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Inferences on the Weibull Exponentiated Exponential Distribution and Applications

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Abstract: In this article, an alternative method of defining the probability density function of Generalized Weibull-exponential distributions is proposed. Based on the method, the distribution can also be called Weibull exponentiated exponential distribution. This distribution includes the exponential, Weibull and exponentiated exponential distributions as special cases. Comprehensive mathematical treatment of the distribution is provided. The quantile function, mode, characteristic function, moment generating function among other mathematical properties of the distribution were derived. The parameters of the distribution were estimated by applying the Maximum Likelihood Procedure.The elements of the Fisher Information Matrix is also provided. Finally, a data set is fitted to the model and its sub-models. It is observed that the new distribution is more flexible and can be used quiet effectively in analysing real life data in place of exponential, Weibull and exponentiated exponential distributions.

Keywords: T-X Family, Exponentiated Exponential Distribution, Order Statistics, Shannon Entropy and Likelihood Ratio Test

1. Introduction

The knowledge of appropriate distributional assumptions in parametric statistical inferences and modeling is of paramount important [1]. Several existing distributions have been used in different areas of environmental sciences, actuarial sciences, engineering, medical sciences, survival analysis, computer science, economics and social sciences in modeling lifetime data and making inferences [1, 2, 3]. However, most of data generated from these areas are characterized by exhibiting a non-monotonic failure rate and varied degree of skewness and kurtosis [1, 2]. Hence, modeling data with the existing distributions may produce an inappropriate parametric fit.

To overcome these problems, distributions with heavy tails, tractable cumulative distribution function that will ease simulation, monotonic and non-monotonic failure rates and can modeled data with varied degree of skewness and kurtosis should be used in modeling data and making inferences. Hence, the statistical literature have been flooded with barrage of methods for developing new distributions that can be used in modeling lifetime data from different areas and that will provide greater flexibility and efficiency. These include the Beta generalized exponential distribution by [4], Beta Burr type V distribution by [5], the Weibull Burr type X distribution by [6], Weibull Burr III distribution by [7] to mention but a few.

A four parameter distribution called the generalized Weibull exponential distribution was developed by [8]. This distribution was developed using the Exponentiated $T-X$ family of distribution developed by [9] and taking exponential distribution as the baseline distribution. This study aim at using the $T-X$ family proposed by [10] and taking the exponentiated exponential distribution as the baseline distribution in order to re-derive the *pdf* and *cdf* of the Generalized Weibull-Exponential distribution (GWE) proposed by [9]. However, following [10], we will prefer to call this distribution Weibull-exponentiated exponential distribution. The study also aimed at providing useful expansion of the *pdf* and extensively studying some statistical properties of the distribution such as mode, characteristic function, moment generating function, moments, order statistics and Shannon entropy. Furthermore, confidence interval and hypothesis testing on the parameters of the distribution were discussed. Finally, a data set was fitted to the distribution and its performance was compared with that of its sub-models.

Ref [11] introduced and studied the exponentiated exponential distribution. Studies have shown that this distribution can serve as an alternative to the two parameter Weibull and two parameter Gamma distributions in many situations (see [4, 12], for more details). The cumulative distribution function of the exponentiated exponential distribution along side with its probability density function (pdf) are respectively given by:

$$G(x) = (1-\exp(-bx))^{a} \tag{1}$$

and

$$g(x) = ab\exp(-bx)(1-\exp(-bx))^{a-1} \tag{2}$$

where $a,\ b > 0$ are the shape and scale parameters respectively. The exponentiated exponential distribution similar to the gamma and Weibull distributions extends the exponential distribution when the shape parameter of the distributions takes the value one but in different ways. Hence, it serves as an alternative to the gamma and Weibull distributions [4, 12]. Unlike the exponential distribution that has a constant failure rate, the exponentiated exponential distribution have a non-decreasing failure rate when the shape parameter is greater than one and a non-increasing failure rate when its shape parameter is less than one, while it is constant when it takes the value one. The exponentiated exponential distribution has a unique mode when its shape parameter takes the value of at least one. The distribution has received a great attention by researchers. For example, various properties and comparison with other distributions have been provided by different researchers such as: [13], [12, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25] and [26] to mention but a few. Also, this distribution have been extended by different researcher. Few of these extensions include: Bivariate generalized exponential distribution by [27], the beta generalized exponential distribution by [4], transmuted exponentiated exponential distribution by [28], Bivariate discrete generalized exponential distribution by [29], Odds generalized exponential-exponential distribution by [30], bivariate generalized exponential distribution based on copula functions by [31], exponentiated generalized exponential distribution by [32].

The rest of the paper is organized as follows. In Section 2 the *pdf* of Weibull Exponentiated Exponentiated distribution was define using $T-X$ family and taking exponentiated exponential distribution as baseline distribution. An outline of some special cases of the distribution, the graphs of probability density function (*pdf*), cumulative distribution function (*cdf*), survival and hazard functions of the distribution are also given in this section. In section 3, useful expansion and some mathematical properties such as quantile function, mode, skewness and kurtosis, characteristic function, moment generating function, order statistics and Shannon entropy were derived. Estimation of the unknown parameters by method of maximum likelihood and information criterion are given in section 4. Analysis of a data set to show the effectiveness of the model over its sub-models is given in section 5 and we conclude in section 6.

2. The Weibull Exponentiated Exponential Distribution

Ref [10] proposed a family of distributions called transformed-transformer $(T-X)$ family of distributions. This family of distribution is an extension of the beta-G family of distribution developed by [2]. The cdf and pdf of the $T-X$ family are respectively given by:

$$F(x) = \int_{0}^{-log\left(\overline{G}(x)\right)} q(t)\,dt = Q\left[-log\left(\overline{G}(x)\right)\right]$$

and

$$f(x) = \frac{g(x)}{\overline{G}(x)} q\left(-log\left(\overline{G}(x)\right)\right) = h(x)\, q\left(-log\left(\overline{G}(x)\right)\right)$$

where $Q(t)$ is the cdf of the random variable T, $h(x)$ is the hazard function for random variable X with *cdf* $G(x)$ and $\overline{G}(x) = 1-G(x)$

