International Journal of Theoretical and Applied Mathematics
2019; 5(6): 118-124
doi: 10.11648/j.ijtam.20190506.16
http://www.sciencepublishinggroup.com/j/ijtam
ISSN: 2575-5072 (Print); ISSN: 2575-5080 (Online)

Strongly $\mathcal{P}$-projective Modules and $\mathcal{P}$-projective Complexes

Liang Yan

College of Mathematics and Physics Science, Hunan University of Arts and Science, Changde, P. R. China

Email address:

lanyuyanliang@163.com

To cite this article:

Liang Yan. Strongly $\mathcal{P}$-projective modules and $\mathcal{P}$-projective Complexes. *International Journal of Theoretical and Applied Mathematics*. Vol. 5, No. 6, 2019, pp. 118-124. doi: 10.11648/j.ijtam.20190506.16

Received: October 30, 2019; **Accepted:** November 26, 2019; **Published:** December 19, 2019

Abstract: In this paper we first study the properties of Strongly $\mathcal{P}$-projective modules, and obtain some equivalent conclusions about Strongly $\mathcal{P}$-projective modules, it is proved that a finitely generated right R-module N is strongly $\mathcal{P}$-projective if and only if $Ext^i(N,R) = 0$ for all $i \geq 1$ over left noetherian and right perfect ring, a $\mathcal{P}$-projective right R-module N is strongly $\mathcal{P}$-projective if and only if the first syzygy of N is strongly $\mathcal{P}$-projective. Then we extend the notion of $\mathcal{P}$-projective modules to that of $\mathcal{P}$-projective complexes. We study the relationships between $\mathcal{P}$-projective complexes and $\mathcal{P}$-projective modules, it is proved that a complex C is $\mathcal{P}$-projective if and only if every C^i is $\mathcal{P}$-projective for every integer i if and only if $Ext^1(C,P) = 0$ for every projective complex P if and only if for every exact sequence $0 \to A \to P \to C \to 0$ with P projective, $A \to P$ is a projective preenvelope of A. Some characterizations of $\mathcal{P}$-projective complexes also obtained.

Keywords: Strongly $\mathcal{P}$-projective Module, $\mathcal{P}$-projective Module, $\mathcal{P}$-projective Complex

1. Introduction

Throughout this paper, R is an associative ring with identity. By module we mean right R-module. $\mathcal{C}$ will be an abelian category of complexes of right R-modules. This category has enough projectives and injectives. This can be seen from the fact that any complex of the form

$$\cdots \longrightarrow 0 \longrightarrow M \xrightarrow{id} M \longrightarrow 0 \longrightarrow \cdots$$

with M projective (injective) is projective (injective). For objects C and D of $\mathcal{C}$, $Hom(C,D)$ is the abelian group of morphisms from C to D in $\mathcal{C}$ and $Ext^i(C,D)$ for $i \geq 0$ will denote the groups we get from the right derived functor of Hom.

In this paper, a complex

$$\cdots \longrightarrow C^{-1} \xrightarrow{\delta^{-1}} C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} \cdots$$

will be denoted C. We will use subscripts to distinguish complexes. So if $\{C^i\}_{i\in I}$ is a family of complexes, C_i will be

$$\cdots \longrightarrow C_i^{-1} \xrightarrow{\delta^{-1}} C_i^0 \xrightarrow{\delta^0} C_i^1 \xrightarrow{\delta^1} \cdots$$

Given a module M, we will denote by $\overline{M}$ the complex

$$\cdots \longrightarrow 0 \longrightarrow M \xrightarrow{id} M \longrightarrow 0 \longrightarrow \cdots$$

with the M in the -1 and 0th position. Also we mean by $\underline{M}$ the complex with M in the 0th place and 0 in the other places. Given a complex C and an integer m, $C[m]$ denotes the complex such that $C[m]^n = C^{m+n}$ and whose boundary operators are $(-1)^m\delta^{m+n}$. If C is a complex we let $Z(C)$ and $B(C)$ be the subcomplex of cycles and boundaries of C and we let $H(C) = Z(C)/B(C)$.

If $f : C \to D$ is a map of complexes, we can form $M(f)$, the mapping cone of f, $M(f)$ is a complex such that $M(f)^n = D^n \oplus C^{n+1}$ is mapped to $(\delta^n(x) + f(x), -\delta^{n+1}(y))$. It is easy to check that there is an exact sequence of complexes $0 \to D \to M(f) \to C[1] \to 0$.

$\underline{Ext}^i(C,D)$ is the complex

$$\cdots \longrightarrow Ext^i(C, D[n-1]) \longrightarrow Ext^i(C, D[n]) \longrightarrow Ext^i(C, D[n+1]) \longrightarrow \cdots,$$

With boundary operator induced by the boundary operator of D[9]. let $\mathcal{F}$ be a class of objects of abelian category $\mathcal{C}$. We donote by ${}^{\perp}\mathcal{F}$ the left orthogonal class of $\mathcal{F}$, where ${}^{\perp}\mathcal{F} = \{G : Ext^1(G, F) = 0, \text{ for all } F \in \mathcal{F}\}$, and denote by $\mathcal{F}^{\perp}$ the right orthogonal class of $\mathcal{F}$, where $\mathcal{F}^{\perp} = \{G : Ext^1(F, G) = 0, \text{for all } F \in \mathcal{F}\}$.

