International Journal of Theoretical and Applied Mathematics
2017; 3(6): 203-209
http://www.sciencepublishinggroup.com/j/ijtam
doi: 10.11648/j.ijtam.20170306.15
ISSN: 2575-5072 (Print); ISSN: 2575-5080 (Online)

On the Equilibrium Without Loss in the Discrete Time Models of Economic Dynamics

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To cite this article:
Sabir Isa Hamidov. On the Equilibrium Without Loss in the Discrete Time Models of Economic Dynamics. *International Journal of Theoretical and Applied Mathematics*. Vol. 3, No. 6, 2017, pp. 203-209. doi: 10.11648/j.ijtam.20170306.15

Received: August 8, 2017; **Accepted**: November 9, 2017; **Published**: December 13, 2017

Abstract: The model of economic dynamics with a fixed budget is considered. The conditions are derived under which the model with a fixed budget has an equilibrium state with the equilibrium prices. The necessary and sufficient conditions for the existence of equilibrium prices are found.

Keywords: Production Function, Equilibrium State, Growth Rate

1. Introduction

Let's consider a model defined at the moment t by the productive mapping $a(x)$ [1-3, 10]:

$$a(x) = \left\{ \tilde{x} = (\tilde{x}^{1\cdot}, \dots, \tilde{x}^{n\cdot}) \in (R_+^n)^n \middle| 0 \le \sum_{i=1}^{n} \tilde{x}^{i\cdot} \le \right.$$

$$\le \sum_{i=1}^{n} B^k \cdot x^{k\cdot} + \left(F^1(x^{1\cdot}), \dots, F^n(x^{n\cdot})\right), x^{k\cdot}$$

$$\left. = (x^{k1}, \dots, x^{kn}), k = \overline{1,n} \right\},$$

where $x = (x^{1\cdot}, \dots, x^{n\cdot}) \in (R_+^n)^n, B^k$ is a diagonal matrix, the main diagonal of which has a form

$$(v^{k1}, \dots, v^{kn}), v^{ki} \in [0,1] \; (k, i = \overline{1,n}),$$

$$F^j(x) = \min_{i=\overline{1,n}} \frac{x^i}{c^{ij}}, c^{ij} > 0 \; (i,j = \overline{1,n}).$$

The productive mapping a^k of the branch k is in the form

$$a^k(x) = \langle 0, B^k x + (0, \dots, 0, F^k(x), 0, \dots, 0) \rangle \; (x \in R_+^n).$$

Let's define the conditions under which the model with fixed budgets

$$M = \{y, U(\ell), \Omega\}$$

has an equilibrium state $(P, x^1, \dots, x^n, y)$ with equilibrium prices $P = (P^1, \dots, P^n)$. Here as usual $U(\ell) = \left(U^1(\ell,\cdot), \dots, U^n(\ell,\cdot)\right)$, in its turn $U^i(\ell, x) = [\ell, B^k x] + \ell^i \cdot F^k(x)$; $\Omega = (\lambda^1, \dots, \lambda^n)$ is a budget vector, $\ell = (\ell^1, \dots, \ell^n)$ is a prices vector.

Let $I = \{1, 2, \dots, n\}$.

The following sets are introduced in [4]

$$I_1(x) = \{i \in I | x^i = 0\}.$$

$$I_2(x) = \{i \in I | x^i > 0\},$$

$$R^k(x) = \left\{ i \in I \middle| \frac{x^i}{c^{ik}} = \min \frac{x^j}{c^{jk}} \right\}, (k \in I),$$

$$Q^k(x) = I \backslash R^k(x), k \in I.$$

The problem of $k-th$ consumer is as follows:

$$U^k(\ell, x) \to \max, x \in V = \{x \ge 0, [P, x] = 1\}. \quad (1)$$

Let $\bar{x}^{k\cdot}$ be a maximum point in (1) $(k \in I)$.

2. Main Part

Definition. The equilibrium $(P, \bar{x}^{1\cdot}, \dots, \bar{x}^{n\cdot}, \Omega, y)$ is called equilibrium without loss if for all $k \in I$ is valid

$$R^k(\bar{x}^{k\cdot}) = I.$$

Definition. The prices $P = (P^1, \dots, P^n)$ defined in the equilibrium without loss is called an equilibrium prices without loss.

Note that due to the consequence of the lemma 1 [4] we have $I_1(\bar{x}^{k\cdot}) = \emptyset$.

First, consider the consumer problem without loss, and then the equilibrium without loss.

Note that the solution of the $k-th$ problem without loss has the following property

$$\bar{x}^{k\cdot} = \gamma^k \cdot c^{\cdot k} \ (k \in I), \tag{2}$$

where $\gamma^k \geq 0$ is some constant that is equal to the value of the $k-th$ production function in the point $\bar{x}^{k\cdot}$.