If the random variable T follows the Weibull distribution with parameters θ and ϕ, then $q(t) = \theta\phi t^{\theta-1}\exp\left(-\phi t^{\theta}\right)$ and hence, the *cdf* and *pdf* of the Weibull X-family are respectively given by:

$$F(x) = 1-\exp\left(-\phi\left[-log\overline{G}(x)\right]^{\theta}\right) \tag{3}$$

and

$$f(x) = \frac{\phi\theta g(x)}{\overline{G}(x)}\left[-log\overline{G}(x)\right]^{\theta-1}\exp\left(-\phi\left[-log\overline{G}(x)\right]^{\theta}\right) \tag{4}$$

where $G(x)$ and $g(x)$ are the *cdf* and *pdf* of any baseline distribution, θ and ϕ are additional and scale shape parameters respectively. Substituting equations (1) and (2) in (3) and (4) yields the *cdf* and *pdf* of the WEE distribution given by:

$$F(x;\ a,\ b,\ \theta,\ \phi) = 1-\exp\left[-\phi\left(-\ell n\left\{1-(1-\exp(-bx))^{a}\right\}\right)^{\theta}\right] \tag{5}$$

and

$$f\left(x;\ a,\ b,\ \theta,\ \phi\right)=\frac{ab\theta\phi\exp\left(-bx\right)}{1-\left(1-\exp\left(-bx\right)\right)^{a}}\left[-\ell n\left\{1-\left(1-\exp\left(-bx\right)\right)^{a}\right\}\right]^{\theta-1}\left(1-\exp\left(-bx\right)\right)^{a-1}$$
$$\exp\left[-\phi\left(-\ell n\left\{1-\left(1-\exp\left(-bx\right)\right)^{a}\right\}\right)^{\theta}\right] \tag{6}$$

respectively. Where $x>0, a>0, b>0, \theta>0$ and $\phi>0$, θ and b are scale parameters and θ and a are shape parameters. The Weibull Exponentiated Exponential (WEE) distribution extends the Weibull distribution, as well as the exponentiated exponential and exponential distributions. The survival function, hazard function, cumulative hazard function and reverse hazard rate function of the WEE distribution are respectively given by:

$$S\left(t\right)=\exp\left[-\phi\left(-\ell n\left\{1-\left(1-\exp\left(-bt\right)\right)^{a}\right\}\right)^{\theta}\right]$$

$$h\left(t\right)=\frac{ab\theta\phi\exp\left(-bx\right)\left(1-\exp\left(-bx\right)\right)^{a-1}}{1-\left(1-\exp\left(-bt\right)\right)^{a}}\left[-\ell n\left\{1-\left(1-\exp\left(-bx\right)\right)^{a}\right\}\right]^{\theta-1}$$

$$H\left(t\right)=\phi\left(-\ell n\left\{1-\left(1-\exp\left(-bt\right)\right)^{a}\right\}\right)^{\theta}$$

and

$$r\left(t\right)=\frac{ab\theta\phi\exp\left(-bt\right)\left(1-\exp\left(-bt\right)\right)^{a-1}\left[-\ell n\left\{1-\left(1-\exp\left(-bt\right)\right)^{a}\right\}\right]^{\theta-1}}{1-\left(1-\exp\left(-bt\right)\right)^{a}}\frac{\exp\left[-\phi\left(-\ell n\left\{1-\left(1-\exp\left(-bt\right)\right)^{a}\right\}\right)^{\theta}\right]}{1-\exp\left[-\phi\left(-\ell n\left\{1-\left(1-\exp\left(-bt\right)\right)^{a}\right\}\right)^{\theta}\right]}$$

Theorem 1: $\int\limits_{0}^{\infty} f\left(x\right)dx=1$

Proof

$$\int\limits_{0}^{\infty}\frac{ab\theta\phi\left(1-\exp\left(-bx\right)\right)^{a-1}}{1-\left(1-\exp\left(-bx\right)\right)^{a}}\left[-\ell n\left\{1-\left(1-\exp\left(-bx\right)\right)^{a}\right\}\right]^{\theta-1}\exp\left(-bx\right)\exp\left[-\phi\left(-\ell n\left\{1-\left(1-\exp\left(-bx\right)\right)^{a}\right\}\right)^{\theta}\right]dx$$

letting $m=\phi\left(-\ell n\left\{1-\left(1-\exp\left(-bx\right)\right)^{a}\right\}\right)^{\theta}$ gives:

$$\int\limits_{0}^{\infty} f\left(x\right)dx=\int\limits_{0}^{\infty}\exp\left(-m\right)dm=1$$

Theorem 2: $\underset{x\to 0}{lt}\, f\left(x;\, a,\, b,\, \theta,\, \phi\right)=0$ and $\underset{x\to\infty}{lt}\, f\left(x;\, a,\, b,\, \theta,\, \phi\right)=0$.

Proof

This is straight forward.

Hence, this clearly confirms that the distribution in (6) has at least a mode since $\underset{x\to 0}{lt}\, f\left(x;\, a,\, b,\, \theta,\, \phi\right)=0$ and $\underset{x\to\infty}{lt}\, f\left(x;\, a,\, b,\, \theta,\, \phi\right)=0$.

Sub-Models

One good characteristic of the WEE distribution is that it contains several well known distributions as a special case.

1. when $a=1$ or $a=b=1$, the pdf in (6) reduces to Weibull distribution
2. if $\theta=\phi=1$, it reduces to exponentiated exponential distribution
3. it reduces to exponential distribution when $a=\theta=1$ and
4. to exponential exponentiated exponential when $\theta=1$

The shape of the pdf, cdf, survival and hazard functions for some selected parameter values are illustrated in figures 1a, 1b, 2a and 2b respectively.

3. Mathematical Properties

In this section, some important mathematical properties of the WEE distribution such as alternative formula for the pdf of the WEE distribution, Quantile function, Characteristic function, moment generating function, Moments, kurtosis, Shannon entropy and order statistics will be studied.

3.1. Useful Expansions

In the following, an alternative formula for the pdf of the WEE distribution given in equation (6) is obtain using power series expansion, binomial series expansion and following [33] and [34]. Using power series, the pdf in equation (6) can be written as:

$$f\left(x;\ a,\ b,\ \theta,\ \phi\right)=\frac{ab\theta\phi\exp\left(-bx\right)\left(1-\exp\left(-bx\right)\right)^{a-1}}{1-\left(1-\exp\left(-bx\right)\right)^{a}}\sum_{r=0}^{\infty}\frac{\left(-1\right)^{r}\phi^{r}}{r!}\left[-\ell n\left\{1-\left(1-\exp\left(-bx\right)\right)^{a}\right\}\right]^{\theta r+\theta-1}$$

recall that the binomial series expansion of $\left(1-x\right)^{-1}=\sum_{s=0}^{\infty}x^{s}$. Hence,

$$\left[1-\left(1-\exp\left(-bx\right)\right)^{a}\right]^{-1}=\sum_{s=0}^{\infty}\left(1-\exp\left(-bx\right)\right)^{as},$$

which makes the *pdf* to be written as

$$f\left(x;\ a,\ b,\ \theta,\ \phi\right)=ab\theta\phi\exp\left(-bx\right)\sum_{r,s=0}^{\infty}\frac{\left(-1\right)^{r}\phi^{r}}{r!}\left(1-\exp\left(-bx\right)\right)^{as+a-1}\left[-\ell n\left\{1-\left(1-\exp\left(-bx\right)\right)^{a}\right\}\right]^{\theta r+\theta-1}$$

