CCL}}}\, p_{21} \right].$$

If, for example, $z_{\mathrm{CCL}} = \frac{2}{3}$ holds and $p_{11} = 1$ is chosen without restricting the generality, then $p_{21} = \frac{1}{4\,c}$ and $r = \frac{1}{2}$, for example, fulfill the above conditions. For $\underline{K}$ and $\underline{M}$, it follows from this, but for general r,

$$\underline{K} = \frac{1}{r} \left[\frac{16}{45} \quad \frac{1}{10\,c} \right],$$

$$\underline{M} = \frac{1}{r}\, \frac{1}{10\,c}.$$

If the controlled system model is then extended by the dead time element, Eq. (82) immediately provides the extended controller matrix

$$\underline{K}_{\text{ext}} = \frac{1}{r}\begin{bmatrix}\frac{16}{45} & \frac{1}{10\,c}\end{bmatrix}\begin{bmatrix}\frac{2}{3} & 0 & \frac{1}{3}\\ c & 1 & \frac{c}{5}\end{bmatrix} = \frac{1}{r}\begin{bmatrix}\frac{91}{270} & \frac{1}{10\,c} & \frac{187}{1350}\end{bmatrix}.$$

Finally, if a controller integral-action component with the eigenvalue $z_{\text{I}} = \frac{4}{5}$ assigned to it is added, it follows from Eqs. (17) and (18)

$$\underline{K}_{\text{P}} = \begin{bmatrix}\frac{1}{2}+\frac{503}{1350\,r} & \frac{1}{c}\left(\frac{1}{6}+\frac{121}{675\,r}\right) & \frac{187}{1350\,r}+\frac{1}{5}\end{bmatrix},$$

$$\underline{K}_{\text{I}} = -\frac{1}{50\,r\,c}.$$

Figure 3 shows the transient response that is achieved. Here, a speed setpoint step from 0 to $2\pi \cdot 100$ rpm is specified at time $t = 5$ ms with vanishing initial state variable values. The step height was intentionally chosen so large that the manipulated variable of the speed controller, i.e. the torque setpoint $T_{\text{e,ref}}$, reaches the limit $T_{\text{e,max}} = 8$ Nm at the start of the transient response. In addition to the uncorrected speed setpoint ω_{ref}, the actual speed value ω and the unlimited torque setpoint $T_{\text{e,ref}}$, Figure 3 also shows the corrected speed setpoint $\omega_{\text{ref,corr}}$, the saturated torque setpoint $T_{\text{e,ref,sat}}$ and the output variable $v_{\text{I},\omega}$ of the speed controller integrator. With regard to the latter, it should be noted that Figure 3 is based on the fact that the control difference is first multiplied by $-\underline{K}_{\text{I}}$ and only then integrated and fed to the manipulated variable determination with a positive sign. In addition to the aforementioned value $r = \frac{1}{2}$ for the weighting factor, Figure 3 also shows curves for other values of the weighting factor r in order to demonstrate its influence on the control quality. All diagrams are based on the moment of inertia $J = 0{,}01$ kgm^2 and the sampling time $T_s = 100$ µs.

As can be clearly seen in Figure 3, the control loop dynamics continue to increase as the weighting factor r decreases. At $r = 0.05$, however, there appears a clear torque ripple. But, at $r = 0.05$, the previously mentioned sufficient stability condition is no longer fulfilled. Since this is only a sufficient condition, stable operation cannot be excluded, which is also shown in principle in Figure 3d.