Let C be a object of $\mathcal{C}$, Recalled that a morphism $f : C \to F$ with $F \in \mathcal{F}$ is called an $\mathcal{F}$-preenvelope of C if for any morphism $g : C \to F^{'}$ with $F^{'} \in \mathcal{F}$, there is a morphism $\theta : F \to F^{'}$ such that $\theta f = g$, moreover, when $F^{'} = F$ and $g = f$ the only such θ are automorphisms of F, then $f : C \to F$ is called an $\mathcal{F}$-envelope of C, A monomorphism $f : C \to F$ is said to be a special $\mathcal{F}$-preenvelope of C if $coker(f) \in^{\perp} \mathcal{F}$. Dually, we have the concepts of (special) $\mathcal{F}$-precover and $\mathcal{F}$-cover. A pair $(\mathcal{F}, \mathcal{G})$ is called a cotorsion theory, if $\mathcal{F}^{\perp} = \mathcal{G}$ and ${}^{\perp}\mathcal{G} = \mathcal{F}$. A cotorsion theory $(\mathcal{F}, \mathcal{G})$ is called perfect if every object of $\mathcal{C}$ has a $\mathcal{F}$-cover and a $\mathcal{G}$-envelope. A cotorsion theory $(\mathcal{F}, \mathcal{G})$ is called hereditary if $\mathcal{G}$ is resolving.

Recalled that a complex P is projective if and only if it is exact and every $Ker\delta^i$ is a projective module for all $i \in \mathcal{Z}$, a complex E is called a #-projective complex if all terms E^i are projective.

A module M is said to be $\mathcal{P}$-projective if it is a coker of a projective preenvelope [3]. These modules were discovered when studying projective (pre)envelopes. We note that the notion of P-projectivity is dual to that of copure injectiveness defined by Enochs and Jenda in [4].

A module M is said to be copure flat if it is flat with respect to the exact sequence $0 \to A \to B \to B/A \to 0$ with B/A injective[5]. It is easy to see that M is copure flat if and only if $Tor_1(M, E) = 0$ for all injective right R-module. We will say that M is strongly copure flat if $Tor_i(M, E) = 0$ for all injective right R-module and all $i \geq 1$.

In section 2, we introduce the strongly $\mathcal{P}$-projective R-module, some characterizations of strongly $\mathcal{P}$-projective R-module are given. We also study $\mathcal{P}$-projective dimensions which is based on a similar idea due to [5].

The main purpose of section 3 is to extend the notions of $\mathcal{P}$-projective modules to that of copure injective $\mathcal{P}$-projective complexes . A complex C is said to be $\mathcal{P}$-projective if given any morphism $f : A \to B \to 0$ with $ker f$ projective and any morphism $g : C \to B$, there exists a homomorphism $h : C \to A$ such that the following diagram commutative

$$\begin{tikzcd} C \arrow[d, dashed, "h"'] \arrow[dr, "g"] & & \\ A \arrow[r, "f"] & B \arrow[r] & 0. \end{tikzcd} \qquad (1)$$

We first obtain a number of characterizations of $\mathcal{P}$-projectivity of complexes. Then It is natural to consider the relationships of $\mathcal{P}$-projectivity of a complex C and $\mathcal{P}$-projectivity of all R-modules C^i for $i \in \mathbb{Z}$.

We prove that a complex C is $\mathcal{P}$-projective if and only if every C^i is $\mathcal{P}$-projective for $i \in \mathbb{Z}$. Some characterizations of $\mathcal{P}$-projective complexes also obtained.

2. Strongly $\mathcal{P}$-projective Modules

In this section we introduce the definitions of strongly $\mathcal{P}$-projective modules, and give some characterizations of strongly $\mathcal{P}$-projective modules.

A right R-module N is said to be $\mathcal{P}$-projective if it is a coker of projective (pre)envelope [3]. A right R-module N is $\mathcal{P}$-projective if and only if $Ext^1(N, P) = 0$ for any projective right R-module P. We shall say a right R-module N is strongly $\mathcal{P}$-projective if $Ext^i(N, P) = 0$ for any projective right R-module P and all $i \geq 1$.

Remark 2.1 (1)Projective module $\Rightarrow$ Gorenstein projective module $\Rightarrow$ (strongly) $\mathcal{P}$-projective module.

(2) The class of (strongly) $\mathcal{P}$-projective modules is closed under extensions, direct sums and direct summands.

(3) It is easy to see that N is $\mathcal{P}$-projective if and only if given any homomorphism $f : A \to B \to 0$ with $ker f$ projective and any homomorphism $g : N \to B$, there exists a homomorphism $h : N \to A$ such that the following diagram commutative

$$\begin{tikzcd} N \arrow[d, dashed, "h"'] \arrow[dr, "g"] & & \\ A \arrow[r, "f"] & B \arrow[r] & 0. \end{tikzcd} \qquad (2)$$

Proposition 2.2 The following are equivalent for a right R-module N:

(1)*N is projective.*

(2)*N is $\mathcal{P}$-projective and $pd(N) \leq 1$.*

(3)*N is strongly $\mathcal{P}$-projective and $pd(N) \leq 1$.*

Proof. $(1) \Rightarrow (2)$ is trivial.

