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On the Tractability of Transmuted Type I Generalized Logistic Distribution with Application

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Abstract: Transmutation of baseline distributions has gained popularity in the last decade and many authors have studied the some transmuted distributions such as exponential, Weilbul, gamma, Pareto, normal and many more. This article will focus on the transmutation of type I generalized logistic distribution using quadratic rank transmutation map to develop a transmuted type I generalized logistic distribution. The quadratic rank transmutation map enables the introduction of extra parameter into its parent model to enhance more flexibility in the analysis of data in various disciplines such as biological sciences, actuarial science, finance and insurance. The graphs of the probability density function (pdf) and cumulative distribution function (cdf) of the model for different values of parameters are illustrated respectively. The mathematical properties such as moment generating function, quantile, median and characteristic function of this distribution are discussed. The probability density functions of the minimum and maximum order statistics of the transmuted type I generalized logistic distribution are established and the relationships between the probability density functions of the minimum and maximum order statistics of the parent model and the probability density function of the transmuted type I generalized logistic distribution are considered. The parameter estimation is done by the method of maximum likelihood estimation. The flexibility of the model in statistical data analysis and its applicability is demonstrated by using the model to fit relevant data. The study is concluded by demonstrating the performance of transmuted type I generalized logistic distribution over its parent model.

Keywords: Logistic Distribution, Parameter Estimation, Order Statistics, Transmutation

1. Introduction

Logistic distribution, like normal distribution, has been used over the years to model real life data in various fields of human endeavors such as medicine, Economics, Demography to mention a few.

This distribution has been generalized by many authors over the years to enhance its flexibility in data analysis. The distribution in its simplest form has its probability density function (pdf) and cumulative distribution function (cdf) as

$$g(x) = \frac{e^{-\left(\frac{x-\mu}{\sigma}\right)}}{\sigma\left(1+e^{-\left(\frac{x-\mu}{\sigma}\right)}\right)^2}, -\infty < x < \infty$$

and

$$G(x) = \left(1+e^{-\left(\frac{x-\mu}{\sigma}\right)}\right)^{-1}, -\infty < x < \infty \qquad (1)$$

respectively where $\mu \leq x$ is the location parameter and $\sigma > 0$ is the scale parameter.

Balakrishnan and Leung [1] generalized (1) into type I, type II and type III generalized logistic distribution while type I is positively skewed, type II is negatively skewed and type III is symmetric in nature. The probability density function and cumulative distribution function of type I generalized logistic distribution are given by

$$g(x) = \frac{be^{-x}}{(1+e^{-x})^{b+1}}, -\infty < x < \infty, b > 0$$

and

$$G(x) = (1+e^{-x})^{-b}, -\infty < x < \infty, \mathrm{b} > 0 \qquad (2)$$

respectively with $b > 0$ is the shape parameter.

Many works have been done on the transmutation of baseline distributions in view to ensuring better goodness of fit in data analysis from various disciplines. Shaw et al [2] suggested a quadratic transmutation map and applied it to probability models such as exponential, uniform and normal distribution. Aryal et al [3] worked on the transmuted extreme value distribution, Aryal et al [4] on transmuted Weilbul distribution, Aryal et al [5] on transmuted log-logistic, Merovci et al [6] on generalized transmuted family of distributions, Merovci et al [7] on transmuted Lindley-geometric Distribution, Merovci et al [8] on transmuted generalized Rayleigh distribution, Merovci et al [9] on transmuted Pareto distribution, Merovci et al [10] on transmuted Lindley distribution, AL-Kadim et al [11] on Cubic transmuted Weilbul distribution, Granzoto et al [12] on cubic transmuted distributions, Rahman et al [13] generalized the work of Shaw et al [1], Adeyinka et al [14] on four parameters transmuted generalized distribution, Adeyinka et al [15] on transmuted half logistic distribution, Adeyinka et al [16] on transmuted type I generalized half logistic distribution to mention a few.

The transmutation of (1) has been considered by Adeyinka [17]. The properties, estimation issue and the applicability of the model in the analysis of data have been established.

