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A Comparative Study of Homotopy Perturbation Aboodh Transform Method and Homotopy Decomposition Method for Solving Nonlinear Fractional Partial Differential Equations

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Abstract: In this paper, we present the solution of nonlinear fractional partial differential equations by using the Homotopy Perturbation Aboodh Transform Method (HPATM) and Homotopy Decomposition Method (HDM). The Two methods introduced an efficient tool for solving a wide class of linear and nonlinear fractional differential equations. The results shown that the (HDM) has an advantage over the (HPATM) that it takes less time and using only the inverse operator to solve the nonlinear problems and there is no need to use any other inverse transform as in the case of (HPATM).

Keywords: Homotopy Decomposition Method, Integral Transforms, Nonlinear Fractional Differential Equation, Aboodh Transform

1. Introduction

In recent years, fractional calculus has been increasingly used for numerous applications in many scientific and technical fields such as medical sciences, biological research, as well as various chemical, biochemical and physical fields. Nonlinear partial differential equations appear in many branches of physics, engineering and applied mathematics. It has turned out that many phenomena in engineering, physics and other sciences can be described very successfully by models using mathematical tools from fractional calculus [1-3]. For better understanding of a phenomenon described by a given nonlinear fractional partial differential equation, the solutions of differential equations of fractional order are much involved. Fractional derivatives provide more accurate models of real world problems than integer order derivatives. Because of their many applications in scientific fields, fractional partial differential equations [9-11, 24-26] are found to be an effective tool to describe certain physical phenomena, such as diffusion processes, electrical and rheological materials properties and viscoelasticity theories.

In recent years, many research workers have paid attention to study the solutions of nonlinear fractional differential equations by using various methods

Among these numerical methods, the Variational Iteration Method (VIM), Adomian Decomposition Method (ADM) [12-13], and the Differential Transform Method (ADM) are the most popular ones that are used to solve differential and integral equations of integer and fractional order. The Homotopy Perturbation Method (HPM) [4-6] is a universal approach which can be used to solve both fractional ordinary differential equations FODEs as well as fractional partial differential equations FPDEs. This method the HPM, was originally proposed by He [7, 8]. The HPM is a coupling of homotopy and the perturbation method. The Homotopy decomposition method (HDM) was recently proposed by [14-15] to solve the groundwater flow equation and the modified fractional KDV equation [17]. The Homotopy decomposition method [16] is actually the combination of perturbation method and Adomian decomposition method. Recently, Khalid Aboodh, has introduced a new integral transform, named the Aboodh transform [18-23], and it has further applied to the solution of ordinary and partial differential equations. In this paper, the main objective is to introduce a comparative study of nonlinear fractional partial differential equations by using the Homotopy Perturbation Aboodh Transform Method (HPATM) which is the coupling of the Aboodh transform and the HPM using He's polynomials. And the Homotopy Decomposition Method (HDM) which is the coupling of Adomian decomposition method and HPM.

2. Fundamental Facts of the Aboodh Transformation Method

A new transform called the Aboodh transform defined for function of exponential order we consider functions in the set A, defined by:

$$A = \{f(t): \exists\, M, k_1, k_2 > 0, |f(t)| < M e^{-vt} \tag{1}$$

For a given function in the set M must be finite number, k_1, k_2 may be finite or infinite. Aboodh transform which is defined by the integral equation

$$A[f(t)] = K(v) = \frac{1}{v}\int_0^{\infty} f(t)e^{-vt}\,dt, t \geq 0, k_1 \leq v \leq k_2 \tag{2}$$

The following results can be obtained from the definition and simple calculations

1) $A[t^n] = \frac{n!}{v^{n+2}}$
2) $A[f\,'(t)] = vK(v) - \frac{f(0)}{v}$
3) $A[f\,''(t)] = v^2K(v) - \frac{f'(0)}{v} - f(0)$.
4) $A\left[f^{(n)}(t)\right] = v^nK(v) - \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{v^{2-n+k}}$.