$(2) \Rightarrow (1)$ Let N be a $\mathcal{P}$-projective module and $pd(N) \leq 1$. Then there exists an exact sequence $0 \to K \to P \to N \to 0$. Note that K is projective since $pd(N) \leq 1$. So $Ext^1(N, K) = 0$, the above exact sequence splits. Thus N is projective.

$(1) \Rightarrow (3)$ is trivial.

$(3) \Rightarrow (1)$ is similar to $(2) \Rightarrow (1)$.

Proposition 2.3 Let R be a left coherent and right perfect ring. Then the following results are true:

(1) *Every (strongly) $\mathcal{P}$-projective right R-module is (strongly) copure flat.*

(2)*Every finitely presented (strongly) copure flat right R-module is (strongly) $\mathcal{P}$-projective.*

Proof. (1) Let E be an injective left R-module, then E^+ is flat since R is left coherent ring, and so E^+ is projective since R is right perfect ring. Thus (1) follows from the standard isomorphism $Ext^1(N, M^+) \cong Tor_1(N, M)^+$, where N is any left R-module and M is any right R-module.

(2) Let N be a finitely presented (strongly) copure flat right R-module. Then there exists a standard isomorphism $Ext^i(N,P)^+ \cong Tor_i(M,P^+)$. Note that any projective right R-module P is flat since R is a right perfect ring, and so P^+ is injective. Thus the result holds.

Proposition 2.4 Let R be a left noetherian and right perfect ring. Then a finitely generated right R-module N is strongly $\mathcal{P}$-projective if and only if $Ext^i(N,R) = 0$ for all $i \geq 1$.

Proof. If N is strongly $\mathcal{P}$-projective, then $Ext^i(N,R) = 0$ for all $i \geq 1$ since R is projective as a right R-module. Conversely, Let P be a projective right R-module, P is flat since R is a right perfect ring. By Theorem 4.34 of [11], P is a direct limit of finitely generated free modules, $Ext^i(N,P) = 0$ by Lemma 3.1.16 of [6] since R is left noetherian and N is finitely generated.

Corollary 2.5 Let R be a left noetherian and right perfect ring. Then a finitely generated right R-module N is $\mathcal{P}$-projective if and only if $Ext^1(N,R) = 0$.

Let $\mathcal{P}_\mathcal{P}$ (resp., $\mathcal{SP}_\mathcal{P}$) denote the class of $\mathcal{P}$-projective (resp., strongly $\mathcal{P}$-projective) right R-modules. *Proposition 2.6 Let R be a left coherent and right perfect ring, then:*

(1) *$(\mathcal{P}_\mathcal{P}, {\mathcal{P}_\mathcal{P}}^\perp)$ is a perfect cotorsion theory.*

(2) *$(\mathcal{SP}_\mathcal{P}, {\mathcal{SP}_\mathcal{P}}^\perp)$ is a perfect hereditary cotorsion theory.*

Proof. Since every projective right R-module P is pure injective over a left coherent and right perfect ring, $(\mathcal{P}_\mathcal{P}, {\mathcal{P}_\mathcal{P}}^\perp)$ is a perfect cotorsion theory by Theorem 2.8 of [14], and $(\mathcal{SP}_\mathcal{P}, {\mathcal{SP}_\mathcal{P}}^\perp)$ is a perfect cotorsion theory by Corollary 3.2.12 of [10].

Let $\mathcal{P}$ denote the class of projective right R-module. We have the following : *Proposition 2.7 The following are equivalent:*

(1) *R is a QF ring.*

(2) *Every module is $\mathcal{P}$-projective.*

(3) *Every quotient of a $\mathcal{P}$-projective module is $\mathcal{P}$-projective.*

(4) *$(\mathcal{P}_\mathcal{P}, \mathcal{P})$ is a cotorsion theory.*

Proof. (1) $\Leftrightarrow$ (2) follows from the fact that R is a QF ring if and only if every projective module is injective (we also can see Remark 2. 3 of [3]) .

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (2) We simply note that every module is a quotient of a $\mathcal{P}$-projective module since every projective module is $\mathcal{P}$-projective.

(1) $\Rightarrow$ (4) is clear.

(4) $\Rightarrow$ (1) R is a QF ring since every injective module is projective by (4).

An exact sequence $\cdots \to A_2 \to A_1 \to A_0 \to N \to 0$ where $A_0 \to N$, $A_1 \to ker(A_0 \to N)$, $A_{n+1} \to ker(A_n \to A_{n-1})$ are strongly $\mathcal{P}$-projective precovers is called a $\mathcal{P}$-projective resolution of N. If there exists a $\mathcal{P}$-projective resolution $0 \to A_n \to \cdots \to A_2 \to A_1 \to A_0 \to N \to 0$, we say that N has $\mathcal{P}$-projective dimension ($\mathcal{P}pd$) $\leq n$. Let $\mathcal{P}pd(R) = sup\{\mathcal{P}pd(N) \mid N \text{ is a right } R\text{-module}\}$, we call $\mathcal{P}pd(R)$ the $\mathcal{P}$-projective dimension of R.