Indeed from the conditions $R^k(\bar{x}^{k\cdot}) = I, I_1(\bar{x}^{k\cdot}) = \emptyset$ we get that

$$\frac{\bar{x}^{k1}}{c^{1k}} = \dots = \frac{\bar{x}^{kn}}{c^{nk}} = \gamma^k, \bar{x}^{ki} > 0 \ (k, i \in I).$$

Consider the problem of kth consumer without loss. To analyze the problem (1) we apply the necessary and sufficient conditions for the extremum, wherein in the point $\bar{x}$ the maximum is reached if and only if when

$$(U^k)'(\bar{x}, g) \leq 0 \ for \ all \ g \in G_{\bar{x}}(V),$$

where

$$G_{\bar{x}}(V) = \{g \in R^n \mid [P, g] = 0, g^i \geq 0 \ \forall \ i \in I_1(\bar{x})\}.$$

As is known [2], $(U^k)'(\bar{x}, g) = q^k(g)$, where

$$q^k(g) = [\ell_\nu^k, g] + \ell^k \cdot \min_{i \in R^k(\bar{x})} \frac{g^i}{c^{ik}} \ (k \in I).$$

Then in our case (without loss) we get that the necessary and sufficient optimality conditions for $\bar{x}$ of the branch k take the forms

$$q^k(g) = [\ell_\nu^k, g] + \ell^k \cdot \min_{i \in I} \frac{g^i}{c^{ik}} \leq 0 \ \forall \ g \in \Omega = \{g \in R^n \mid [P, g] = 0\}. \tag{3}$$

Lemma 1. The number $\mu^k \ (k \in I)$, defined in the lemma 4 [4] in the case of without loss $(R^k(\bar{x}) = I)$ coincides with the maximal growth rate of the total wealth of the $k-th$ branch and is equal to

$$\mu^k = \frac{\ell^k + [\ell_\nu^k, c^{\cdot k}]}{[P, c^{\cdot k}]} \ (k \in I), \tag{4}$$

where $\ell_\nu^k = (\ell^1 \cdot \nu^{k1}, \dots, \ell^n \cdot \nu^{kn})$.

Proof. In the case, when $R^k(x) = I$, as follows from lemma 4 [4] the number μ^k is in the form of (4).

From the other hand let $\bar{x}$ be a maximum point in the problem (1) of the kth consumer that is indeed similar to the following relations

$$\max_{x \geq 0} \frac{U^k(\ell, x)}{[P, x]} = \frac{U^k(\ell, \bar{x})}{[P, \bar{x}]} \ (k \in I),$$

where the functions $U^k(\ell, x)$ are defined in the introduction.

From the definition of μ^k and (2) we get that

$$\max_{x \geq 0} \mu^k(x) = \mu^k,$$

where μ^k is of the form (4).

Lemma is proved.

Theorem 1. Let strongly positive vector $P = (P^1, \dots, P^n)$ be given, index $k \in I$ and the number μ^k is defined by (4). The vector $\bar{x}$ is a solution of the problem (1), satisfying the relation

$$R^k(\bar{x}) = I.$$

Then and only then

$$\ell^j \cdot \nu^{kj} \leq \mu^k \cdot P^j \ \forall \ j \in I, \tag{5}$$

$$P \in \frac{1}{\mu^k} \cdot (\ell_\nu^k + \partial\tilde{q}^k), \tag{6}$$

where $\ell_\nu^k, \partial\tilde{q}^k$ are defined in the lemma 3 [4] for the case $R^k(\bar{x}) = I$.

The proof follows from the consequence of 1 [4], theorem 1 [4] and lemma 1

Remark. By given $\mu^k \ (k \in I)$, the equality (3) may be considered as a system of n linear equation with respect to variables- coordinates of equilibrium vector of prices P without loss

$$[P, c^{\cdot k}] = \frac{1}{\mu^k} \cdot \left[\ell^k + [\ell_\nu^k, c^{\cdot k}]\right] \ (k \in I),$$

where $c^{\cdot k} = (c^{1k}, \dots, c^{nk}), \ell_\nu^k = (\ell^1 \cdot \nu^{k1}, \dots, \ell^n \cdot \nu^{kn})$ and in contrary by the given prices P from the equality (3) are defined uniquely the maximal growth rate μ^k of the total wealth of the $k-th$ branch $(k \in I)$.

Let's consider the following problem. Let $\nu^{ki} \geq 0, c^{ik} > 0 \ (i, k \in I)$ be given. By which $\ell^i > 0, \mu^k > 0 \ (i, k \in I)$ there exists the vector $P = (P^1, \dots, P^n)$, that for some $\lambda^1, \dots, \lambda^n, y$ is an equilibrium prices without loss in the model M, defined by the set $\{\ell, \lambda^1, \dots, \lambda^n, y\}$.