3. Fundamental Facts of the Fractional Calculus

Definition 3.1. A real function $f(x), x > 0$, is said to be in the space C_μ, $\mu \in \mathbb{R}$ if there exists a real number $p > \mu$, such that $f(x) = x^p h(x)$, where $h(x) \in [0, \infty)$ and it is said to be in space C_μ^m if $f^{(m)} \in C_m$, $m \in \mathbb{N}$.

Definition 3.2. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_m$, $\mu \geq -1$, is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-1)^{\alpha-1} f(t)dt, \alpha > 0, x > 0 \tag{3}$$

$$J^\alpha f(x) = f(x)$$

Let's consider some of properties for operator J^α (e.g., [1-3]):

If $f \in C_m$, $\mu \geq -1, \alpha, \beta \geq 0$ and $\gamma > -1$ then $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x), J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x), J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$

Lemma 3.1. If $m-1 < \alpha \leq m, m \in \mathbb{N}$ and $f \in C_m$, $\mu \geq -1$ then $D^\alpha J^\alpha f(x) = f(x)$ and,

$$J^\alpha D_0^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0)\frac{x^k}{k!}, x > 0 \tag{4}$$

Definition 3.3. (Partial Derivatives of Fractional order). Assume now that $f(x)$ is a function of n variables x_i, $i = 1, \dots, n$ also of class C on $D \in \mathbb{R}_n$. As an extension of definition 3.3 we define partial derivative of order α for $f(x)$ respect to x_i

$$a\partial_{\underline{x}}^\alpha f = \frac{1}{\Gamma(m-\alpha)}\int_0^{x_i}(x_i-1)^{m-\alpha-1}\,\partial_{x_i}^\alpha f(x_j)\Big|_{x_j=t} dt \tag{5}$$

If it exists, where $\partial_{x_i}^\alpha$ is the usual partial derivative of integer order m.

Theorem 3.1. If $K(v)$ is Aboodh transform of $f(x)$, then as is known that the Aboodh transform of derivative with integral order can be expressed by

$$A[f\,'(t)] = vK(v) - \frac{f(0)}{v}.$$

Proof. Let us take the Aboodh transform $f\,'(t) = \frac{d}{dt}f(t)$, using integration by parts we get

$$A\left[\frac{d}{dt}f(t)\right] = \frac{1}{v}\int_0^\infty \frac{d}{dt}f(t)e^{-vt}\,dt$$

$$= \lim_{p\to\infty}\frac{1}{v}\int_0^p \frac{d}{dt}f(t)e^{-vt}\,dt$$

$$= \lim_{p\to\infty}\left\{\left[\frac{1}{v}f(t)e^{-vt}\right]_0^p + \frac{1}{v}\int_0^p f(t)e^{-vt}\,dt\right\}$$

$$= v\,K(v) - \frac{f(0)}{v} \tag{6}$$

Equation (6) gives us the proof of Theorem3.1. When we continue in the same manner, we get the Aboodh transform of the second order derivative as follows

$$A\left[\frac{d^2}{dt^2}f(t)\right] = A\left[\frac{d}{dt}\left(\frac{d}{dt}f(t)\right)\right]$$

$$= v\,A\left[\frac{d}{dt}f(t)\right] - \left.\frac{\frac{d}{dt}f(t)}{v}\right|_{t=0}$$

$$= v\,A\left[v\,K(v) - \frac{f(0)}{v}\right] - \left.\frac{\frac{d}{dt}f(t)}{v}\right|_{t=0}$$

$$= v^2K(v) - \frac{f'(0)}{v} - f(0)$$

If we go on the same way, we get the Aboodh transform of the nth order derivative as follows

$$A\left[f^{(n)}(t)\right] = v^n K(v) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{v^{2-n+k}} \text{ for } n \geq 1 \quad (7)$$

or

$$A\left[f^{(n)}(t)\right] = v^n \left[K(v) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{v^{2+k}}\right]. \quad (8)$$