Proposition 2.8 Let R be a left coherent and right perfect ring. Then the following are equivalent for a right R-module:

(1) *$\mathcal{P}pd(N) \leq n$.*

(2) *$Ext^{n+i}(N,P) = 0$ for all projective right R-module P and all $i \geq 1$.*

(3) *Every nth syzygy of N is strongly $\mathcal{P}$-projective.*

Proof. (1) $\Leftrightarrow$ (2) Let $0 \to K \to A_{n-1} \to \cdots \to A_2 \to A_1 \to A_0 \to N \to 0$ be an exact sequence with A_0, A_1, A_2, $\cdots$, A_{n-1} strongly $\mathcal{P}$-projective. Then $Ext^i(K,P) \cong Ext^{n+i}(N,P)$. Thus the result follows.

(2) $\Leftrightarrow$ (3) is similar to (1) $\Leftrightarrow$ (2).

Remark 2.9 We note that the $\mathcal{P}$-projective dimension of a right R-module N can be considered as the largest positive integer n such that $Ext^n(N,P) \neq 0$ for some projective module P. Taking this as a definition of $\mathcal{P}$-projective dimension, we may drop the left coherent and right perfect conditions in Proposition above.

Corollary 2.10 Let R be a ring. Then a $\mathcal{P}$-projective right R-module N is strongly $\mathcal{P}$-projective if and only if the first syzygy of N is strongly $\mathcal{P}$-projective. Corollary 2.11 if $pd(N) < \infty$, then $\mathcal{P}pd(N) = pd(N)$.

Proof. $\mathcal{P}pd(N) \leq pd(N)$ follows from Proposition 2.8 since projective modules are $\mathcal{P}$-projective. Suppose $pd(N) = n$, then there exists a right R-module A such that $Ext^n(N,A) \neq 0$. For right R-module A, there is an exact sequence $0 \to K \to P \to A \to 0$ with P projective. Applying $Hom(N,-)$ to this exct sequence, we get a long exact sequence $\cdots \to Ext^n(N,P) \to Ext^n(N,A) \to Ext^{n+1}(N,K) \to \cdots$, then $Ext^{n+1}(N,K) = 0$ since $pd(N) = n$. But $Ext^n(N,A) \neq 0$, so $Ext^n(N,P) \neq 0$. Thus $\mathcal{P}pd(N) \geq n = pd(N)$ by Remark 2.9. □

Proposition 2.12 Let R be a left noetherian and right perfect ring, N a finitely generated right R-module and n a nonnegative integer. Then $\mathcal{P}pd(N) \leq n$ if and only if $Ext^{n+i}(N,R) = 0$ for all $i \geq 1$.

Proof. Similar to the proof of Proposition 2.4.

The following proposition is Dual to the equivalences (1)-(3) of [5].

Proposition 2.13 The following are equivalent for a left and right noetherian ring R:

(1) *R is n-Gorenstein.*

(2) *$\mathcal{P}pd(N) \leq n$ for all R-modules (left and right) N.*

(3) *Every nth syzygy of N is strongly $\mathcal{P}$-projective.*

Corollary 2.14 The following are equivalent for a two-sided noetherian and perfect ring R:

(1) *R is 1-Gorenstein.*

(2) *$\mathcal{P}pd(N) \leq 1$ for all R-module (left and right) N.*

(3) *Every $\mathcal{P}$-projective R-module (left and right) is strongly $\mathcal{P}$-projective.*

(4) *Every submodule of a strongly $\mathcal{P}$-projective R-module (left and right) is strongly $\mathcal{P}$-projective.*

Proof. (1) $\Leftrightarrow$ (2) follows from the Proposition above.

(2) $\Rightarrow$ (3) Let N be a $\mathcal{P}$-projective R-module. Then the first syzygy of N is strongly $\mathcal{P}$-projective by Proposition 2.8, and so N is strongly $\mathcal{P}$-projective by Corollary 2.10.

(3) $\Rightarrow$ (2) Let N be an R-module. Then there exists an exact sequence $0 \to K \to P \to N \to 0$ with P projective. Let $K \to P'$ be a projective preenvelope which is monic since K is a submodule of projective. Note that $L = coker(K \to$

$P')$ is $\mathcal{P}$-projective by the definition of $\mathcal{P}$-projective modules, and so L is strongly $\mathcal{P}$-projective by (3). Hence K is strongly $\mathcal{P}$-projective, as desired.

$(2) \Rightarrow (4)$ Let M be a submodule of a strongly $\mathcal{P}$-projective R-module N. Applying the functor $Hom(-,P)$ to the exact sequence $0 \to M \to N \to N/M \to 0$, we obtain an exact sequence $0 = Ext^i(N,P) \to Ext^i(M,P) \to Ext^{1+i}$ for all $i \geq 1$. But the last term is zero by (2), so $Ext^i(M,P) = 0$ for all $i \geq 1$, as desired.

$(4) \Rightarrow (2)$ is obvious.