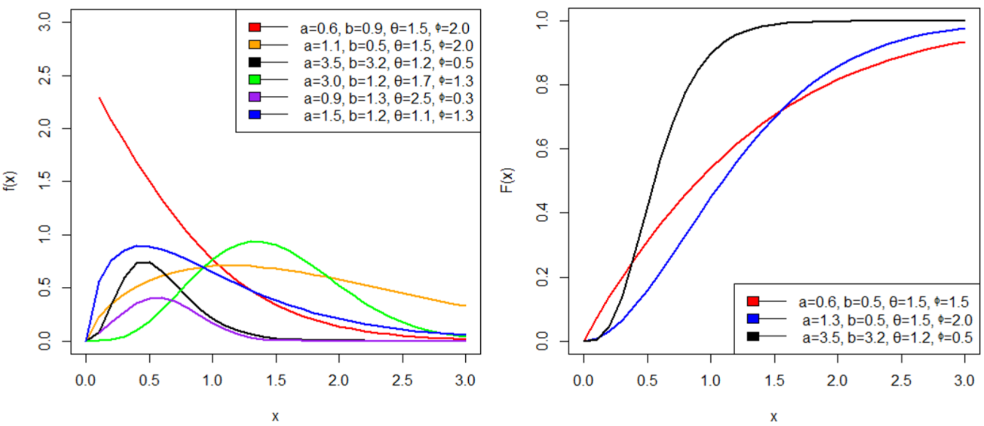
following [33] and [34],

$$\left[-\ell n\left(1-x\right)\right]^{n}=\sum_{t=0}^{\infty}\sum_{u=0}^{t}\frac{\left(-1\right)^{t+u}\binom{(t-n)}{t}\binom{t}{u}}{n-u}p_{u,t}x^{n+t}$$

where $p_{u,t}$ can recursively be calculated as: $p_{u,t}=\frac{1}{t}\sum_{v=0}^{t}\left[t-v\left(u+1\right)\right]c_{v}p_{u,t-v}$ for $t=1,2,\cdots$ with $p_{u,0}=1$ and $c_{t}=\frac{(-1)^{t+1}}{t+1}$. Hence, the *pdf* in (6) is expressed as:

$$f\left(x;\ a,\ b,\ \theta,\ \phi\right)=a\theta\phi\sum_{s,r,t=0}^{\infty}\sum_{u=0}^{t}\Omega_{srtu}g\left(x/\lambda,b\right) \tag{7}$$

where $\Omega_{srtu}=\frac{(-1)^{r+t+u}\phi^{r}}{r!(\theta r+\theta-u-1)}\binom{t+1-\theta r-\theta}{t}\binom{t}{u}p_{u,t}\frac{\theta r-\theta-1}{a[\theta(r+1)+s]+t}$ $\lambda=a\left[\theta\left(r+1\right)+s\right]+t$ and $g\left(x/\lambda,b\right)$ is the *pdf* of the exponentiated exponential distribution with parameters λ and b. Hence, several mathematical properties of the WEE distribution can easily be derived from that of the exponentiated exponential distribution.



(a) plot of the pdf of WEE distribution (b) Plot of the cdf of WEE distribution

Figure 1. *The pdf and cdf plots of the WEE distribution for some selected values.*

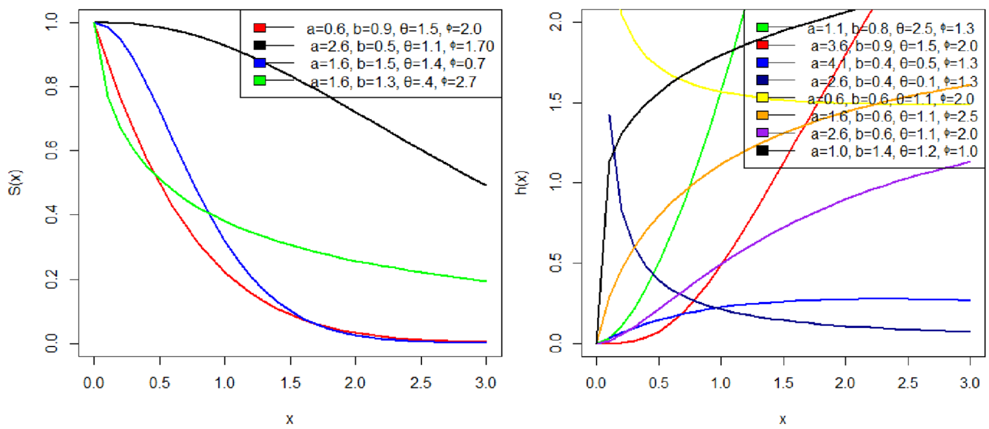
3.2. Quantile Function and Simulation

Random realizations from a given distribution are generated using the quantile function. The quantile function of the WEE distribution is given by:

$$Q\left(u\right)=-{}^{1}\!/_{b}\,\ell n\left\{1-\left[1-\exp\left(-\left(\ell n\left(1-u\right)^{-1/\phi}\right)^{1/\theta}\right)\right]^{1/a}\right\} \tag{8}$$

where u is a random number generated from uniform distribution. That is $u\sim U\left(0,1\right)$.

The first, second (median) and third quantiles are obtain by letting $u=0.25,0.50$ and 0.75 respectively. For instance, if we let $u=0.50$, we obtain the median of the WEE distribution given by:



(a) Plot of the survival function of WEE distribution (b) Plot of the Hazard function of WEE distribution

Figure 2. *The survival and hazard functions plots of the WEE distribution for some selected values.*

$$Median = \ell n \left\{ 1 - \left[1 - \exp\left(- \left({}^{1}\!/_{\phi} \ell n\,(2) \right)^{{}^{1}\!/_{\theta}} \right) \right]^{{}^{1}\!/_{a}} \right\}^{-{}^{1}\!/_{b}}$$

To obtain random realizations from the WEE distribution,

1. Generate a random variable U from $U\,(0,1)$.
2. Return $X = -{}^{1}\!/_{b} \ell n \left\{ 1 - \left[1 - \exp\left(- \left(\ell n\,(1-u)^{-{}^{1}\!/_{\phi}} \right)^{{}^{1}\!/_{\theta}} \right) \right]^{{}^{1}\!/_{a}} \right\}$.
3. The numbers obtained are said to follow the WEE distribution.