Theorem 3.2. If $K(v)$ is Aboodh transform of (t), one can take into consideration the Aboodh transform of the Riemann-Liouville derivative as follow

$$A[D^\alpha f(t)] = v^\alpha \left[K(v) - \sum_{k=1}^{n} \frac{D^{\alpha-k} f(0)}{v^{\alpha-k+2}}\right] ; -1 < n-1 \leq \alpha < n \quad (9)$$

Proof. Let's consider

$$A[D^\alpha f(t)] = v^\alpha K(v) - \sum_{k=0}^{n-1} v^k [D^{\alpha-k-1} f(0)]$$

$$= v^\alpha K(v) - \sum_{k=0}^{n} v^{k-1} [D^{\alpha-k} f(0)] = v^\alpha K(v) - \sum_{k=1}^{n} v^{k-2} [D^{\alpha-k} f(0)]$$

$$= v^\alpha K(v) - \frac{1}{v^{-k+2}} \sum_{k=1}^{n} [D^{\alpha-k} f(0)] = v^\alpha K(v) - \sum_{k=0}^{n} \frac{1}{v^{\alpha-k+2-\alpha}} [D^{\alpha-k} f(0)]$$

$$= v^\alpha K(v) - \sum_{k=1}^{n} v^\alpha \frac{1}{v^{\alpha-k+2}} [D^{\alpha-k} f(0)]$$

Therefore, we get the Aboodh transformation of fractional order of $f(t)$ as follows

$$A[D^\alpha f(t)] = v^\alpha \left[K(v) - \sum_{k=1}^{n} \left(\frac{1}{v}\right)^{\alpha-k+2} [D^{\alpha-k} f(0)]\right] \quad (10)$$

Definition 3.4. The Aboodh transform of the Caputo fractional derivative by using Theorem3.2 is defined as follows

$$A[D_t^\alpha f(t)] = v^\alpha A[f(t)] - \sum_{k=0}^{m-1} v^{k-\alpha-2} f^{(k)}(0), \quad m-1 < \alpha < m \quad (11)$$

4. Basic Idea

4.1. Basic Idea of HPATM

To illustrate the basic idea of this method, we consider a general form of nonlinear non homogeneous partial differential equation as the follow

$$D_t^\alpha u(x,t) = L(u(x,t)) + N(u(x,t)) + f(x,t), \alpha > 0 \quad (12)$$

with the following initial conditions

$$u(x,0) = g_k, k = 0, \dots, n-1, D_0^n u(x,0) = 0 \text{ and } = [\alpha]. \quad (13)$$

Where D_t^α denotes without loss of generality the Caputo fraction derivative operator, f is a known function, N is the general nonlinear fractional differential operator and L represents a linear fractional differential operator.

Taking Aboodh transform on both sides of equation (12), we get

$$A[D_t^\alpha u(x,t)] = A[L(u(x,t))] + A[N(u(x,t))] + A[f(x,t)] \quad (14)$$

Using the differentiation property of Aboodh transform and above initial conditions, we have

$$A[u(x,t)] = \frac{1}{v^\alpha} A[L(u(x,t))] + \frac{1}{v^\alpha} A[N(u(x,t))] + g(x,t) \quad (15)$$

Operating with the Aboodh inverse on both sides of equation (15) gives the solution

$$u(x,t) = G(x,t) + A^{-1}\left[\frac{1}{v^\alpha} A[L(u(x,t))] + \frac{1}{v^\alpha} A[N(u(x,t))]\right] \quad (16)$$

Where $G(x,t)$ represents the term arising from the known function $f(x,t)$ and the initial condition. Now, we apply the homotopy perturbation method