This work intends to study the transmutation of type I generalized logistic distribution for the purpose of achieving better performance of the model in the analysis of data in various disciplines.

2. Methodology

2.1. Derivation of Transmuted Type I Generalized Logistic Distribution

If a random variable X has the type I generalized logistic distribution with probability density function and its cumulative distribution function given in (2). The corresponding transmuted type I generalized logistic distribution, using the quadratic rank transmutation map,

$$F(x) = (1+x)G(x) - \lambda G^2(x), |\lambda| \leq 1 \tag{3}$$

is given by

$$F(x) = \frac{(1+\lambda)(1+e^{-x})^b - \lambda}{(1+e^{-x})^{2b}}, -\infty < x < \infty, b > 0 \tag{4}$$

and the corresponding pdf is obtained by differentiating (4) with respect to x to have

$$f(x) = \frac{be^{-x}\{(1+\lambda)(1+e^{-x})^b - 2\lambda\}}{(1+e^{-x})^{2b+1}}, -\infty < x < \infty, b > 0 \tag{5}$$

Where $b > 0$ is the shape parameter and λ is the transmutation parameter.

2.2. Graphical Illustration of the Model

Figures 1 and 2 illustrate the graphs of pdf and cdf of transmuted type I generalized logistic distribution for some selected values of parameters b and λ respectively.

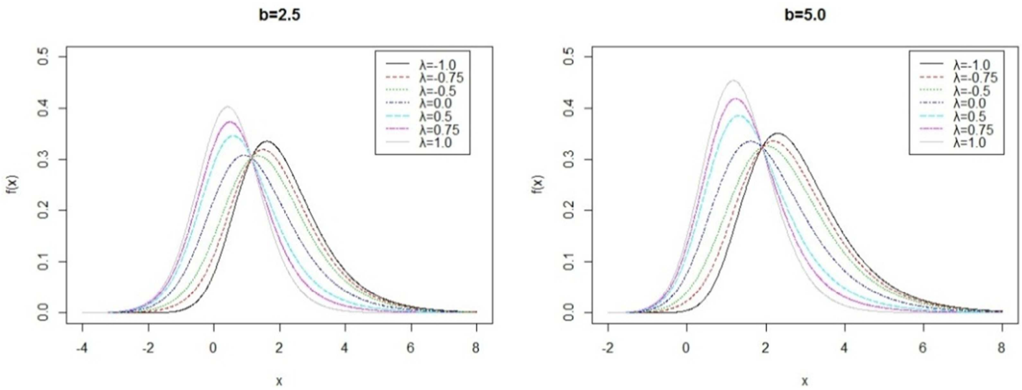

Figure 1. *The probability distribution function of transmuted type I generalized logistic distribution.*

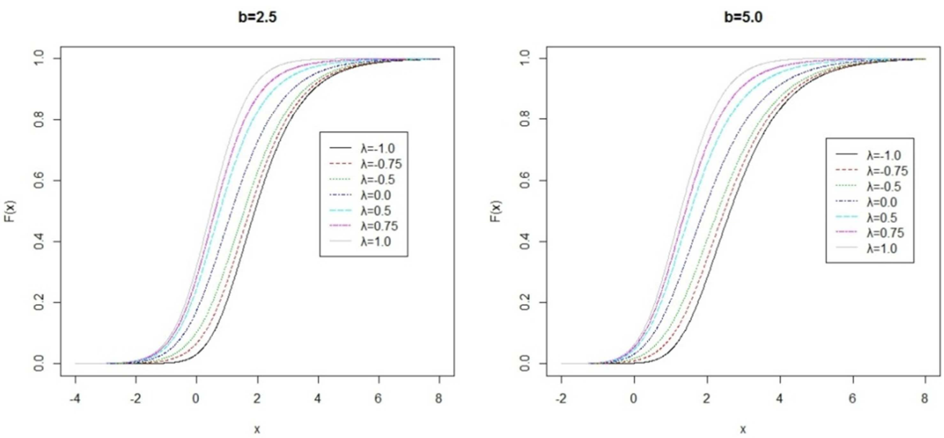

Figure 2. *The cumulative distribution function of transmuted type I generalized logistic distribution.*