3.3. Mode

Let X be a random variable with pdf $f\,(x)$, then the mode of X is defined as:

$$\frac{d}{dx}\left(\ell n\,(f\,(x))\right) = 0 \tag{9}$$

Hence, if $X \sim WEE\,(a,b,\theta,\phi)$, then the mode of WEE distribution is obtain by substituting equation (6) in (9) Which gives:

$$\begin{aligned} Mode &= \frac{d}{dx}\Big[\ell n\,(a) + \ell n\,(b) + \ell n\,(\theta) + \ell n\,(\phi) - bx + (a-1)\,\ell n\left(1-e^{-bx}\right) - \ell n\left(1-\left(1-e^{-bx}\right)^{a}\right) \\ &+ (\theta-1)\,\ell n\left(-\ell n\left(1-\left(1-e^{-bx}\right)^{a}\right)\right) - \phi\left(-\ell n\left(1-\left(1-e^{-bx}\right)^{a}\right)\right)^{\theta}\Big] \end{aligned}$$

and simplifying it gives:

$$\begin{aligned} Mode &= -b + \frac{b\,(a-1)\,e^{-bx}}{1-e^{-bx}} + \frac{abe^{-bx}\left(1-e^{-bx}\right)^{a-1}}{1-\left(1-e^{-bx}\right)^{a}} - \frac{ab\,(\theta-1)\,e^{-bx}\left(1-e^{-bx}\right)^{a-1}}{\left[1-\left(1-e^{-bx}\right)^{a}\right]\ell n\left[1-\left(1-e^{-bx}\right)^{a}\right]} \\ &- \frac{ab\theta\phi e^{-bx}\left(1-e^{-bx}\right)^{a-1}\left[-\ell n\left(1-\left(1-e^{-bx}\right)^{a}\right)\right]^{\theta-1}}{1-\left(1-e^{-bx}\right)^{a}} = 0 \end{aligned} \tag{10}$$

Numerical methods such as bisection or fixed point methods should be used to solve this equation since obtaining an explicit solution of (10) is very difficult. However, it is easily observed that this mode reduces to that of Weibull distribution with parameters θ and ϕ when $a = b = 1$. Also when $a = 1$, the mode of the random variable X reduces to that of Weibull distribution with parameters θ and $b^{\theta}\phi$. When $\theta = \phi = 1$, it reduces to that of exponentiated exponential distribution.

3.4. Skewness and Kurtosis

The quantile function of the WEE distribution exist in a closed form. This make it easy in evaluating Galton's coefficient of skewness and Moor's coefficient of Kurtosis proposed by [35] and [36] respectively. These coefficients are defined by:

$$S_k = \frac{Q\left(3/4\right) + Q\left(3/8\right) + Q\left(1/4\right) - 2Q\left(1/2\right)}{Q\left(3/4\right) - Q\left(1/4\right)} \tag{11}$$

$$M_u = \frac{Q\left(7/8\right) + Q\left(3/8\right) + Q\left(1/8\right) - 2Q\left(5/8\right)}{Q\left(3/4\right) - Q\left(1/4\right)} \tag{12}$$

Using the quantile function in (8), the coefficient of skewness and kurtosis for the WEE distribution can easily be evaluated.

3.5. Characteristic Function

Let X be a random variable that follows the WEE distribution with pdf given in (6). The characteristic function of X defined as $\varphi_t(x) = E\left(e^{itx}\right)$ is obtain as:

$$\varphi_t(x) = \int_0^\infty e^{itx} \frac{ab\theta\phi \exp(-bx)\left(1-\exp(-bx)\right)^{a-1}}{1-\left(1-\exp(-bx)\right)^a} \left[-\ell n\left\{1-\left(1-\exp(-bx)\right)^a\right\}\right]^{\theta-1} \exp\left[-\phi\left(-\ell n\left\{1-\left(1-\exp(-bx)\right)^a\right\}\right)^\theta\right] dx$$

let $u = \phi\left(-\ell n\left\{1-\left(1-\exp(-bx)\right)^a\right\}\right)^\theta$. This gives:

$$\varphi_t(x) = \int_0^\infty e^{-u} \sum_{s=0}^{\infty} \binom{\frac{it}{b}+s-1}{s} \left(1-\exp\left(-\left(\frac{u}{\phi}\right)^{\frac{1}{\theta}}\right)\right)^{\frac{s}{a}} du$$

applying binomial series expansion to $\left(1-\exp\left(-\left(\frac{u}{\phi}\right)^{\frac{1}{\theta}}\right)\right)^{\frac{s}{a}}$ gives:

$$\varphi_t(x) = \sum_{s,v=0}^{\infty} \frac{(-1)^v \left(\frac{s}{a}\right)!}{\left(\frac{s}{a}-v\right)!s!v!} \left(\frac{it}{b}\right)_s \int_0^\infty e^{-u} e^{-v\left(\frac{u}{\phi}\right)^{\frac{1}{\theta}}} du$$

where $\left(\frac{it}{b}\right)_s = \frac{it}{b}\left(\frac{it}{b}+1\right)\left(\frac{it}{b}+2\right)\cdots\left(\frac{it}{b}+s-1\right)$. Hence, the characteristic function is given by:

$$\varphi_t(x) = \sum_{s,v,w=0}^{\infty} \frac{(-1)^{v+w}\left(\frac{s}{a}\right)! v^w \phi^{-\frac{w}{\theta}}}{\left(\frac{s}{a}-v\right)!s!v!w!} \Gamma\left(\frac{w}{\theta}+1\right)\left(\frac{it}{b}\right)_s \tag{13}$$

when power series expansion is applied to $e^{-v\left(\frac{u}{\phi}\right)^{\frac{1}{\theta}}}$.

3.6. Moment Generating Function

If X is a random variable that follows the WEE distribution with parameters a, b, θ and ϕ. Then the moment generating function of X defined by $M_X(t) = E\left(e^{tx}\right)$ is obtain as follows:

$$M_X(t) = \int_0^\infty e^{tx} \frac{ab\theta\phi \exp(-bx)\left(1-\exp(-bx)\right)^{a-1}}{1-\left(1-\exp(-bx)\right)^a} \left[-\ell n\left\{1-\left(1-\exp(-bx)\right)^a\right\}\right]^{\theta-1} \exp\left[-\phi\left(-\ell n\left\{1-\left(1-\exp(-bx)\right)^a\right\}\right)^\theta\right] dx$$

letting $u = \phi\left(-\ell n\left\{1-\left(1-\exp(-bx)\right)^a\right\}\right)^\theta$ and following the steps in section (3.5), the moment generating function is obtain as:

$$M_X(t) = \sum_{s,v,w=0}^{\infty} \frac{(-1)^{v+w}\left(\frac{s}{a}\right)! v^w \phi^{-\frac{w}{\theta}}}{\left(\frac{s}{a}-v\right)!s!v!w!} \Gamma\left(\frac{w}{\theta}+1\right)\left(\frac{t}{b}\right)_s \tag{14}$$

where $\left(\frac{t}{b}\right)_s = \frac{t}{b}\left(\frac{t}{b}+1\right)\left(\frac{t}{b}+2\right)\cdots\left(\frac{t}{b}+s-1\right)$.