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t). \quad (17)$$

And the nonlinear term can be decomposed as

$$Nu(x,t) = \sum_{n=0}^{\infty} p^n H_n(u) \quad (18)$$

Where $H_n(u)$ are He's polynomial and given by

$$H_n(u_0, u_1, u_2 \dots u_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left[N\left(\sum_{i=0}^{\infty} p^i u_i(x,t)\right)\right]_{p=0}, \quad n = 0,1,2,\dots \quad (19)$$

Substituting equations (18) and (17) in equation (16) we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = G(x,t) + p\left[A^{-1}\left[\frac{1}{v^\alpha} A[L(\sum_{n=0}^{\infty} p^n u_n(x,t))] + \frac{1}{v^\alpha} A[N(\sum_{n=0}^{\infty} p^n u_n(x,t))]\right]\right] \quad (20)$$

which is the coupling of the Aboodh transform and the homotopy perturbation method using He's polynomials and after comparing the coefficient of like powers of p, we obtain the following approximations

$$p^0 : u_0(x,t) = G(x,t),$$

$$p^1 : u_1(x,t) = A^{-1}\left[\frac{1}{v^\alpha} A[L(u_0(x,t)) + H_0(u)]\right],$$

$$p^2 : u_2(x,t) = A^{-1}\left[\frac{1}{v^\alpha} A[L(u_1(x,t)) + H_1(u)]\right],$$

$$p^3 : u_3(x,t) = A^{-1}\left[\frac{1}{v^\alpha} A[L(u_2(x,t)) + H_2(u)]\right],$$

$$p^n : u_n(x,t) = A^{-1}\left[\frac{1}{v^\alpha}A[L(u_{n-1}(x,t)) + H_{n-1}(u)]\right], \quad (21)$$

Hence, the solution can be expressed in the form

$$u(x,t) = \lim_{p\to 1} u_n(x,t) = u_0(x,t)+u_1(x,t) + u_2(x,t) + \cdots \quad (22)$$

By virtue of (21) the solution (22) is converges very rapidly

4.2. Basic Idea of HDM

The method consists of first step to transform the fractional partial differential equation to the fractional partial integral equation which applying the inverse operator D_t^α to the both sides of equation (12), finally, solution $u(x,t)$ can be written in the form

$$u(x,t) = \sum_{j=1}^{n-1}\frac{g_j}{\Gamma(\alpha-j+1)}t^j + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left[L\big(u(x,\tau)\big) + N\big(u(x,\tau)\big) + f(x,\tau)\right]d\tau \quad (23)$$

Other side using the following

$$\sum_{j=1}^{n-1}\frac{f_j(x)}{\Gamma(\alpha-j+1)}t^{\alpha-j} = f(x,t) \text{ or } \sum_{j=1}^{n-1}\frac{g_j}{\Gamma(\alpha-j+1)}t^j = f(x,t)$$

We have

$$u(x,t) = T(x,t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left[L\big(u(x,\tau)\big) + N\big(u(x,\tau)\big) + f(x,\tau)\right]d\tau \quad (24)$$

In the method of homotopy decomposition, the basic assumption is that the solutions can be written as a power series in p

$$u(x,t,p) = \sum_{n=0}^{\infty} p^n u_n(x,t) \quad (25)$$

$$u(x,t) = \lim_{p\to 1} u(x,t,p) \quad (26)$$

and the nonlinear term can be decomposed as

$$Nu(x,t) = \sum_{n=0}^{\infty} p^n H_n(u) \quad (27)$$

where $p \in (0,1]$ is an embedding parameter and the He's polynomials that can be generated by

$$H_n(u_0, u_1, u_2 \dots u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i u_i(x,t)\right)\right]_{p=0}, \quad n = 0,1,2,\dots \quad (28)$$

The homotopy decomposition method is obtained by the graceful coupling of homotopy technique with Abel integral and can be written as

$$\sum_{n=0}^{\infty} p^n u_n(x,t) - T(x,t) = \frac{p}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}[f(x,\tau) + L(\sum_{n=0}^{\infty} p^n u_n(x,\tau)) + N(\sum_{n=0}^{\infty} p^n u_n(x,\tau))]d\tau \quad (29)$$

Comparing the terms of same powers of gives solutions of various orders with the first term

$$u_0(x,t) = T(x,t) \quad (30)$$

we include that the term is the Taylor series of the exact solution of equation (12) of order $n-1$.