3. $\mathcal{P}$-projective Complexes

Definition 3.1 A complex C is said to be $\mathcal{P}$-projective if given any morphism $f : A \to B \to 0$ with $ker f$ projective and any morphism $g : C \to B$, there exists a morphism $h : C \to A$ such that the following diagram commutative

$$\begin{tikzcd} C \arrow[d, dashed, "h"'] \arrow[dr, "g"] & & \\ A \arrow[r, "f"] & B \arrow[r] & 0. \end{tikzcd} \tag{3}$$

Proposition 3.2 The following are equivalent for a complex C:

(1) *C is $\mathcal{P}$-projective.*

(2) *$Ext^1(C,P) = 0$ for every projective complex.*

(3) *For every exact sequence $0 \to A \to P \to C \to 0$ with P projective, $A \to P$ is a projective preenvelope of A.*

(4) *C is a $coker$ of a projective preenvelope.*

(5) *For any short exact sequence $0 \to A \to B \to C \to 0$ and any projective P, the sequence $Hom(B,P) \to Hom(A,P) \to 0$ is exact.*

(6) *$Hom(C,-)$ exact any short exact sequence $0 \to P \to A \to B \to 0$ with P projective.*

Proof. $(1) \Rightarrow (2)$ Let $0 \to P \to E \to N \to 0$ be a short injective resolution of projective complex P. Then we have a long exact sequence $0 \to Hom(C,P) \to Hom(C,E) \to Hom(C,N) \to Ext^1(C,P) \to Ext^1(C,E) = 0$. But $0 \to Hom(C,P) \to Hom(C,E) \to Hom(C,N) \to 0$ is exact by definition of $\mathcal{P}$-projective modules. So $Ext^1(C,P) = 0$.

$(2) \Rightarrow (1)$ is straightforward.

$(2) \Rightarrow (3)$ Let P' be projective, then there is an exact sequence $0 \to Hom(P,P') \to Hom(A,P') \to Ext^(C,P') = 0$ by (2), so $A \to P$ is a projective preenvelope.

$(3) \Rightarrow (4)$ is obvious.

$(4) \Rightarrow (2)$ Let C be a $coker$ of a projective preenvelope $f : A \to P$ with P projective. then there is an exact sequence $0 \to K \to P \to C \to 0$ with $K = im(f)$. For each projective module P' we have a long exact sequence $Hom(P,P') \to Hom(K,P') \to Ext^1(C,P') \to Ext^1(P,P') = 0$. But $Hom(P,P') \to Hom(K,P') \to 0$ is exact by (4). Hence $Ext^1(C,P') = 0$.

$(2) \Rightarrow (5)$ is obvious.

$(5) \Rightarrow (2)$ There exists an exact sequence $0 \to K \to P \to C \to 0$ with P projective. Then for any projective P', we have the exact sequence $Hom(P,P') \to Hom(K,P') \to Ext^1(C,P') \to Ext^1(P,P') = 0$. But $Hom(P,P') \to Hom(K,P') \to 0$ is exact, so $Ext^1(C,P') = 0$.

$(2) \Rightarrow (6)$ is obvious.

$(6) \Rightarrow (2)$ Let P be a projective. Then there is an exact sequence $0 \to P \to E \to L \to 0$, and we have a long exact $0 \to Hom(C,P) \to Hom(C,E) \to Hom(C,L) \to Ext^1(C,P) \to Ext^1(C,E) = 0$. But $0 \to Hom(C,P) \to Hom(C,E) \to Hom(C,L) \to 0$ is exact by (6), so $Ext^1(C,P) = 0$. □ *Remark 3.3 (1)Projective complex $\Rightarrow$ Gorenstein projective complex $\Rightarrow$ $\mathcal{P}$-projective complex.*

(2) The class of $\mathcal{P}$-projective complexes is closed under extensions, direct sums and direct summands.

(3) If C is a $\mathcal{P}$-projective complex, then $C[i]$ is also a $\mathcal{P}$-projective complex for all $i \in \mathbb{Z}$.

It is natural to consider the relationships of $\mathcal{P}$-projectivity of a complex C and $\mathcal{P}$-projectivity of all modules C^i for $i \in \mathbb{Z}$. Next we give the following results.

Theorem 3.4 The following are equivalent for a complex C:

(1) *C is $\mathcal{P}$-projective.*

(2) *$Ext^1(C, \overline{P}[n]) = 0$ for any projective module P.*

(3) *Every C^i is $\mathcal{P}$-projective for $i \in \mathbb{Z}$ and $Hom^{\cdot}(C,P)$ is exact for each projective complex P.*

(4) *Every C^i is $\mathcal{P}$-projective for $i \in \mathbb{Z}$.*

Proof. $(1) \Rightarrow (2)$ For any projective module P, $\overline{P}[n]$ is projective, so $Ext^1(C, \overline{P}[n]) = 0$ by Proposition 3.2.

$(2) \Rightarrow (1)$ Note that any projective complex is the direct product of complexes

$$\overline{P_i}[i] = \cdots \longrightarrow 0 \longrightarrow P_i \xrightarrow{id} P_i \longrightarrow 0 \longrightarrow \cdots$$

with P_i projective. It is easy to check that $Ext^1(C,P) = 0$ for any projective complex P. So C is $\mathcal{P}$-projective.