3.7. Moments and Cumulants

The kth moment about the origin denoted by $\mu_k' = E\left(X^k\right)$ can easily be obtain from the moment generating function. This is done by differentiating the moment generating function k-times and letting $t = 0$. That is $\mu_k' = \frac{d^k}{dt^k}\left(M_X(t)\right)|_{t=0}$. The first and second moments of the WEE distribution are given by:

$$\mu_1' = \sum_{s,v,w=0}^{\infty} \frac{(-1)^{v+w}\left(\frac{s}{a}\right)! v^w \phi^{-\frac{w}{\theta}}}{\left(\frac{s}{a}-v\right)! v! w!} \frac{\Gamma\left(\frac{w}{\theta}+1\right)}{sb} \tag{15}$$

and

$$\begin{aligned} \mu_2' &= 2\sum_{s,v,w=0}^{\infty} \frac{(-1)^{v+w}\left(\frac{s}{a}\right)! v^w \phi^{-\frac{w}{\theta}}}{\left(\frac{s}{a}-v\right)! v! w!} \\ \Gamma & \left(\frac{w}{\theta}+1\right) \frac{\psi(s)-\psi(1)}{sb^2} \end{aligned} \tag{16}$$

respectively. where $\psi(.)$ is digamma functions. Digamma function is referred to as the derivative of the logarithm of gamma function. The variance of the WEE distribution is obtain using equations (15) and (16) by using the relation $var(x) = \mu_2' - \left(\mu_1'\right)^2$. The kth central moments denoted by μ_k is related to the kth non-central moments by:

$$\mu_k = \sum_{y=0}^{k} (-1)^y \binom{k}{y} \left(\mu_1'\right)^k \mu_{k-y}' \tag{17}$$

The kth kumulant denoted by κ_k is obtain by:

$$\kappa_k = \mu_k' - \sum_{p=0}^{k-1} \binom{k-1}{p-1} \kappa_p \mu_{k-p}' \tag{18}$$

The first three cumulants are given by:

$$\kappa_1 = \mu_1' \tag{19}$$

$$\kappa_2 = \mu_2' - \left(\mu_1'\right)^2 \tag{20}$$

$$\kappa_3 = \mu_3' - 3\mu_2'\mu_1' + \left(\mu_1'\right)^3 \tag{21}$$

It is important to note that coefficients of skewness and kurtosis can easily be evaluated from moments about the origin using appropriate relations.

3.8. Order Statistics

The probability density function (pdf) of the ith order statistics from an independent random sample with cumulative distribution function $(cdf) F(x)$ and p$df\, f(x)$ is defined by:

$$f_{i:n}(x) = \frac{n!}{(i-1)!\,(n-i)!} f(x) \left(F(x)\right)^{i-1} \left(1-F(x)\right)^{n-i}$$

This can easily be written as:

$$f_{i:n}(x) = \sum_{y=0}^{n-i} \frac{(-1)^y \Gamma(n+1)}{\Gamma(i)\Gamma(n-i-y+1)} f(x) \left(F(x)\right)^{y+i-1} \tag{22}$$

From equation (5),

$$\begin{aligned} [F(x)]^{y+i-1} &= \left\{1-\exp\left[-\phi\left(-\ell n\left\{1-\left(1-\exp(-bx)\right)^a\right\}\right)^{\theta}\right]\right\}^{y+i-1} \\ &= \sum_{z=0}^{\infty} \frac{(-1)^z \Gamma(y+1)}{\Gamma(y+i-z)\Gamma(z+1)} \exp\left\{-z\phi\left[-\ell n\left\{1-\left(1-\exp(-bx)\right)^a\right\}\right]^{\theta}\right\} \end{aligned}$$

Substituting this and (6) in (14) gives the ith order statistics for the WEE distribution as:

$$\begin{aligned} f_{i:n}(x) = \sum_{y=0}^{n-i}\sum_{z=0}^{\infty} & \frac{(-1)^{y+z}\Gamma(n+1)\Gamma(y+i)}{(z+1)\Gamma(i)\Gamma(y+1)\Gamma(z+1)\Gamma(n-i-y+1)\Gamma(y+i-z)} \frac{ab\theta\lambda\exp(-bx)\left(1-\exp(-bx)\right)^{a-1}}{1-\left(1-\exp(-bx)\right)^a} \\ & \left[-\ell n\left\{1-\left(1-\exp(-bx)\right)^a\right\}\right]^{\theta-1} \exp\left[-\lambda\left(-\ell n\left\{1-\left(1-\exp(-bx)\right)^a\right\}\right)^{\theta}\right] \end{aligned}$$

where $\lambda = (z+1)\phi$. Hence, the pdf of the ith order statistics is given by:

$$f_{i:n}(x) = \sum_{y=0}^{n-i}\sum_{z=0}^{\infty} \Omega_i f_i(x) \tag{23}$$

where $\Omega_i = \frac{(-1)^{y+z}\Gamma(n+1)\Gamma(y+i)}{(z+1)\Gamma(i)\Gamma(y+1)\Gamma(z+1)\Gamma(n-i-y+1)\Gamma(y+i-z)}$ and $f_i(x)$ is the pdf of WEE distribution with parameters a, b, θ and λ. Thus, the density function of the WEE order statistics is a linear mixture of WEE distribution. Based on equation (23), some structural properties of the ith order statistics such as characteristic function, moment generating function, moments etc. can be deduced from that of WEE distribution. For example, the moment generating function of $X_{i:n}$ can easily be obtain from the moment generating function in equation (14) with new parameters a, b, θ and λ.

3.9. Shannon Entropy

The entropy of a random variable X, is defined as a measure of uncertainty about the outcome of a random experiment. Different entropies such as Shannon and Rényi entropies have been studied and discussed by researchers. The Shannon entropy of a random variable X denoted by η_X is defined by:

$$\eta_X = E\left[-\ell n\left(f\left(x\right)\right)\right]$$

If X is a random variable that follows the WEE distribution with parameters a, b, θ and ϕ. Then, the Shannon entropy is obtain as follows:

$$\begin{aligned}\eta_X &= E\left(-\ell n\left(a\right)\right) - E\left(\ell n\left(b\right)\right) - E\left(\ell n\left(\theta\right)\right) - E\left(\ell n\left(\phi\right)\right) + E\left(bx\right) - E\left(\left(a-1\right)\ell n\left(1-e^{-bx}\right)\right) - \\ &E\left(\left(\theta-1\right)\ell n\left(-\ell n\left(1-\left(1-e^{-bx}\right)^a\right)\right)\right) + E\left(\phi\left[-\ell n\left(1-\left(1-e^{-bx}\right)^a\right)\right]^\theta\right) + \left(\ell n\left(1-\left(1-e^{-bx}\right)^a\right)\right)\end{aligned}$$