5. Applications

Example 5.1. Let's consider the following one dimensional fractional heat like problem:

$$D_t^\alpha u(x,t) = \frac{1}{2}x^2 u_{xx}(x,t), 0 < x < 1, 0 < \alpha \le 1, t > 0 \quad (31)$$

with the boundary conditions

$$u(0,t) = 0, u(1,t) = e^t$$

and initial condition;

$$u(x,0) = x^2$$

5.1. Application Method of Homotopy Perturbation Aboodh Transform

Applying the steps involved in HPATM as presented in section 4.1 to equation (31) we obtain the following:

$$p^0 : u_0(x,t) = x^2$$

$$p^1 : u_1(x,t) = A^{-1}\left[\frac{1}{v^\alpha}A\left[\frac{1}{2}x^2 u_0(x,t)_{xx}\right]\right] = A^{-1}\left[\frac{1}{v^\alpha}A[x^2]\right] = A^{-1}\left[\frac{x^2}{v^{\alpha+2}}\right] = \frac{x^2t^\alpha}{\alpha!} = \frac{x^2t^\alpha}{\Gamma(\alpha+1)},$$

$$p^2 : u_2(x,t) = A^{-1}\left[\frac{1}{v^\alpha}A\left[\frac{1}{2}x^2 u_1(x,t)_{xx}\right]\right] = A^{-1}\left[\frac{1}{v^\alpha}A\left[\frac{x^2t^\alpha}{\Gamma(\alpha+1)}\right]\right] = A^{-1}\left[\frac{x^2}{(v^{2\alpha+2})\Gamma(\alpha+1)}\right] = \frac{x^2t^{2\alpha}}{\Gamma(2\alpha+1)},$$

Proceeding in a similar manner, we have:

$$p^3 : u_3(x,t) = A^{-1}\left[\frac{1}{v^\alpha}A\left[\frac{1}{2}x^2 u_2(x,t)_{xx}\right]\right] = \frac{x^2t^{3\alpha}}{\Gamma(3\alpha+1)},$$

$$p^n : u_n(x,t) = A^{-1}\left[\frac{1}{v^\alpha}A\left[\frac{1}{2}x^2 u_n(x,t)_{xx}\right]\right] = \frac{x^2t^{n\alpha}}{\Gamma(n\alpha+1)},$$

Therefore, the solution $u(x,t)$ can be written in the form

$$u(x,t) = x^2\left(1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots + \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} + \cdots\right) \quad (32)$$

This is an equivalent form to the exact solution in closed form

$$u(x,t) = x^2 E_{1,\alpha}(t^\alpha) \quad (33)$$

where $E_{1,\alpha}(\,)$ is the Mittag-Leffler function

5.2. Application the Method of Homotopy Perturbation Adomain Decomposition

Applying the steps involved in HDM as presented in section 4.2 to equation (31) we obtain the following

$$\sum_{n=0}^{\infty} p^n u_n(x,t) - x^2 = \frac{p}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}[x^2(\sum_{n=0}^{\infty} p^n u_n(x,\tau)_{xx})]d\tau \quad (34)$$

Comparing the terms of the same powers of p we obtain

$$u_0(x,t) = x^2$$

$$u_1(x,t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}[x^2(u_0(x,\tau)_{xx})]d\tau = \frac{x^2 t^\alpha}{\Gamma(\alpha+1)},$$

$$u_2(x,t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}[x^2(u_1(x,\tau)_{xx})]d\tau = \frac{x^2 t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$u_3(x,t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}[x^2(u_2(x,\tau)_{xx})]d\tau = \frac{x^2 t^{3\alpha}}{\Gamma(3\alpha+1)},$$

$$u_n(x,t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}[x^2(u_{n-1}(x,\tau)_{xx})]d\tau = \frac{x^2 t^{n\alpha}}{\Gamma(n\alpha+1)},$$

