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An Empirical Examination of the Asymptotic Normality of the *k*th Order Statistic

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Abstract: In this paper, the equivalence of the sample *p*th quantile of a distribution and the *k*th order statistic of a random sample obtained from the distribution is reviewed. Based on the review, a new corollary on the almost sure convergence of the *k*th order statistic to the *p*th quantile was obtained without proof. Through an extensive Monte Carlo simulation, the extreme as well as the central *k*th order statistics of five different continuous distributions were obtained at different sample sizes and the asymptotic normality of the order statistics were investigated with the use of the Anderson – Darling (AD) statistic for normality test. The result showed among other things that asymptotic normality holds only for the central order statistics.

Keywords: Asymptotic Normality, Inverse Distribution Function, *k*th Order Statistic, Monte Carlo Simulation, *p*th Quantile, Test for Normality

1. Introduction

Let $x_1, x_2, \ldots, x_n$ be a set of n observations obtained from a known distribution $F(x)$. In the probability plot of this set of observations, emphasis is always made on the orderedness of the set in comparison with their corresponding expected ordered observations. These expected ordered observations (usually called expected order statistics) are obtained as the population *p*th quantiles of the distribution; $p \in (0, 1)$. The *p*th quantile of a distribution, $F(x)$, also known as the inverse distribution function, $F^{-1}(p)$ (Jones [1]), is denoted by ξ_p and defined by

$$\xi_p = F^{-1}(p) = \inf\{x : F(x) \geq p\};\ p \in (0, 1) \tag{1}$$

It exists for both discrete and continuous distributions. If $F(x)$ is continuous, ξ_p can be simplified by

$$\xi_p = F^{-1}(p) = \inf\{x : F(x) = p\};\ p \in (0, 1) \tag{2}$$

Xu and Miao [2] state that the *p*th quantile of a distribution, ξ_p, can be estimated by either the sample *p*th quantile of the distribution or the appropriate *k*th order statistic of a sample drawn from the distribution. This amounts to estimating a population parameter by either of two statistics which are of different concepts. The sample *p*th quantile of a distribution, $F(x)$, denoted by $\hat{\xi}_{np}$, is obtained as the inverse of the sample distribution function, denoted by $F_n(x)$, which is also called empirical distribution function (EDF). That is, precisely for $p \in (0, 1)$,

$$\hat{\xi}_{np} = F_n^{-1}(p) = \inf\{x : F_n(x) \geq p\} \tag{3}$$

For a random variable X with a distribution function $F(x)$, the sample distribution function $F_n(x)$ which is also known as the EDF is given as

$$F_n(x) = n^{-1}\sum_{j=1}^{n} I(X_j \leq x) \tag{4}$$

where $I(X_j \leq x)$ is an indicator function given by $I(X_j \leq x) = \begin{cases} 0 \text{ if } X_j > x \\ 1 \text{ if } X_j \leq x \end{cases}$. Let $\sum_{j=1}^{n} I(X_j \leq x)$ which is the number of observations in the random sample that are less than or equal to x be k. Then $F_n(x)$ = k/n. Hence, $p \in (0, 1)$ can be approximated by k/n such that the sample *p*th quantile of a distribution can be given as

$$\hat{\xi}_{np} = F_n^{-1}\left(\frac{k}{n}\right) = \inf\left\{x : F_n(x) \geq \frac{k}{n}\right\} \tag{5}$$

Also, suppose the random sample of n independent observations, $x_1, x_2, \ldots, x_n$, from a distribution $F(x)$ whose density function is $f(x)$ is arranged in an increasing order of magnitude $X_{(1)} \le X_{(2)} \le \ldots \le X_{(n)}$ such that $X_{(k)}$; $k = 1, 2, \ldots, n$ is the *k*th smallest observation in the sample. In its simplest sense, $X_{(k)}$ is known as the *k*th order statistic of the distribution obtained from the random sample that is drawn from the distribution. Severini [3] classified these *k*th order statistics, $k = 1, 2, \ldots, n$ into two, namely: central order statistics and extreme order statistics. The extreme order statistics are those *k*th order statistics such as the minimum and the maximum (i.e. $X_{(1)}$ and $X_{(n)}$ respectively) while the central order statistics are those other than the extremes, such as the median.