Note that it follows from the lemma 3 [4] that in the case of lossless $(R^k(\bar{x}) = I)$ subdifferential $\partial\tilde{q}^k$ takes the form [5, 6]

$$\partial\tilde{q}^k = \left\{f = \ell^k \cdot (f^1, \dots, f^n) \mid \exists \ \alpha^i \geq 0: \textstyle\sum_{i \in I} \alpha^i = 1, \ f^i = \frac{\ell^k \cdot \alpha^i}{c^{ik}} \ \forall \ i \in I\right\}. \tag{7}$$

It can be shown that for each set $\mu^1 > 0, \dots, \mu^n > 0$ there are weights $\lambda^1, \dots, \lambda^n$ such that the growth rate in the model defined by the weights $\lambda^1, \dots, \lambda^n$, coincides up to those cells in which $\lambda^i = 0$ and the growth rate does not depend on the choice of the equilibrium price P.

Lemma 2. Let the numbers $\ell^i > 0, \mu^k > 0, \nu^{ji} \geq 0, c^{ij} > 0 \ (i, j, k \in I)$ be given and $(P, x^{1\cdot}, \dots, x^{n\cdot})$ is an equilibrium without loss in the model with utility functions U^j, defined in the introduction section by the budgets $\lambda^i = [P, x^j]$ and distributed by the vector $y = \sum_{i=1}^n x^{i\cdot} \ (j \in I)$. The relation (6) is fulfilled for $\forall \ k \in I$ then and only then when for any

$v^{ji} \geq 0$ and u^j $(i, j \in I)$, satisfying the relations

$$\sum_{j\in I} \mu^j \cdot (v^{ji} + u^j \cdot c^{ij}) = 0 \; \forall \, i \in I, \tag{8}$$

the inequality below is valid

$$\sum_{j\in I}\left(\sum_{i\in I} v^{ji} \cdot \ell^i \cdot v^{ji} + u^j \cdot \left(\ell^j + \sum_{i\in I} \ell^i \cdot v^{ji} \cdot c^{ij}\right)\right) \leq 0. \tag{9}$$

Proof. It follows from (7) that

$$\partial\tilde{q}^k = \left\{(\beta^1, \dots, \beta^n) \mid \sum_{i\in I} c^{ik} \cdot \beta^i = \ell^k, \beta^i \geq 0\right\} \; (k \in I).$$

Denote the set of the form $\frac{1}{\mu^k}\left(\ell_v^k + \partial\tilde{q}^k\right)$ by Φ^k, that considering lemma 3 [4] (in the case $R^k(\bar{x}) = I$) indeed has a form

$$\Phi^k = \left\{\frac{1}{\mu^k} \cdot (\gamma^1, \dots, \gamma^n) \mid \gamma^i = \ell^i \cdot v^{ki} + \beta^i, \sum_{i\in I} c^{ik} \cdot \beta^i = \ell^k, \beta^i \geq 0\right\} =$$

$$= \left\{\frac{1}{\mu^k} \cdot (\gamma^1, \dots, \gamma^n) \mid \sum_{i\in I} c^{ik} \cdot (\gamma^i - \ell^i \cdot v^{ki}) = \ell^k, \gamma^i \geq \ell^i \cdot v^{ki}\right\} =$$

$$= \left\{\frac{1}{\mu^k} \cdot \gamma \mid \sum_{i\in I} c^{ik} \cdot \gamma^i = \ell^k + \sum_{i\in I} \ell^i \cdot v^{ki} \cdot c^{ik}, \gamma^i \geq \ell^i \cdot v^{ki}\right\} \; (k \in I).$$

Since for all $k \in I$ due the conditions of the lemma3 [4] (6) is fulfilled we have

$$P \in \bigcap_{k\in I} \Phi^k, where\ P = (P^1, \dots, P^n).$$

Thus

$$\mu^j \cdot P^i \geq \ell^i \cdot v^{ji} \; (i, j \in I),$$

$$\mu^j \cdot \sum_{i\in I} P^i \cdot c^{ij} = \ell^j + \sum_{i\in I} \ell^i \cdot v^{ji} \cdot c^{ij} \; (j \in I).$$

This system may be written as a system of inequalities

$$\begin{cases} \mu^j \cdot P^i \geq \ell^i \cdot v^{ji} \; (i, j \in I), \\ \mu^j \cdot \sum_{i\in I} P^i \cdot c^{ij} \geq \ell^j + \sum_{i\in I} \ell^i \cdot v^{ji} \cdot c^{ij} \; (j \in I), \\ -\mu^j \cdot \sum_{i\in I} P^i \cdot c^{ij} \geq -(\ell^j + \sum_{i\in I} \ell^i \cdot v^{ji} \cdot c^{ij}) \; (j \in I). \end{cases} \tag{10}$$

Let's introduce the denotations

$$\begin{cases} f^{ji} = e^i \cdot \mu^j, \\ f^j = \mu^j \cdot (c^{1j}, c^{2j}, \dots, c^{nj}), \\ f^{n+j} = -f^j, \\ \beta^{ji} = \ell^i \cdot v^{ji}, (i, j \in I), \\ \beta^j = \ell^j + \sum_{i\in I} \ell^i \cdot v^{ji} \cdot c^{ij}, \\ \beta^{n+j} = -\beta^j, \end{cases} \tag{11}$$

where e^i is $i - th$ ort in the space R_+^n.