Hence, the asymptotic solution can expressed by

$$u(x,t) = x^2\left(1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots + \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} + \cdots\right) \quad (35)$$

$$\lim_{\substack{n\to\infty \\ \alpha\to 1}} u_n(x,t,\alpha) = x^2 e^t$$

This is the exact solution of equation (31) when $\mathrm{n} = 1$.

Example 5.2. Let's consider the following three dimensional fractional heat like equation

$$D_t^\alpha u(x,y,z,t) = (xyz)^4 + \frac{1}{36}\left(x^2 u_{xx} + y^2 u_{yy} + z^2 u_{zz}\right), 0 < x,y,z < 1, 0 < \alpha \le 1 \quad (36)$$

With the initial condition

$$u(x,y,z,t) = 0.$$

5.3. Application the Method of Homotopy Perturbation Adomain Decomposition

Applying the steps involved in HPATM as presented in section 4.1 to equation (36) we obtain the following

$$p^0 : u_0(x,y,z,t) = (xyz)^4$$

$$p^1 : u_1(x,y,z,t) = A^{-1}\left[\frac{1}{v^\alpha} A\left[\frac{1}{36}\left(x^2 u_{0_{xx}} + y^2 u_{0_{yy}} + z^2 u_{0_{zz}}\right)\right]\right] = \frac{(xyz)^4 t^\alpha}{\Gamma(\alpha+1)},$$

$$p^2 : u_2(x,y,z,t) = A^{-1}\left[\frac{1}{v^\alpha} A\left[\frac{1}{36}\left(x^2 u_{1_{xx}} + y^2 u_{1_{yy}} + z^2 u_{1_{zz}}\right)\right]\right] = \frac{(xyz)^4 t^{2\alpha}}{\Gamma(2\alpha+1)}$$

Proceeding in a similar way, we have

$$p^3 : u_3(x,y,z,t) = \frac{(xyz)^4 t^{3\alpha}}{\Gamma(3\alpha+1)},$$

$$p^n : u_n(x,y,z,t) = \frac{(xyz)^4 t^{n\alpha}}{\Gamma(n\alpha+1)},$$

Therefore, the solution $u(x,t)$ can be written in the form

$$u(x,y,z,t) = (xyz)^4\left(1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots + \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} + \cdots\right). \quad (37)$$

5.4. Application the Method of Homotopy Perturbation Adomain Decomposition

Applying the steps involved in HDM as presented in section 4.2 to equation (36) we obtain the following

$$\sum_{n=0}^{\infty} p^n u_n(x,y,z,t) = \frac{p}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} \left[(xyz)^4\left(\frac{1}{36}\left(x^2 \sum_{n=0}^{\infty} p^n u_n(x,y,z,t)_{xx} + y^2 \sum_{n=0}^{\infty} p^n u_n(x,y,z,t)_{yy} + z^2 \sum_{n=0}^{\infty} p^n u_n(x,y,z,t)_{zz}\right)\right)\right] d\tau \quad (38)$$

$$u_0(x,y,z,t) = 0$$

$$u_1(x,y,z,t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}(xyz)^4 d\tau = \frac{(xyz)^4 t^\alpha}{\Gamma(\alpha+1)},$$

$$\vdots$$

$$u_n(x,y,z,t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left[(xyz)^4 \frac{1}{36}\left(x^2 u_{n-1_{xx}} + y^2 u_{n-1_{yy}} + z^2 u_{n-1_{zz}}\right)\right] d\tau = \frac{(xyz)^4 t^{n\alpha}}{\Gamma(n\alpha+1)},$$

Therefore, the approximate solution of equation for the first N can be expressed by

$$u_n(x,y,z,t) = \sum_{n=1}^{N} \frac{(xyz)^4 t^{n\alpha}}{\Gamma(n\alpha+1)},$$

when $N \to \infty$ the solution can be expressed by

$$u_n(x,y,z,t) = \sum_{n=0}^{\infty} \frac{(xyz)^4 t^{n\alpha}}{\Gamma(n\alpha+1)} - (xyz)^4 = (xyz)^4[E_\alpha(t^\alpha) - 1]$$

where $E_\alpha(t^\alpha)$ is the generalized Mittag-Leffler function.