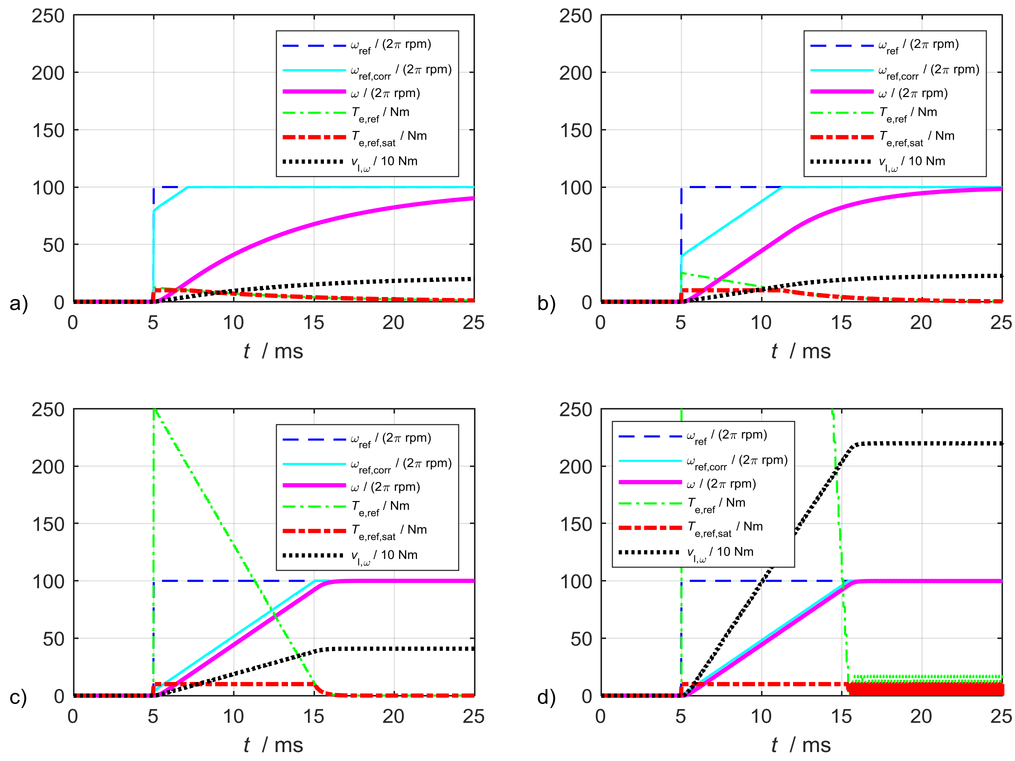

Figure 3. *Transient response of the relevant variables of the speed control loop when a speed setpoint step is specified with a limited torque setpoint; a)* $r = 10$, *b)* $r = 5$, *c)* $r = 0{,}5$, *d)* $r = 0{,}05$.

7. Conclusions

In the paper it was shown by means of a clearly simplified proof that for linear time-invariant controlled systems with limited manipulated variables, the stability of the anti-windup measures of a PI-state controller in the limiting case depends exclusively on the stability of the P-state-controlled system, provided that the PI-state controller was designed in such a way that it produces the same control behavior as a P-state controller in the unlimited case. This statement was proven in the article for both continuous-time and discrete-time controllers. Furthermore, it was shown that even in linear time-invariant controlled systems with dead times in the manipulated variable paths, the stability of the anti-windup measures can be traced back to the stability of the anti-windup measures of a corresponding dead-time-free system if the state controller used for the dead-time-controlled system is derived from the state controller for the dead-time-free system via the setting rules presented. The handling of the necessary design steps was demonstrated using an example from the field of electrical drives, where a controlled system with dead time and manipulated variable saturation is controlled by a PI-state controller in discrete time.

Abbreviations

ADE: Additional dynamic element
LMI: Linear matrix inequalities
P(-state controller): proportional acting (state controller)
PI(-state controller): proportional and integral acting (state controller)

Author Contributions

Uwe Nuss is the sole author. The author read and approved the final manuscript.

Data Availability Statement

The data supporting the outcome of this research work has been reported in this manuscript.

Conflicts of Interest

The author declares no conflicts of interest.

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Biography

Uwe Nuss teaches and researches since 2003 at Offenburg University of Applied Sciences. He is a lecturer in the fields of electrical drive technology, power electronics and control engineering at the Department of Electrical Engineering, Medical Engineering and Computer Science. He received his doctoral degree from Karlsruhe University (TH), Germany, in 1989, where he joined the electrotechnical institute and researched about inverter-based reactive power compensation. In 1994, he was habilitated at Karlsruhe University (TH), where he was awarded a teaching qualification on the subject of control of electrical drives. From 1994 to 2003, Prof. Nuss worked for a medium-sized company, where he was responsible for software development for the control of three-phase drives.

Research Field

Uwe Nuss: control of electrical drives, state control theory, continuous-time systems, discrete-time systems, power electronics.