3. Result

3.1. Moment Generating Function and Quantiles

The moment generating function of transmuted type I generalized logistic random variable X, is given by

$$M_X(\mathrm{t}) = \mathrm{E}[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x)dx \quad (6)$$

where $f(x)$ is the pdf of transmuted type I generalized logistic distribution. This implies

$$(1+\lambda)\int_{-\infty}^{\infty} \frac{e^{-x(1-t)}}{(1+e^{-x})^{b+1}}dx - 2\lambda \int_{-\infty}^{\infty} \frac{e^{-x(1-t)}}{(1+e^{-x})^{2b+1}}dx \quad (7)$$

Setting $y = e^{-x}, x = -\ln y$ and $dx = -\frac{1}{y}dy$. By substituting these into (7) it becomes

$$(1+\lambda)\int_{0}^{\infty} \frac{y^{-t}}{(1+y)^{b+1}}dy - 2\lambda \int_{0}^{\infty} \frac{y^{-t}}{(1+y)^{2b+1}}dy \quad (8)$$

Setting $u = \frac{1}{1+y}, y = \frac{1-u}{u}$ and $dy = -\frac{1}{u^2}dy.$

By substituting these into (8) it becomes

$$(1+\lambda)\int_0^1 (1-u)^{-t} u^{b+t-1}\, du - 2\lambda \int_0^1 (1-u)^{-t} u^{2b+t-1} \quad (9)$$

The integral in (9) is a beta function given by

$$B(a,b) = \int_0^1 x^{a-1}\,(1-x)^{b-1}dx$$

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \text{ and } \Gamma(n) = (n-1)$$

Using these functional relationships (9) becomes

$$(1+\lambda)\frac{\Gamma(b+t)\Gamma(1-t)}{\Gamma(b)} - \lambda \frac{\Gamma(2b+t)\Gamma(1-t)}{\Gamma(2b)} \quad (10)$$

Similarly, the characteristic function transmuted type I generalized logistic distribution is given by

$$Q_X(it) = (1+\lambda)\frac{\Gamma(b+it)\Gamma(1-it)}{\Gamma(b)} - \lambda \frac{\Gamma(2b+it)\Gamma(1-it)}{\Gamma(2b)} \quad (11)$$

The mean, variance, skewness and kurtosis of the transmuted type I generalized logistic distribution can be obtained from (10).

The q^{th}-quantile of the transmuted type I generalized logistic distribution is given by

$$F_X(x_q) = q \quad (12)$$

Where $F_X(x)$ is the cdf of transmuted type I generalized logistic distribution. This implies

$$\frac{(1+\lambda)(1+e^{-x_q})^b - \lambda}{(1+e^{-x_q})^{2b}} = q \quad (13)$$

By solving for x_q in (13) it becomes

$$x_q = -ln\left\{\left[\frac{1+\lambda-\sqrt{(1+\lambda)^2-4\lambda q}}{2q}\right]^{1/b} - 1\right\} \quad (14)$$

The median of transmuted type I generalized logistic distribution is obtained by setting $q = 0.5$ in (14) to have

$$x_{0.5} = -ln\left\{\left(1+\lambda-\sqrt{1+\lambda^2}\right)^{1/b} - 1\right\} \quad (15)$$

3.2. Random Number Generation

Random numbers can be generated from the transmuted type I generalized logistic distribution by using the method of inversion as follows:

$$F_X(x) = u \quad (16)$$

Where $u \sim U(0,1)$ i.e u is uniformly distributed on (0,1) and $F_X(x)$ is the cdf of transmuted type I generalized logistic distribution. Therefore, (16) implies

$$\frac{(1+\lambda)(1+e^{-x})^b - \lambda}{(1+e^{-x})^{2b}} = u \quad (17)$$

By solving (17) for x it becomes

$$x = -ln\left\{\left[\frac{1+\lambda-\sqrt{(1+\lambda)^2-4\lambda u}}{2u}\right]^{1/b} - 1\right\} \quad (18)$$

Random numbers can be generated from (18) when the value of parameter$s\ b\ and\ \lambda$ are known.