$(1) \Rightarrow (3)$ Let $0 \to P \to A \to C^i \to 0$ be any exact sequence of modules with P projective. Then we get the following pullback diagram:

$$\begin{tikzcd} & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\ 0 \arrow[r] & 0 \arrow[r] \arrow[d] & C^{i-2} \arrow[r, "id"] \arrow[d] & C^{i-2} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & P \arrow[r] \arrow[d, "id"] & B \arrow[r] \arrow[d] & C^{i-1} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & P \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & C^{i} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & 0 \arrow[r] \arrow[d] & C^{i+1} \arrow[r, "id"] \arrow[d] & C^{i+1} \arrow[r] \arrow[d] & 0 \\ & \vdots & \vdots & \vdots & \end{tikzcd} \tag{4}$$

Then $Ext^1(C, \overline{P}[-i]) = 0$ since $\overline{P}[-i]$ is projective. So the above exact sequence of complexes is split. Thus the exact sequence $0 \to P \to A \to C^i \to 0$ is split, $Ext^1(C^i, P) = 0$, so C^i is $\mathcal{P}$-projective. For any projective complex P, the short exact sequence of complexes $0 \to P[n] \to M(f) \to C[1] \to 0$ is split for any $n \in \mathbb{Z}$ and any map $f : C \to p[n]$ by (1). So f is homotopic to zero by lemma 2.3.2 of [9]. It is easy to check that $Hom^{\cdot}(C, P)$ is exact.

$(3) \Rightarrow (4)$ is obvious.

$(4) \Rightarrow (2)$ Let

$$0 \longrightarrow \overline{P}[n] \xrightarrow{f} A \xrightarrow{g} C \longrightarrow 0$$

be any exact sequence of complexes, and we consider the following commutative diagram:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & A^{-n-2} \arrow[r, "g^{-n-2}"] \arrow[d, "\delta_A^{-n-2}"] & C^{-n-2} \arrow[r] \arrow[d, "\delta_C^{-n-2}"] & 0 \\
0 \arrow[r] & P \arrow[r, "f^{-n-1}"] \arrow[d, "id"] & A^{-n-1} \arrow[r, "g^{-n-1}"] \arrow[d, "\delta_A^{-n-1}"] & C^{-n-1} \arrow[r] \arrow[d, "\delta_C^{-n-1}"] & 0 \\
0 \arrow[r] & P \arrow[r, "f^{-n}"] \arrow[d] & A^{-n} \arrow[r, "g^{-n}"] \arrow[d, "\delta_A^{-n}"] & C^{-n} \arrow[r] \arrow[d, "\delta_C^{-n}"] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & A^{-n+1} \arrow[r, "g^{-n+1}"] \arrow[d, "\delta_A^{-n+1}"] & C^{-n+1} \arrow[r] \arrow[d, "\delta_C^{-n+1}"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd} \tag{5}$$

Note that $Ext^1(C^{-n}, P) = 0$ since C^{-n} is $\mathcal{P}$-projective. So $f^{-n} : P \to A^{-n}$ splits, there exists a morphism $h^{-n} : A^{-n} \to P$ such that $h^{-n}f^{-n} = id_P$. We define $h^{-n-1} : A^{-n-1} \to P$ as $h^{-n-1} = h^{-n}\delta_A^{-n-1}$ and $h^i = 0$ for $i \neq -n$, $-n-1$. Then $h = \{h^i\}_{i\in\mathbb{Z}}$ is the morphism from A to $\overline{P}[n]$ such that $hf = id_{\overline{P}[n]}$. So the sequence

$$0 \longrightarrow \overline{P}[n] \xrightarrow{f} A \xrightarrow{g} C \longrightarrow 0$$

splits, and so $Ext^1(C, \overline{P}[n]) = 0$.

Corollary 3.5 The following are equivalent for a module M:

(1) *M is $\mathcal{P}$-projective.*

(2) *$\underline{M}[n]$ is $\mathcal{P}$-projective for all $n \in \mathbb{Z}$.*

(3) *$\overline{M}[n]$ is $\mathcal{P}$-projective for all $n \in \mathbb{Z}$.*

Corollary 3.6 The following are equivalent for a complex C:

(1) *C is $\mathcal{P}$-projective.*

(2) *Every exact sequence $0 \to P \to A \to C \to 0$ with P projective splits.*

(3) *$\underline{Ext}(C, P) = 0$ for any projective complex P.*

Recalled that a complex C is said to be #-projective if every C^i is projective for all $i \in \mathbb{Z}$. We have the following Proposition.