Rewriting the system (10) in new denotation considering (11) we obtain

$$\begin{cases} [f^{ji}, P] \geq \beta^{ji} \; (i, j \in I), \\ [f^j, P] \geq \beta^j \; (j \in I), \\ [f^{n+j}, P] \geq \beta^{n+j} \; (j \in I). \end{cases} \tag{12}$$

Then, by the theorem of [7] for the compatibility of the system (12) is necessary and sufficient that for any $v^{ji} \geq 0, v^j \geq 0, v^{n+j} \geq 0$ $(i, j \in I)$ from the equality

$$\sum_{i,j\in I} v^{ji} \cdot f^{ji} + \sum_{j\in I} v^j \cdot f^j + \sum_{j\in I} v^{n+j} \cdot f^{n+j} = 0$$

follow the inequality

$$\sum_{i,j\in I} v^{ji} \cdot \beta^{ji} + \sum_{j\in I} v^j \cdot \beta^j + \sum_{j\in I} v^{n+j} \cdot \beta^{n+j} \leq 0.$$

In our case necessary and sufficient conditions have a form

$$\sum_{j\in I} \mu^j \cdot (v^{ji} + (v^j - v^{n+j}) \cdot c^{ij}) = 0 \text{ for } \forall \, i \in I,$$

$$\sum_{j\in I}\left(\sum_{i\in I} v^{ji} \cdot \ell^i \cdot v^{ji} + (v^j - v^{n+j}) \cdot \left(\ell^j + \sum_{i\in I} \ell^i \cdot v^{ji} \cdot c^{ij}\right)\right) \leq 0. \tag{13}$$

Let $u^j = v^j - v^{n+j}, \forall \, i \in I$.

Then it follows from (13) that u^j are such that

$$\sum_{j\in I} \mu^j \cdot u^j \cdot c^{ij} \leq 0 \; for \; \forall \, i \in I.$$

Substituting u^j $(j \in I)$ into (13) we get (8), (9).

Lemma is proved.

Consider the matrix $C = (c^{ij})_{i,j=1}^n$. Let $|C|$ be its determinant. It is valid Lemma 3. Let the numbers $v^{ji} \geq 0, c^{ij} > 0$ $(i, j \in I)$ be given and $|C| \neq 0$. The following conditions are equivalent

1. numbers $v^{ji} \geq 0, u^j$ $(i, j \in I)$ are such that (8) and (9) are fulfilled;
2. numbers ℓ^i, μ^j $(i, j \in I)$ are such that for $\forall \, i, j \in I$ is valid

$$\ell^i \cdot v^{ji} + \frac{1}{|C|} \cdot \sum_{i\in I} (-1)^{i+k+1} \cdot \frac{\mu^j}{\mu^k} \left(\ell^k + \sum_{m\in I} \ell^m \cdot v^{km} \cdot c^{mk}\right) \left|C_i^k\right| \leq 0, \tag{14}$$

where C_i^k is $(n-1) \times (n-1)$ matrix obtained from the matrix C by removing $k - th$ column and $i - th$ row.

Proof. The system of equalities (8) we rewrite as follows

$$\sum_{j\in I}\mu^j\cdot u^j\cdot c^{ij}=-\sum_{j\in I}\mu^j\cdot v^{ji},\forall\, i\in I. \qquad (15)$$

Consider the equation

$$A\cdot u=-b, \qquad (16)$$

where

$$A=\begin{pmatrix}\mu^1\cdot c^{11} & \mu^2\cdot c^{12} & \dots & \mu^n\cdot c^{1n}\\ \mu^1\cdot c^{21} & \mu^2\cdot c^{22} & \dots & \mu^n\cdot c^{2n}\\ \dots & \dots & \dots & \dots\\ \mu^1\cdot c^{n1} & \mu^2\cdot c^{n2} & \dots & \mu^n\cdot c^{nn}\end{pmatrix}, b=\begin{pmatrix}b^1\\ b^2\\ \dots\\ b^n\end{pmatrix},$$

$$b^m=\sum_{k\in I}\mu^k\cdot v^{km}\ (m\in I).$$

Note that

$$|A|=\prod_{j\in I}\mu^j\cdot|C|,$$

where $|C|$ is a determinant of the matrix.

As is known [8] the equation (16) has a solution (note that $|C|\neq 0$):

$$u^j=-\frac{|A^j|}{|A|}\ (j\in I),$$

where A^j is $n\times n$ matrix obtained from the matrix A by replacing $j-th$ column by the row b.