3.3. Order Statistics

We know that if $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$ denotes the order statistics of a random sample $X_1, X_2 \dots X_n$ from a continuous population with cdf $F_X(x)$ and pdf $f_X(x)$, David [18] gave the probability density function of $X_{(r)}$ as

$$f_{X_{(r)}}(x) = \frac{1}{B(r,n-r+1)}[F(x)]^{r-1}[1-F(x)]^{n-r} f(x) \quad (19)$$

where $r = 1,2, \dots n$.

We have from (2) the pdf of the r^{th} order of the logistic random variable $X_{(r)}$ given by

$$g_{X_{(r)}}(x) = \frac{be^{-x}\{(1+e^{-x})^b - 1\}^{n-r}}{B(r,n-r+1)(1+e^{-x})^{bn+1}} \quad (20)$$

Therefore the pdf of the nth order logistic statistic $X_{(n)}$ is given by

$$g_{X_{(n)}}(x) = \frac{nb\, e^{-x}}{(1+e^{-x})^{bn+1}} \quad (21)$$

and the pdf of the 1^{st} order logistic distribution statistic $X_{(1)}$ is given by

$$g_{X_{(1)}}(x) = \frac{nbe^{-x}\{(1+e^{-x})^b - 1\}^{n-1}}{(1+e^{-x})^{bn+1}} \quad (22)$$

Note that in a particular case of $n = 2$, (21) yields

$$g_{X_{(2)}}(x) = \frac{2be^{-x}}{(1+e^{-x})^{2b+1}} \quad (23)$$

and (22) yields

$$g_{X_{(1)}}(x) = \frac{2be^{-x}\{(1+e^{-x})^b - 1\}}{(1+e^{-x})^{2b+1}} \quad (24)$$

It can be observed that $max(X_1, X_2)$ and $min(X_1, X_2)$ in (23) and (24) are special cases of (5) for $\lambda = -1$ and $\lambda = 1$ respectively.

The distribution of the order statistics for the transmuted type I generalized logistic variable.

The pdf of the r^{th} order statistics for the transmuted type I generalized logistic distribution is given by

$$f_{X_{(r)}}(x) = \frac{be^{-x}\{(1+\lambda)(1+e^{-x})^b - 2\lambda\}\{(1+\lambda)(1+e^{-x})^b - \lambda\}^{r-1}\{(1+e^{-x})^{2b} - [(1+\lambda)(1+e^{-x})^b - \lambda]\}^{n-r}}{B(r, n-r+1)(1+e^{-x})^{2bn+1}} \quad (25)$$

Therefore the pdf of the largest order statistic $X_{(n)}$ is obtained when $r = n$ in (25) and it is given by

$$f_{X_{(n)}}(x) = \frac{nb\, e^{-x}\{(1+\lambda)(1+e^{-x})^b - 2\lambda\}\{(1+\lambda)(1+e^{-x})^b - \lambda\}^{r-1}}{(1+e^{-x})^{2bn+1}} \quad (26)$$

and the pdf of the smallest order statistic $X_{(1)}$ is obtained when $r = 1$ in (25) and it is given by

$$f_{X_{(1)}}(x) = \frac{nbe^{-x}\{(1+\lambda)(1+e^{-x})^b - 2\lambda\}\{(1+e^{-x})^{2b} - [(1+\lambda)(1+e^{-x})^b - \lambda]\}^{n-1}}{(1+e^{-x})^{2bn+1}} \quad (27)$$

3.4. Estimation of Parameters

Given a sample $X_1, X_2 \dots X_n$ of size n from transmuted type I generalized logistic distribution with the probability density function given by

$$f(x; b, \lambda) = \frac{be^{-x}\{(1+\lambda)(1+e^{-x})^b - 2\lambda\}}{(1+e^{-x})^{2b+1}}, -\infty < x < \infty, b > 0$$

where $b > 0$ is the shape parameter and $|\lambda| \leq 1$ is the transmutation parameter.