Proposition 3.7 The following are equivalent for a complex C:

(1) *C is $\mathcal{P}$-projective.*

(2) *$Ext^1(C, P) = 0$ for every $Hom(Q, -)$ exact bounded #-projective complex P whenever Q is a $\mathcal{P}$-projective module.*

(3) *$Ext^1(C, P) = 0$ for every $Hom(Q, -)$ exact bounded above #-projective complex P whenever Q is a $\mathcal{P}$-projective module.*

Proof. $(1) \Rightarrow (3)$ Let P be a $Hom(Q, -)$ exact bounded above #-projective complex whenever Q is a $\mathcal{P}$-projective module, without loss of generality, we may assume that $P^i = 0$ for $i > 0$. Let

$$0 \longrightarrow P \xrightarrow{f} A \xrightarrow{g} C \longrightarrow 0$$

be any exact sequence of complexes. Then we consider the following commutative diagram:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & P^{-2} \arrow[r, "f^{-2}"] \arrow[d, "\delta_P^{-2}"] & A^{-2} \arrow[r, "g^{-2}"] \arrow[d, "\delta_A^{-2}"] & C^{-2} \arrow[r] \arrow[d, "\delta_C^{-2}"] & 0 \\
0 \arrow[r] & P^{-1} \arrow[r, "f^{-1}"] \arrow[d, "\delta_P^{-1}"] & A^{-1} \arrow[r, "g^{-1}"] \arrow[d, "\delta_A^{-1}"] & C^{-1} \arrow[r] \arrow[d, "\delta_C^{-1}"] & 0 \\
0 \arrow[r] & P^{0} \arrow[r, "f^{0}"] \arrow[d] & A^{0} \arrow[r, "g^{0}"] \arrow[d, "\delta_A^{0}"] & C^{0} \arrow[r] \arrow[d, "\delta_C^{0}"] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & A^{1} \arrow[r, "g^{1}"] \arrow[d] & C^{1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd} \tag{6}$$

We see that every exact sequence

$$0 \longrightarrow P^i \xrightarrow{f^i} A^i \xrightarrow{g^i} C^i \longrightarrow 0$$

is split since C^i is $\mathcal{P}$-projective by Theorem 3.4. So there exists $h^i : A^i \to P^i$ such that $h^i f^i = id_{P^i}$ for all $i \leq 0$. Now Let $\alpha^0 = h^0$.

Since $coker f^{-1} \cong C^{-1}$, it follows that

$$Hom(coker f^{-1}, P^{-1}) \to Hom(coker f^{-1}, P^0) \to 0$$

is exact. Note that $(\delta_P^{-1}h^{-1} - \alpha^0\delta_A^{-1})f^{-1} = \delta_P^{-1}h^{-1}f^{-1} - \alpha^0\delta_A^{-1}f^{-1} = \delta_P^{-1} - \alpha^0 f^0 \delta_P^{-1} = \delta_P^{-1} - \delta_P^{-1} = 0$, so $\overline{\delta_P^{-1}h^{-1} - \alpha^0\delta_A^{-1}} \in Hom(coker f^{-1}, P^0)$. Thus there exists $\overline{\gamma^{-1}} \in Hom(coker f^{-1}, P^{-1})$ such that $\overline{\delta_P^{-1}h^{-1} - \alpha^0\delta_A^{-1}} = \delta_P^{-1}\overline{\gamma^{-1}}$, where $\gamma^{-1} \in Hom(A^{-1}, P^{-1})$ and $\gamma^{-1}(P^{-1}) = 0$. let $\alpha^{-1} = h^{-1} - \gamma^{-1}$, then $\delta_P^{-1}\alpha^{-1} = \delta_P^{-1}h^{-1} - \delta_P^{-1}\gamma^{-1} = \alpha^0\delta_A^{-1}$, and $\alpha^{-1}f^{-1} = h^{-1}f^{-1} - \gamma^{-1}f^{-1} = h^{-1}f^{-1} = id_{P^{-1}}$ since $\gamma^{-1}f^{-1} = 0$.

It follows that

$$Hom(cokerf^{-2}, P^{-2}) \to$$

$$Hom(cokerf^{-2}, P^{-1}) \to Hom(cokerf^{-2}, P^{0})$$

is exact since $cokerf^{-2} \cong C^{-2}$. Note that $(\delta_P^{-2}h^{-2} - \alpha^{-1}\delta_A^{-2})f^{-2} = \delta_P^{-2}h^{-2}f^{-2} - \alpha^{-1}\delta_A^{-2}f^{-2} = \delta_P^{-2} - \alpha^{-1}f^{-1}\delta_P^{-2} = \delta_P^{-2} - \delta_P^{-2} = 0$, then $\overline{(\delta_P^{-2}h^{-2} - \alpha^{-1}\delta_A^{-2})} \in Hom(cokerf^{-2}, P^{-1})$, but $\delta_P^{-1}\overline{(\delta_P^{-2}h^{-2} - \alpha^{-1}\delta_A^{-2})} = 0$, so there exists $\overline{\gamma^{-2}} \in Hom(cokerf^{-2}, P^{-2})$ such that $\delta_P^{-2}\overline{\gamma^{-2}} = \overline{(\delta_P^{-2}h^{-2} - \alpha^{-1}\delta_A^{-2})}$, where $\gamma^{-2} \in Hom(A^{-2}, P^{-2})$ and $\gamma^{-2}(P^{-2}) = 0$. Now let $\alpha^{-2} = h^{-2} - \gamma^{-2}$, then $\delta_P^{-2}\alpha^{-2} = \delta_P^{-2}h^{-2} - \delta_P^{-2}\gamma^{-2} = \alpha^{-1}\delta_A^{-2}$, and $\alpha^{-2}f^{-2} = h^{-2}f^{-2} - \gamma^{-2}f^{-2} = h^{-2}f^{-2} = id_{P^{-2}}$ since $\gamma^{-2}f^{-2} = 0$.