Expanding the determinant of the matrix A^j over the element of the $j-th$ column (t refers the column b), we get

$$|A^j|=\frac{1}{\mu^j}\cdot\prod_{j\in I}\mu^k\cdot|C^j| =\frac{1}{\mu^j}\cdot\prod_{j\in I}\mu^k\cdot\sum_{m\in I}(-1)^{m+j}\cdot b^m\cdot\left|C_m^j\right|\ (j\in I),$$

here C_m^j is $(n-1)\times(n-1)$ matrix obtained from the matrix C by removing $j-th$ column and $m-th$ row.

Substituting the values of $|A|,|A^j|$ into the solution of the equation (16), we obtain

$$u^j=\frac{1}{\mu^j\cdot|C|}\cdot\sum_{m\in I}(-1)^{m+j+1}\cdot\left(\sum_{k\in I}\mu^k\cdot v^{km}\right)\cdot\left|C_m^j\right|\ (j\in I). \qquad (17)$$

Note that (16) is indeed matrix form of the system of equations (15) relatively u.

Then

$$\sum_{j\in I}\left(\sum_{i\in I}v^{ji}\cdot\ell^i\cdot\nu^{ji}+u^j\cdot\left(\ell^j+\sum_{i\in I}\ell^i\cdot\nu^{ji}\cdot c^{ij}\right)\right)=$$

$$=\sum_{j\in I}\left[\sum_{i\in I}v^{ji}\cdot\ell^i\cdot\nu^{ji}+\frac{1}{\mu^j\cdot|C|}\cdot\left(\ell^j+\sum_{i\in I}\ell^i\cdot\nu^{ji}\cdot c^{ij}\right)\times\right. \qquad (18)$$

$$\left.\times\sum_{m\in I}(-1)^{m+j+1}\cdot\left(\sum_{k\in I}\mu^k\cdot v^{km}\right)\cdot\left|C_m^j\right|\right],$$

where $|C|\neq 0$.

The expression (17) after some transformations may be reduced to the form

$$\sum_{j\in I}\sum_{i\in I}d^{ji}\cdot v^{ji}=[d,v], \qquad (19)$$

where $d=(d^{11},d^{12},\dots,d^{nn}), v=(v^{11},v^{12},\dots,v^{nn})$, and d^{ji} are corresponding coefficients at v^{ji}, depending on $\ell^i,\nu^{ji},c^{ij},\mu^j$:

$$d^{ji}=d^{ji}(\ell^i,\nu^{ji},c^{ij},\mu^j)\ (i,j\in I).$$

To reduce the expression (19) to the form (19), it is necessary in (18) pass from v^{km} to v^{ji}. To do this we accept reindexing "k" ↔ "j","m" ↔ "i":

$$\sum_{j\in I}\frac{1}{\mu^j\cdot|C|}\cdot\left(\ell^j+\sum_{i\in I}\ell^i\cdot\nu^{ji}\cdot c^{ij}\right)\cdot\sum_{m\in I}(-1)^{m+j+1}\cdot\left(\sum_{k\in I}\mu^k\cdot v^{km}\right)\cdot\left|C_m^j\right|=$$

$$=\sum_{k\in I}\frac{1}{\mu^j\cdot|C|}\cdot\left(\ell^k+\sum_{m\in I}\ell^m\cdot\nu^{km}\cdot c^{mk}\right)\cdot\sum_{m\in I}(-1)^{i+k+1}\cdot\left(\sum_{j\in I}\mu^j\cdot v^{ji}\right)\times$$

$$\times\left|C_i^k\right|=\sum_{j\in I}\frac{1}{|C|}\cdot\sum_{k\in I}\frac{\mu^j}{\mu^k}\left(\ell^k+\sum_{m\in I}\ell^m\cdot\nu^{km}\cdot c^{mk}\right)\sum_{i\in I}(-1)^{i+k+1}\cdot\left|C_i^k\right|\cdot v^{ji}=$$

$$=\sum_{j\in I}\sum_{i\in I}\frac{1}{|C|}\cdot\sum_{k\in I}(-1)^{i+k+1}\cdot\frac{\mu^j}{\mu^k}\cdot\left(\ell^k+\sum_{m\in I}\ell^m\cdot\nu^{km}\cdot c^{mk}\right)\cdot\left|C_i^k\right|\cdot v^{ji}.$$

Then (18) turns to

$$\sum_{j\in I}\sum_{i\in I}\left(\ell^i\cdot\nu^{ji}+\frac{1}{|C|}\cdot\sum_{k\in I}(-1)^{i+k+1}\times\frac{\mu^j}{\mu^k}\cdot\left(\ell^k+\sum_{m\in I}\ell^m\cdot\nu^{km}\cdot c^{mk}\right)\cdot\left|C_i^k\right|\cdot v^{ji}\right).$$