We obtain the Shannon entropy as:

$$\eta_X = -\ell n\left(ab\theta\phi\right) + b\mu_X - \frac{a-1}{a}\sum_{s=1}^{\infty}\sum_{v=0}^{\infty}\frac{\left(-1\right)^v s^{v-1}}{v!\phi^{\frac{v}{\theta}}}\Gamma\left(\frac{v}{\theta}+1\right) - \frac{\theta-1}{\theta}\left(\psi\left(1\right)-\ell n\left(\phi\right)\right) + \frac{1}{\phi^{1/\theta}}\Gamma\left(1/\theta+1\right) + 1$$

since

$$E\left[\ell n\left(-\ell n\left(1-\left(1-e^{-bx}\right)^a\right)\right)\right] = \frac{1}{\theta}\left(\psi\left(1\right)-\ell n\left(\phi\right)\right), E\left\{\left[-\ell n\left(1-\left(1-e^{-bx}\right)^a\right)\right]^\theta\right\} = \frac{1}{\phi},$$

and

$$E\left[\ell n\left(1-\left(1-e^{-bx}\right)^a\right)\right] = \frac{1}{\phi^{1/\theta}}\Gamma\left(1/\theta+1\right)$$

4. Estimation of the Parameters of the WEE Distribution

In this section, the unknown parameters of the WEE distribution will be estimated using the Maximum Likelihood Estimation technique.

Let $x_1, x_2, \cdots, x_n$ be a random sample of size n drawn from the WEE distribution with parameters a, b, θ and ϕ. The likelihood is defined by:

$$\begin{aligned}&L\left(x;\ a,\ b,\ \theta,\ \phi\right) = \prod_{i=1}^{n} f\left(x_i;\ a,\ b,\ \theta,\ \phi\right) \\ &= \prod_{i=1}^{n}\frac{ab\theta\phi\exp\left(-bx_i\right)\left(1-\exp\left(-bx_i\right)\right)^{a-1}}{1-\left(1-\exp\left(-bx_i\right)\right)^a}\left[-\ell n\left\{1-\left(1-\exp\left(-bx_i\right)\right)^a\right\}\right]^{\theta-1}\exp\left[-\phi\left(-\ell n\left\{1-\left(1-\exp\left(-bx_i\right)\right)^a\right\}\right)^\theta\right] \\ &= a^n b^n \theta^n \phi^n \exp\left(-b\sum_{i=1}^{n}x_i\right)\exp\left[-\phi\sum_{i=1}^{n}\left(-\ell n\left\{1-\left(1-\exp\left(-bx_i\right)\right)^a\right\}\right)^\theta\right]\prod_{i=1}^{n}\frac{\left(1-\exp\left(-bx_i\right)\right)^{a-1}}{1-\left(1-\exp\left(-bx_i\right)\right)^a}\left[-\ell n\left\{1-\left(1-\exp\left(-bx_i\right)\right)^a\right\}\right]^{\theta-1}\end{aligned} \tag{24}$$

Taking the natural logarithm of (24) yields the log likelihood of WEE distribution given by:

$$\begin{aligned}\ell\left(x;\ a,\ b,\ \theta,\ \phi\right) &= n\log\left(a\right) + n\log\left(b\right) + n\log\left(\theta\right) + n\log\left(\phi\right) - b\sum_{i=1}^{n}x_i - \phi\sum_{i=1}^{n}\left(-\log\left\{1-\left(1-\exp\left(-bx_i\right)\right)^a\right\}\right)^\theta \\ &+ \left(a-1\right)\sum_{i=1}^{n}\log\left(\left(1-\exp\left(-bx_i\right)\right)\right) + \left(\theta-1\right)\sum_{i=1}^{n}\log\left[-\log\left\{1-\left(1-\exp\left(-bx_i\right)\right)^a\right\}\right] \\ &- \sum_{i=1}^{n}\log\left[1-\left(1-\exp\left(-bx_i\right)\right)^a\right]\end{aligned} \tag{25}$$

The MLE of the parameters a, b, θ and ϕ denoted by $\widehat{a}, \widehat{b}, \widehat{\theta}$ and $\widehat{\phi}$ are obtained by differentiating (25) partially with respect to

a, b, θ and ϕ. Hence, the normal equations are:

$$\frac{\partial \ell}{\partial a} = \frac{n}{a} - \theta\phi \sum_{i=1}^{n} \frac{(1-\exp(-bx_i))^a \log(1-\exp(-bx_i))}{1-(1-\exp(-bx_i))^a} \left[-\log\left(1-(1-\exp(-bx_i))^a\right)\right]^{\theta-1} + \sum_{i=1}^{n} \log(1-\exp(-bx_i))$$
$$+ \sum_{i=1}^{n} \frac{(1-\exp(-bx_i))^a \log(1-\exp(-bx_i))}{1-(1-\exp(-bx_i))^a} - (\theta-1) \sum_{i=1}^{n} \frac{(1-\exp(-bx_i))^a \log(1-\exp(-bx_i))}{1-(1-\exp(-bx_i))^a \log\left(1-(1-\exp(-bx_i))^a\right)} \tag{26}$$

$$\begin{aligned} \frac{\partial \ell}{\partial b} &= \frac{n}{b} + a\sum_{i=1}^{n} \frac{x_i \exp(-bx_i)}{1-\exp(-bx_i)} - \sum_{i=1}^{n} \frac{x_i}{1-\exp(-bx_i)} \\ &- a\theta\phi \sum_{i=1}^{n} \frac{x_i \exp(-bx_i)(1-\exp(-bx_i))^{a-1}}{1-(1-\exp(-bx_i))^a} \left(-\log\left(1-(1-\exp(-bx_i))^a\right)\right)^{\theta-1} \\ &+ a\sum_{i=1}^{n} \frac{x_i \exp(-bx_i)(1-\exp(-bx_i))^{a-1}}{1-(1-\exp(-bx_i))^a} - a(\theta-1)\sum_{i=1}^{n} \frac{x_i \exp(-bx_i)(1-\exp(-bx_i))^{a-1}}{\left[1-(1-\exp(-bx_i))^a\right] \log\left(1-(1-\exp(-bx_i))^a\right)} \end{aligned} \tag{27}$$

$$\begin{aligned} \frac{\partial \ell}{\partial \theta} &= \frac{n}{\theta} - \phi\sum_{i=1}^{n} \left(-\log\left(1-(1-\exp(-bx_i))^a\right)\right)^{\theta} \log\left(1-(1-\exp(-bx_i))^a\right) \\ &+ \sum_{i=1}^{n} \log\left[-\log\left(1-(1-\exp(-bx_i))^a\right)\right] \end{aligned} \tag{28}$$

and

$$\frac{\partial \ell}{\partial \phi} = \frac{n}{\phi} - \sum_{i=1}^{n} \left(-\log\left(1-(1-\exp(-bx_i))^a\right)\right)^{\theta} \tag{29}$$