Similarly, we can obtain $\alpha^{-i} : A^{-i} \to P^{-i}$ such that $\delta_P^{-i}\alpha^{-i} = \alpha^{-i+1}\delta_A^{-i}$ and $\alpha^{-i}f^{-i} = id_{P^{-i}}$ for $i \geq 3$. Finally, let $\alpha^n = 0$ for $n \geq 1$. Then we obtain a morphism $\alpha = \{\alpha^i\}_{i\in\mathbb{Z}} : A \to P$ such that $\alpha f = id_P$. This implies that the exact sequence

$$0 \longrightarrow P \xrightarrow{f} A \xrightarrow{g} C \longrightarrow 0$$

splits. Thus $Ext^1(C, P) = 0$.

(3) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1) Let P be a projective module, $Ext^1(C, \overline{P}[n]) = 0$ by (2), so C is $\mathcal{P}$-projective by Theorem 3.4.

Proposition 3.8 The following are equivalent:

(1) *R is a QF ring.*

(2) *Every complex is $\mathcal{P}$-projective.*

(3) *Every quotient of a $\mathcal{P}$-projective complex is $\mathcal{P}$-projective.*

Proof. (1) $\Rightarrow$ (2) Let C be a complex, then C^i is $\mathcal{P}$-projective by Proposition 2.7. So C is a $\mathcal{P}$-projective complex by Theorem 3.4.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1) Let M be a quotient of a $\mathcal{P}$-projective module N. Then $\overline{M}$ is a quotient of a $\mathcal{P}$-projective complex $\overline{N}$, so $\overline{M}$ is $\mathcal{P}$-projective, thus M is $\mathcal{P}$-projective by Theorem 3.4. So R is a QF ring by Proposition 2.7.

We conclude the paper with the following Remark. *Remark 3.9 (1) let R be a left coherent and right perfect ring. Then every right R-module has a $\mathcal{P}$-projective cover by Proposition 2.6. So every bounded above complex has a $\mathcal{P}$-projective precover by Proposition 5.3 of [12] since $\mathcal{P}$-projective complexes coincide with #-$\mathcal{P}$-projective complexes by Theorem 3.4.*

(2) If every complex has a projective envelope and we replace "$\mathcal{P}$-projective modules" with "$\mathcal{P}$-projective complexes" in [3], the results still hold by similar proofs.

4. Conclusion

The relationship between module and complex is an important research content in homological algebra. We find a new complex called $\mathcal{P}$-projective complex from $\mathcal{P}$-projective module, and reveal the relationship between them. Then we study the properties and homology dimension of $\mathcal{P}$-projective complex in different rings. This kind of work is very meaningful.

Acknowledgements

The work was supported by the Natural Science Foundation of Hunan Province (grant Nos.2016JJ3095) and the DR. Start -up Funds of Hunan University of Arts and Science(grant Nos. 15BSQD15).

References

[1] F. W. Anderson, K. R. Fuller, Rings and Categories of Modules, second ed., New York, Spring-verlag, 1992.

[2] L. L. Avramov, H.-B.Foxby, Homological dimensions of unbounded complexes. J. Pure Appl. Algebra 71(1991):129-155.

[3] J. L. Chen, P-Projective modules, Communications in Algebra,24(1996): 3,821-831

[4] E. E. Enochs, O. M. G. Jenda, Copure injective modules, Quaest. Math. 14 (1991) 401-409.

[5] E. E. Enochs, O. M. G. Jenda, Copure injective resolutions, flat resolutions and dimensions, Comment. Math. 34 (1993) 203-211.

[6] E. E. Enochs, O. M. G. Jenda, Relative Homological Algebra, Walter de Gruyter, Berlin, New York, 2000.

[7] E. E. Enochs, L. Oyonarte, Covers, Envelopes and Cotorsion Theories. Nova Science Publishers, Inc. New York, 2002.

[8] E. E. Enochs, S. Estrada, A.Iacob, Gorenstein projective and flat complexes over noetherian rings, Math. Nachr.(2012) 1-18.

[9] J. R. Garcı́a Rozas, Covers and Envelopes in the Category of Complexes of Modules. Boca Raton-London-New York-Washington, D. C.: CRC Press, 1999.

[10] R. Göbel, J. Trlifaj, Approximations and Endomorphism Algebras of modules. Berlin-New York: Walter de Gruyter, 2006.

[11] T. Y. Lam, Lectures on modules and rings, New York-Heidelberg-Berlin: Springer-Verlag, 1999.

[12] L. Li, N. Q. Ding, G.Yang, covers and Envelopes by #-$\mathcal{F}$ Complexes. Communications in Algebra 39(2011) 3253-3277.

[13] L. X. Mao, N. Q. Ding, Relative copure injective and copure flat modules. Journal of Pure and Applied Algebra 208 (2007) 635-646.

[14] J. Trlifaj, Cover, Envelope, and Cotorsion Theories; Lecture notes for the workshop, Homological Methods in Module Theory, Cortona, September 10-16, 2000.