Thus comparing the last one with (19), we arrive to

$$d^{ji} = \ell^i \cdot v^{ji} + \frac{1}{|C|} \cdot \sum_{k\in I}(-1)^{i+k+1} \cdot \frac{\mu^j}{\mu^k} \cdot (\ell^k + \sum_{m\in I} \ell^m \cdot v^{km} \cdot c^{mk}) \cdot |C_i^k|. \quad (20)$$

Then the condition (9) is equivalent to

$$[d, v] \le 0,$$

where scalar product $[d, v]$ is defined by the formula (19). It does not exist $v = (v^{11}, v^{12}, \dots, v^{1n})$ such that $[d, v] > 0$ then and only then when тогда и all $d^{ji} \le 0$ for $\forall\, i, j \in I$, where d^{ji} is defined by (20).

Lemma is proved.

It takes place

Theorem 2. Let the numbers $v^{ji} \ge 0$ и $c^{ij} > 0$ $(i, j \in I)$ are such that $\max_{j\in I} v^{ji} > 0$ и $|C| \ne 0$. The equilibrium prices without loss by given v^{ji}, c^{ji} and some $\ell^i > 0$ $(i \in I)$ and $\mu^j > 0$ $(j \in I)$ exist only and only when (14) is satisfied; the coefficients μ^k $(k \in I)$ and equilibrium prices P are related by the formula

$$\mu^k \cdot [P, c^{\cdot k}] = \ell^k + [\ell_v^k, c^{\cdot k}] \ (k \in I), \quad (21)$$

where $c^{\cdot k} = (c^{1k}, \dots, c^{nk}), \ell_v^k = (\ell^1 \cdot v^{k1}, \dots, \ell^n \cdot v^{kn})$.

The proof immediately follows from the lemmas 2 and 3.

Note 1. Note that the condition (14), that is necessary and sufficient condition of existence equilibrium prices without loss does not depend on the vector of distributed resources y.

Note 2. By given $v^{ji} \ge 0, c^{ij} > 0$ $(i, j \in I)$ the parameters ℓ^i, μ^j $(i, j \in I)$ is a solution of the system of inequalities (14).

Example. Rewrite the conditions (14) for the cases $n = 2, n = 3$ и $v^{ji} = 0, \mu^i = \mu^j$ $(i \ne j)$ $(i, j \in I)$.

a) by $n = 2$ we have

$$\frac{1}{|C|} \cdot [-(c^{22} + v^{11} \cdot c^{21} \cdot c^{12}) \cdot \ell^1 + (1 + v^{22} \cdot c^{22}) \cdot c^{21} \cdot \ell^2] \le 0,$$

$$\frac{1}{|C|} \cdot [-(1 + v^{11} \cdot c^{11}) \cdot c^{22} \cdot \ell^1 + (1 + v^{22} \cdot c^{22}) \cdot c^{21} \cdot \ell^2] \le 0,$$

$$\frac{1}{|C|} \cdot [(1 + v^{11} \cdot c^{11}) \cdot c^{12} \cdot \ell^1 - (1 + v^{22} \cdot c^{22}) \cdot c^{11} \cdot \ell^2] \le 0,$$

$$\frac{1}{|C|} \cdot [(1 + v^{11} \cdot c^{11}) \cdot c^{12} \cdot \ell^1 - (c^{11} + v^{22} \cdot c^{12} \cdot c^{21}) \cdot \ell^2] \le 0,$$

where $|C|$ is a determinant of the 2×2 matrix C;

b) by $n = 3$ conditions (14) turn to:

$$\left(v^{11} - \frac{|C_1^1|}{|C|} \cdot (1 + v^{11} \cdot c^{11})\right) \cdot \ell^1 + \frac{|C_1^2|}{|C|} \cdot (1 + v^{22} \cdot c^{22}) \cdot \ell^2 - \frac{|C_1^3|}{|C|} \cdot (1 + v^{33} \cdot c^{33}) \cdot \ell^3 \le 0,$$

$$\frac{1}{|C|} \cdot [-(1 + v^{11} \cdot c^{11}) \cdot |C_1^1| \cdot \ell^1 + (1 + v^{22} \cdot c^{22}) \cdot |C_1^2| \cdot \ell^2 - (1 + v^{33} \cdot c^{33}) \cdot |C_1^3| \cdot \ell^3] \le 0,$$

$$\frac{1}{|C|} \cdot [(1 + v^{11} \cdot c^{11}) \cdot |C_2^1| \cdot \ell^1 - (1 + v^{22} \cdot c^{22}) \cdot |C_2^2| \cdot \ell^2 - (1 + v^{33} \cdot c^{33}) \cdot |C_2^3| \cdot \ell^3] \le 0,$$