The likelihood function of transmuted type I generalized logistic distribution is given by

$$L = b^n e^{-\sum_{i=1}^n x_i} \prod_{i=1}^n \{(1+\lambda)(1+e^{-x_i})^b - 2\lambda\} \,/\, \prod_{i=1}^n (1+e^{-x})^{2b+1} \quad (28)$$

The log-likelihood function of (28) becomes

$$lnL = nlnb - \sum_{i=1}^n x_i - (2b+1)\sum_{i=1}^n ln\,(1+e^{-x_i}) + \sum_{i=1}^n ln\{(1+\lambda)(1+e^{-x_i})^b - 2\lambda\} \quad (29)$$

The maximum likelihood estimate of the parameters b and λ that are contained within the transmuted type I generalized logistic distribution function is obtained by differentiating (29) with respect to parameter b and λ respectively and equating the result to zero to have

$$\frac{\partial lnL}{\partial \lambda} = \sum_{i=1}^n \frac{(1+e^{-x_i})^b - 2}{[(1+\lambda)(1+e^{-x_i})^b - 2\lambda]} = 0$$

$$\frac{\partial lnL}{\partial b} = \frac{n}{b} - 2\sum_{i=1}^n ln\,(1+e^{-x_i}) + b(1+\lambda)\sum_{i=1}^n \frac{(1+e^{-x_i})^{b-1}}{[(1+\lambda)(1+e^{-x_i})^b - 2\lambda]} = 0$$

The maximum likelihood estimator $\hat{\vartheta} = (\hat{b}, \hat{\lambda})'$ of parameters $\vartheta = (b, \lambda)'$ can be obtained by solving this non-linear system of equations. It is usually more convenient to use non-linear optimization algorithms such as quasi-Newton algorithm to numerically maximize the log-likelihood function in (29).

4. Discussion

4.1. Data Presentation

The data used in this work is extracted from Gupta et al [19]. It is a strength data originally considered by Badar and Priest [20]. The data which represent the strength measured in GPA, for single carbon fibers and impregnated 1000-carbon fiber tows. Single fibers were tested under tension at gauge lengths of 1, 10, 20 and 50 mm. Impregnated tows of 1000 fibers were tested at gauge lengths of 20, 50, 150 and 300 mm. For illustrative purposes we are considering the single fibers data set of 10 mm in gauge lengths with sample size 63. The data are presented below.

1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020.

4.2. Data Analysis

A quasi Newton algorithm was implemented in R package and the performance of the models are shown in Table 1. Akaike Information criterion (AIC), Corrected Akaike Information criterion (AICC) and Bayesian Information criterion (BIC) were respectively used to compare the performance of transmuted type I generalized logistic distribution (TGLD) to its parent model (GLD) in (2).

$$AIC = 2k - 2LL$$

$$AICC = AIC + \frac{2k(k+1)}{n-k-1}$$

and

$$BIC = 2\log(n) - 2LL$$

Where k is the number of parameters in the model, n is the sample size and LL is the maximized value of log likelihood function.

Table 1. *Performance of the models.*

Model	Estimates	-LL	AIC	AICC	BIC
TGLD	$\hat{b}$ = 2.76 $\hat{\lambda}$=0.527	107.256	218.512	218.712	218.111
GLD	$\hat{b}$ =0.652	114.315	230.630	230.696	232.229

It can be concluded from the results in table 1 that the transmuted type I generalized logistic distribution (TGLD) performs better than its parent model which is the type I generalized logistic distribution (GLD).

5. Conclusion

In this article, we have introduced a new generalization of type I generalized logistic distribution called transmuted type I generalized logistic distribution. The distribution is generalized by the use of quadratic rank transmutation map. Some statistical properties such as moment generating function, quantile, median and characteristic function of the transmuted type I generalized logistic distribution are established and parameters estimation issues are addressed. The order statistics from the model are considered including the minimum and maximum order statistics and the relationships between the order statistics from its base distribution and the probability density function of the transmuted type I generalized logistic distribution for some selected values of transmutation parameter λ are demonstrated. The subject model is used to fit relevant data to illustrate its applicability. We expect that this study will serve as a reference and help to advance future research in this area and other related disciplines.

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