From equation (29), the MLE of ϕ is obtain as a function of a, b, θ, say $\widehat{\phi}(a, b, \theta)$, where

$$\widehat{\phi}(a, b, \theta) = \frac{n}{\sum\limits_{i=1}^{n} \left(-\log\left(1-(1-\exp(-bx_i))^a\right)\right)^{\theta}}$$

Solving equations (26-28) analytically may be intractable. However, these equations can be solved numerically using iterative procedures such as Newton Raphson method. Statistical software can also be used to maximized the likelihood function. This includes the maxLik package, Adequacy package, fitdist package all in R. For interval estimation and hypothesis testing on the parameters a, b, θ and ϕ of the model, we obtain the Fisher Information Matrix $J(\Phi)$. where $\Phi = (a, b, \theta, \phi)^T$ and

$$J(\Phi) = -\begin{pmatrix} V_{aa} & V_{ab} & V_{a\theta} & V_{a\phi} \\ & V_{bb} & V_{b\theta} & V_{b\phi} \\ & & V_{\theta\theta} & V_{\theta\phi} \\ & & & V_{\phi\phi} \end{pmatrix}$$

The diagonal elements are the variances of the corresponding parameters while the off-diagonals elements are the covariances. The elements of $J(\Phi)$ are given in the appendix. The asymptotic distribution of $\sqrt{n}\left(\widehat{\Phi} - \Phi\right)$ is multivariate normal $N_4\left(0, J(\Phi)^{-1}\right)$. Confidence interval for the parameters a, b, θ and ϕ are constructed using the asymptotic distribution $N_4\left(0, J\left(\widehat{\Phi}\right)^{-1}\right)$, where $J\left(\widehat{\Phi}\right)$ is the total observed information matrix evaluated at $\widehat{\Phi}$.

Hence, the asymptotic $100(1-\gamma)\%$ confidence interval for a, b, θ and ϕ are $\widehat{a} \pm Z_{\gamma/2}\sqrt{Var(\widehat{a})}$, $\widehat{b} \pm Z_{\gamma/2}\sqrt{Var\left(\widehat{b}\right)}$, $\widehat{\theta} \pm Z_{\gamma/2}\sqrt{Var\left(\widehat{\theta}\right)}$ and $\widehat{\phi} \pm Z_{\gamma/2}\sqrt{Var\left(\widehat{\phi}\right)}$ respectively. where $Z_{\gamma/2}$ is the $100(1-\gamma)\%$ quantile of the standard normal distribution.

To test for the goodnesss of fit of the WEE distribution, the likelihood ratio (LR) statistic can be used. It can also be used in comparing this distribution with some of its sub-models such as Weibull , exponentiated exponential distribution and exponential distribution. In constructing LR statistics for testing some of the sub-models of the WEE distribution, the maximum values of the unrestricted and restricted log-likelihoods are computed. For example, the LR statistics can be used to check whether the fitted WEE distribution for a given data set is statistically superior to the fits of its sub-models. In such a situation, we formulate the hypothesis $H_0 : \Phi = \Phi_0$ against $H_1 : \Phi \neq \Phi_0$ using the LR statistics. For instance, the LR statistics for testing $H_0 : a = 1$ against $H_1 : a \neq 1$ which is equivalent to compare the Weibull and WEE distributions is $\tau = 2\left[l\left(\widehat{a}, \widehat{b}, \widehat{\theta}, \widehat{\phi}\right) - l\left(\widetilde{a}, \widetilde{b}, \widetilde{\theta}, \widetilde{\phi}\right)\right]$ Where $\widehat{a}, \widehat{b}, \widehat{\theta}, \widehat{\phi}$ are the $MLEs$ under H_1 and $\widetilde{a}, \widetilde{b}, \widetilde{\theta}, \widetilde{\phi}$ are the estimates under H_0. The statistic τ is asymptotically distributed as χ^2_p and the null hypothesis is rejected when $\tau >$

κ_ε where κ_ε is the upper $100\varepsilon\%$ point of the χ^2_p distribution.

5. Applications

In order to illustrate the flexibility of the WEE distribution, we fit the WEE distribution and its sub-models (Weibull, Exponentiated exponential and Exponential distributions) to a real data set. This data consist of the lifetimes of fifty (50) devices given by [37]. The data have a bathtube-shaped hazard function and is given as follows:
0.1, 0.2, 1.0, 1.0, 1.0, 1.0, 1.0, 2.0, 3.0, 6.0, 7.0 ,11.0, 12.0, 18.0, 18.0, 18.0, 18.0, 18.0, 21.0, 32.0, 36.0, 40.0, 45.0, 46.0, 47.0, 50.0, 55.0, 60.0, 63.0, 63.0, 67.0, 67.0, 67.0, 67.0, 72.0, 75.0, 79.0, 82.0, 82.0, 83.0, 84.0, 84.0, 84.0, 85.0, 85.0, 85.0, 85.0, 85.0, 86.0 and 86.0

The $MLEs$ of the model parameters are determined and some goodness of fit statistics for these distributions are compared. Model selection is carried out based upon Akaike Information Criteria (AIC), Bayesian Information Criteria, Consistent Akaike Information Criteria $(CAIC)$ and Hannan-Quinn Information Criterion $(HQIC)$ statistics. Model with smaller values of these statistics is considered to be the best model.

***Table 1.** Descriptive statistics for fifty lifetime devices.*

N	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.	Sd
50	0.1	13.5	48.5	45.69	81.25	86	32.84

***Table 2.** Maximum Likelihood Estimates, standard error in parenthesis and Information Criteria of the fitted models.*

Models	parameters								
	a	b	θ	ϕ	*LL*	*AIC*	*CAIC*	*BIC*	*HQIC*
WEED	0.3018 (0.0688)	0.0035 (0.0005)	5.536 (0.7058)	1.478 (0.4202)	-56.8199	121.6399	122.5287	120.4357	115.4813
WD			0.9487(0.1186)	0.027 (0.0138)	-241.0019	486.0038	486.2591	485.4017	482.9245
EED	0.7774 (0.1346)	0.0187 (0.0036)			-239.9956	483.9912	484.2465	483.3891	480.9119
ED	0.0219 (0.0031)				-241.0897	484.1794	484.2627	483.8784	482.6398