$$\frac{|C_2^1|}{|C|} \cdot (1 + v^{11} \cdot c^{11}) \cdot \ell^1 + \left(v^{22} - \frac{|C_2^2|}{|C|} \cdot (1 + v^{22} \cdot c^{22})\right) \cdot \ell^2 + \frac{|C_2^3|}{|C|} \cdot (1 + v^{33} \cdot c^{33}) \cdot \ell^3 \le 0,$$

$$\frac{1}{|C|} \cdot [-(1 + v^{11} \cdot c^{11}) \cdot |C_3^1| \cdot \ell^1 + (1 + v^{22} \cdot c^{22}) \cdot |C_3^2| \cdot \ell^2 - (1 + v^{33} \cdot c^{33}) \cdot |C_3^3| \cdot \ell^3] \le 0,$$

$$-\frac{|C_3^1|}{|C|} \cdot (1 + v^{11} \cdot c^{11}) \cdot \ell^1 + \frac{|C_3^2|}{|C|} \cdot (1 + v^{22} \cdot c^{22}) \cdot \ell^2 + \left(v^{33} - \frac{|C_3^3|}{|C|} \cdot (1 + v^{33} \cdot c^{33})\right) \cdot \ell^3 \le 0,$$

where $|C|$ is a determinant of the 3×3 matrix C, C_j^i is 2×2 matrix obtained from C by removing $i-th$ column and j −th row.

Introduce the numbers

$$d_i^{kj} = \begin{cases} \ell^i \cdot v^{ji} + (-1)^{i+j+1} \cdot \dfrac{|C_i^j|}{|C|} \cdot \left(\ell^j + \sum\limits_{m\in I} \ell^m \cdot v^{jm} \cdot c^{mj}\right), if\ k = j, \\ (-1)^{i+k+1} \cdot \dfrac{|C_i^k|}{|C|} \cdot \left(\ell^k + \sum\limits_{m\in I} \ell^m \cdot v^{km} \cdot c^{mk}\right), if\ k \ne j, (i, j, k \in I) \end{cases}$$

and vector $\tilde{d}^{kj}$:

$$\tilde{d}^{kj} = \begin{pmatrix} -d_1^{kj} \\ \vdots \\ \vdots \\ -d_n^{kj} \end{pmatrix} (k, j \in I). \quad (22)$$

Proposition 2. The numbers $\mu^j > 0$ $(j \in I)$, satisfying (14), exist if and only if when there exists the index $k_o \in I$ and $æ^k$ such that

$$æ^{k_o} > 0, \sum_{k=1}^{n^2} æ^k \cdot \tilde{d}^{kj} \ge 0 \ (j \in I), \quad (23)$$

where $\tilde{d}^{kj}$ is defined by (22).

Proof. Necessity. Let there exist the numbers $\mu^j >$

$0\ (j \in I)$, satisfying the condition (14) that may be written as

$$\ell^i \cdot \nu^{ji} \cdot \frac{1}{\mu^j} + \frac{1}{|C|} \cdot \sum_{k\in I} (-1)^{i+k+1} \cdot \left(\ell^k + \sum_{m\in I} \ell^m \cdot \nu^{km} \cdot c^{mk}\right) \cdot \left|C_i^k\right| \cdot \frac{1}{\mu^k} \le 0$$

$$(i, j \in I). \tag{24}$$

Let's rewrite (24) in more proper form

$$\frac{1}{|C|} \cdot \sum_{k\in I\setminus\{j\}} (-1)^{i+k+1} \cdot \left(\ell^k + \sum_{m\in I} \ell^m \cdot \nu^{km} \cdot c^{mk}\right) \cdot \left|C_i^k\right| \cdot \frac{1}{\mu^k} + \tag{25}$$

$$+\left[\ell^i \cdot \nu^{ji} + \frac{1}{|C|} \cdot (-1)^{i+j+1} \cdot \left(\ell^j + \sum_{m\in I} \ell^m \cdot \nu^{jm} \cdot c^{mj}\right) \cdot \left|C_i^j\right|\right] \cdot \frac{1}{\mu^j} \le\le 0\ (i, j \in I).$$

Let $\tilde{\mu} = \left(\frac{1}{\mu^1}, \dots, \frac{1}{\mu^n}\right)$, B be a matrix the elements of which are corresponding coefficients at $\frac{1}{\mu^j}$ $(j \in I)$ in (25).