Note that in case of $\alpha = 1$ we have

$$u(x,y,z,t) = (xyz)^4[e^t - 1] \quad (39)$$

This is the exact solution in case of $\alpha = 1$.

Example 5.3. Consider the following nonlinear time-fractional gas dynamics equations [Kilicman]

$$D_t^\alpha u(x,t) + \frac{1}{2}\left(u_x(x,t)\right)^2 - u(x,t)\left(1 - u(x,t)\right) = 0, 0 < \alpha \le 1 \quad (40)$$

with the initial condition

$$u(x,0) = e^{-x}.$$

5.5. Application the Method of Homotopy Perturbation Adomain Decomposition

Applying the steps involved in HPATM as presented in section 4.1 to equation (40), we obtain the following

$$p^0 : u_0(x,t) = e^{-x}$$

$$p^1 : u_1(x,t) = A^{-1}\left[\frac{1}{v^\alpha}A\left[\frac{1}{2}x^2 u_0(x,t)_{xx}\right]\right] = \frac{e^{-x}t^\alpha}{\Gamma(\alpha+1)},$$

$$p^2 : u_2(x,t) = A^{-1}\left[\frac{1}{v^\alpha}A\left[\frac{1}{2}x^2 u_1(x,t)_{xx}\right]\right] = \frac{e^{-x}t^{2\alpha}}{\Gamma(2\alpha+1)}$$

Proceeding in a similar manner, we have:

$$p^3 : u_3(x,t) = A^{-1}\left[\frac{1}{v^\alpha}A\left[\frac{1}{2}x^2 u_2(x,t)_{xx}\right]\right] = \frac{e^{-x}t^{3\alpha}}{\Gamma(3\alpha+1)},$$

$$p^n : u_n(x,t) = A^{-1}\left[\frac{1}{v^\alpha}A\left[\frac{1}{2}x^2 u_n(x,t)_{xx}\right]\right] = \frac{e^{-x}t^{n\alpha}}{\Gamma(n\alpha+1)},$$

Therefore the solution $u(x,t)$ can be written as

$$u(x,t) = e^{-x}\left(1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots + \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} + \cdots\right) \quad (41)$$

5.6. Application the Method of Homotopy Perturbation Adomain Decomposition

Applying the steps involved in HDM as presented in section 4.2 to equation (40) we obtain the following

$$u_0(x,t) = e^{-x},$$

$$u_1(x,t) = \frac{e^{-x}t^\alpha}{\Gamma(\alpha+1)},$$

$$u_2(x,t) = \frac{e^{-x}t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$\vdots$$

$$u_n(x,t) = \frac{e^{-x}t^{n\alpha}}{\Gamma(n\alpha+1)},$$

Therefore, the solution can be written in the form

$$u(x,t) = e^{-x}\left(1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots + \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} + \cdots\right) \quad (42)$$

6. Conclusion

In the present paper, Homotopy Perturbation Aboodh Transform Method (HPATM) is employed for solving nonlinear fractional partial differential equations; the same problems are solved by Homotopy Decomposition Method (HDM). It is worth mentioning that the (HDM) has an advantage over the (HPATM) that it takes less time and using only the inverse operator to solve the nonlinear problems and there is no need to use any other inverse transform as in the case of (HPATM). The results reveal that the (HDM) is a powerful technique and can be applied to other applications.

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