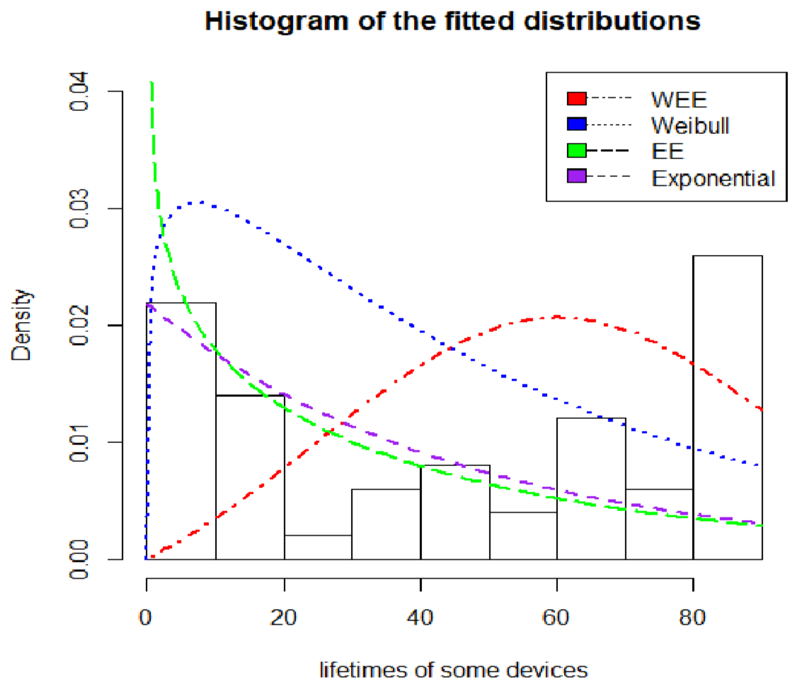


***Figure 3.** Histogram and the fitted WEE, Weibull, Exponentiated exponential and exponential distributions.*

Table 2 gives the estimated values for all the parameters of the WEE distribution and that of its sub-models. The estimates of the loglikelihood and the statistics: AIC, $CAIC$, BIC and $HQIC$ are also given. The WEE distribution have the least estimates of these statistics, thus, WEE distribution provides the best fits to this data and can be consider a very competitive model to other distributions. This is also evident from the histogram of the data set and plot of the fitted WEE density and that of its sub-models as displayed in figure **??**. Hence, WEE distribution could be chosen as the best model based on the data set used.

The asymptotic variance covariance matrix for the estimated parameters is given by:

$$\begin{pmatrix} 0.0047 & 4.930\times10^{-05} & -0.0292 & 0.0436 \\ & 2.378\times10^{-07} & -9.595\times10^{-05} & 0.0007 \\ & & 0.4981 & -0.4925 \\ & & & 0.1765 \end{pmatrix}$$

95% confidence interval for the estimates of the parameters of the WEE distribution a, b, θ and ϕ are: $(0.1670, 0.4365)$, $(0.0025, 0.0044)$, $(4.1527, 6.9193)$ and $(0.6545, 2.3015)$ respectively.

6. Conclusion

The $T - X$ family of distribution proposed by [10] and exponentiated exponential distribution proposed by [13] were used in redefining the pdf of Generalized Weibull-exponential distribution. As mentioned earlier, following [10], we call the resulting pdf Weibull exponentiated exponential distribution. Comprehensive mathematical properties of the distribution were given. Hypothesis testing on the inclusion of additional parameter was discussed. Finally, we fit a data set to the distribution and its sub-models so as to ascertain its performance when compared to the sub-models. It is observed from the performance measures used that the WEE distribution gives a better fit than its sub-models.

Appendix

Let

$$A_0 = x_i^2\,\mathrm{e}^{-bx_i}, A_1 = \left(1-\mathrm{e}^{-bx_i}\right)^a, A_2 = \left(1-\mathrm{e}^{-bx_i}\right)^{a-1}, A_3 = \left(1-\mathrm{e}^{-bx_i}\right)^{a-2}, A_4 = 1-\left(1-\mathrm{e}^{-bx_i}\right)^a,$$
$$B_1 = \left[-\log\left(1-\left(1-\mathrm{e}^{-bx_i}\right)^a\right)\right]^\theta, B_2 = \left[-\log\left(1-\left(1-\mathrm{e}^{-bx_i}\right)^a\right)\right]^{\theta-1}, B_3 = \left[-\log\left(1-\left(1-\mathrm{e}^{-bx_i}\right)^a\right)\right]^{\theta-2},$$
$$B_4 = \log\left(1-\mathrm{e}^{-bx_i}\right) \text{ and } \mathrm{B}_5 = \log\left(1-\left(1-\mathrm{e}^{-\mathrm{bx_i}}\right)^{\mathrm{a}}\right),$$

then the second order derivatives are given as:

$$\frac{\partial^2\ell}{\partial a^2} = -\frac{n}{a^2} - \theta\phi\sum\frac{A_1B_4B_3}{A_4}\left[\frac{A_1}{A_4}(\theta-B_5-1)-B_5\right] + \sum\frac{A_1B_4^2}{A_4^2} - (\theta-1)\sum\frac{A_1B_4^2}{A_4^2B_5^2}(B_5+A_1)$$
$$\frac{\partial^2\ell}{\partial b^2} = -\frac{n}{b^2} - (a-1)\sum\frac{A_0}{{}^{A_5^2}/_{a}} - a\theta\phi\sum\frac{A_0A_3B_3}{A_4}\left[a(\theta-1)A_1e^{-bx_i} - \left(ae^{-bx_i}-1\right)B_4 - \frac{A_1B_4}{A_4}\right]$$
$$+ a\sum\frac{A_0A_3}{A_4^2}\left(ae^{-bx_i}+A_1-1\right) - a(\theta-1)\sum\frac{A_0A_3}{A_4^2B_4^2}\left(\left(ae^{-bx_i}-1\right)B_4 + A_1B_4 - aA_1e^{-bx_i}\right)$$
$$\frac{\partial^2\ell}{\partial\theta^2} = -\frac{n}{\theta^2} + \phi\sum(-B_5)^{\theta+1}\log(-B_5)$$
$$\frac{\partial^2\ell}{\partial\phi^2} = -\frac{n}{\phi^2}$$
$$\frac{\partial^2\ell}{\partial a\partial b} = -\phi\sum\frac{A_1B_4}{A_4}(-B_5)^{\theta-1}(1+\theta\log(-B_5)) - \sum\frac{A_2B_4}{A_4B_5}$$
$$\frac{\partial^2\ell}{\partial a\partial\theta} = -\phi\sum\frac{A_1B_4}{A_4}B_2\left[1+\theta\log(-B_5)\right] - \sum\frac{A_1B_4}{A_4B_5}$$
$$\frac{\partial^2\ell}{\partial a\partial\phi} = -\theta\sum\frac{A_1B_4B_2}{A_4}$$
$$\frac{\partial^2\ell}{\partial b\partial\theta} = -a\phi\sum\frac{x_ie^{-bx_i}A_2B_2}{A_4}(1+\theta\log(-B_5)) - a\sum\frac{x_ie^{-bx_i}A_2}{A_4B_5}$$
$$\frac{\partial^2\ell}{\partial b\partial\phi} = -\theta\sum\frac{x_ie^{-bx_i}A_2B_2}{A_4}\quad,\quad\frac{\partial^2\ell}{\partial\theta\partial\phi} = (-B_5)^{\theta+1}$$

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