Then (25) takes the form

– there exists $\tilde{\mu} \gg 0$ such that $B \cdot \tilde{\mu} \le 0$. Let $D = -B$. Then there exists the vector $\tilde{\mu} \gg 0$ such that $D \cdot \tilde{\mu} \ge 0$, where

$$D = \begin{pmatrix} -D_1 \\ -D_2 \\ \vdots \\ -D_n \end{pmatrix} D_i = \left(d_i^{kj}\right)_{k,j=1}^n (i \in I). \tag{26}$$

For $n^2 \times n$ matrix D introduce the cone

$$K = \{\tilde{\mu} \mid D \cdot \tilde{\mu} \ge 0\}. \tag{27}$$

As follows from (26)

$$K \cap int\, R_+^n \ne \emptyset.$$

Assume the contrary, i. e.

$$K \cap int\, R_+^n = \emptyset. \tag{28}$$

Therefore, these two sets can be identified: a vector$f = (f^1, \dots, f^n)$ such that

$$\begin{cases} [f, \mu] \le 0, \text{if } \mu \in K, \\ [f, \mu] > 0, \text{if } \mu \in R_+^n. \end{cases}$$

From this and the second inequality follows that $f > 0$. Thus there exists the vector $f > 0$ such that $[f, \tilde{\mu}] \le 0$, if $D \cdot \tilde{\mu} \ge 0$ or $[d^i, \tilde{\mu}] \ge 0$ $i = \overline{1, n^2}$, where d^i is a row of the matrix D. Then the vector f takes the form

$$f = \sum_{i=1}^{n^2} æ^i \cdot d^i, æ^i \le 0.$$

Rewrite the cone K as follows

$$K = \{\tilde{\mu} \mid D \cdot \tilde{\mu} \ge 0\ \forall\, i \in I\}.$$

In this case from (26) follows that

$$f \in -K^*,$$

where K^* is an adjoint cone.

The proposition (28) assumes there exists $æ^i \le 0$ for which

$$\sum_{i=1}^{n^2} æ^i \cdot d^i \le 0.$$

In its turn (26) means that there does not exist $æ^i \le 0$ such that $\sum_{i=1}^{n^2} æ^i \cdot d^i \ge 0$, i. e. if $\sum_{i=1}^{n^2} æ^i \cdot d^i \ge 0$, then there exists the index $i_o \in I$ such that $æ^{i_o} > 0$.

Sufficiency. Let (23) be valid. Introduce the cone K by the formula (27) adjoint cone K^*. Making the same reasoning as above, but in reverse order, it is easy to show that there exist $\mu^j > 0$ $(j \in I)$ satisfying (14).

Propostion is proved.

Note. The system of n^2 linear inequalities (24) with respect to the variables $\ell^i, \nu^{ji}, c^{ij}, \mu^j$ may be reduced to the system of n superlinear inequalities with respect to the same variables.

For this purpose let's introduce the denotations

$$d_i^j = (-1)^{i+j+1} \cdot \frac{\left|C_i^j\right|}{|C|} \cdot \left(\ell^j + \sum_{m\in I} \ell^m \cdot \nu^{jm} \cdot c^{mj}\right) (i, j \in I),$$

$$d_i = (d_i^1, \dots, d_i^n)\ (i \in I).$$

Then the inequalities (24) may be rewritten as

$$\begin{cases} [d_1, \tilde{\mu}] \le -\ell^1 \cdot \nu^{1i} \cdot \dfrac{1}{\mu^i}\ \forall\, i \in I, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ [d_n, \tilde{\mu}] \le -\ell^n \cdot \nu^{ni} \cdot \dfrac{1}{\mu^i}\ \forall\, i \in I, \end{cases}$$

where $\tilde{\mu} = \left(\frac{1}{\mu^1}, \dots, \frac{1}{\mu^n}\right)$.

From the last follows that

$$\begin{cases} [d_1, \tilde{\mu}] \le \min\limits_{i\in I}\left(-\ell^1 \cdot \nu^{1i} \cdot \dfrac{1}{\mu^i}\right), \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ [d_n, \tilde{\mu}] \le \min\limits_{i\in I}\left(-\ell^n \cdot \nu^{ni} \cdot \dfrac{1}{\mu^i}\right). \end{cases}$$

Let

$$\tilde{d}_i = -\frac{1}{\ell^i} \cdot d_i\ (i \in I).$$

For this case we obtain that

$$\begin{cases} [\tilde{d}_1, \tilde{\mu}] \ge \max\limits_{i\in I} \nu^{1i} \cdot \dfrac{1}{\mu^i}, \\ \dots\dots\dots\dots\dots\dots \\ [\tilde{d}_n, \tilde{\mu}] \ge \max\limits_{i\in I} \nu^{ni} \cdot \dfrac{1}{\mu^i}. \end{cases}$$

Thus by this way the system of n^2 inequalities (24) is reduced to the system of n superlinear inequalities.

3. Results

In the paper the following results are obtained:

-The necessary and sufficient conditions are derived for the optimality of the branch trajectories;

-The maximal growth rate is defined for the branches in the without loss case;

-The necessary and sufficient condition is derived for the existence of the equilibrium prices without loss;

-The form of the superdifferential is given for the utility function of the consumer;

-The conditions are defined for the reducing the system of n^2 linear inequalities to the system of n superlinear inequalities of the same variables.

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