One area that has extensively employed the use of the population *p*th quantile as the expected value of either the *k*th order statistic or the sample *p*th quantile of a distribution is the probability plots class of tests for multivariate normality (MVN). This class includes the graphical procedures for assessing MVN such as Healy [4], Small [5], Scrucca [6] and the correlation and regression procedures for assessing MVN such as Ahn [7], Singh [8], and Hwu et al [9].

Again, the asymptotic normality of the *k*th order statistic $X_{(k)}$; $k = 1, 2, ..., n$ (or the sample *p*th quantile of a distribution, where $\sqrt{n}\left(\frac{k}{n} - p\right) \to 0$) has been discussed in the literature, especially by Bahadur [10], and Severini [3]. This suggests being true for all *k*th order statistics (including extreme and central order statistics). In this paper, we shall investigate empirically the asymptotic normality of the *k*th order statistic of a sample obtained from a distribution. This shall be preceded however by a review of the equivalence of the sample *p*th quantile of a distribution and the *k*th order statistic of a sample obtained from the distribution.

2. Sample *p*th Quantile of a Distribution and the *k*th Order Statistic

Equivalence of the *k*th order statistic and the sample *p*th quantile of a distribution, where $\frac{k}{n} \to p$ as $n \to \infty$ has been discussed in the literature, especially by Bahadur 10]. Suppose a continuous distribution function, $F(x)$ is at least twice differentiable in some neighbourhood ξ_p with $F'(\xi_p) = f(\xi_p) > 0$. Bahadur [10] showed that the sample *p*th quantile, $\hat{\xi}_{np}$ of the distribution is

$$\hat{\xi}_{np} = \xi_p + \frac{p - F_n(\xi_p)}{f(\xi_p)} + R_n \tag{6}$$

where $R_n = O\left(n^{-3/4} \log n\right)$ as $n \to \infty$. He also showed that for $k = np + o\left(n^{1/2} \log n\right)$ as $n \to \infty$, the *k*th order statistic, $X_{(k)}$, is

$$X_{(k)} = \xi_p + \frac{k/n - F_n(\xi_p)}{f(\xi_p)} + R_n \tag{7}$$

where R_n is as defined in (6). These results in (6) and (7) show that the *k*th order statistic of a sample of size n observations, taken from a distribution $F(x)$ with density function $f(x)$, which is given as $X_{(k)}$, is equivalent to the sample *p*th quantile of the distribution, obtained from the sample, provided $\frac{k}{n} \to p$ as $n \to \infty$. Notice that $\frac{k}{n} \to p$ implies that $k \to np$. But k is an integer. Hence, $k = [np] + \delta_n$ where $[np]$ is the integral part of np such that $\delta_n \to 0$. Using the results of Bahadur [10] given in (6) and (7), Serfling [11] states, as a corollary, that the equivalence of $X_{(k)}$ and $\hat{\xi}_{np}$, obtained from the same sample, depends on the rate of convergence of $\frac{k}{n}$ to p. His corollary is as follows:

Corollary (Serfling [11]): Assume a continuous distribution function, $F(x)$ to be at least twice differentiable in some neighbourhood ξ_p with $F'(\xi_p) = f(\xi_p) > 0$ and suppose that

$$\frac{k}{n} = p + \frac{r}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right) \text{as } n \to \infty \tag{8}$$

Where r is a constant determined by the rate of convergence of $\frac{k}{n}$ to p. Then

$$\sqrt{n}\left(X_{(k)} - \hat{\xi}_{np}\right) \xrightarrow{a.s.} \frac{r}{f(\xi_p)} \tag{9}$$

and

$$\sqrt{n}\left(X_{(k)} - \xi_p\right) \sim N\left(\frac{r}{f(\xi_p)}; \frac{p(1-p)}{f^2(\xi_p)}\right) \tag{10}$$

The result in (9) is true because from (6) and (7),

$$X_{(k)} - \hat{\xi}_{np} = \frac{k/n - F_n(\xi_p)}{f(\xi_p)} - \frac{p - F_n(\xi_p)}{f(\xi_p)} = \frac{1}{f(\xi_p)}\left(\frac{k}{n} - p\right) \tag{11}$$

Using the expression for $\frac{k}{n}$ in (8) and upon simplification, (11) reduces to (9).

From the foregoing results, we have the following corollary:

Corollary 2.1. Suppose a random variable X has a distribution function $F(x)$ whose *p*th quantile is ξ_p, $p \in (0, 1)$. Let $X_{(k)}$ be the kth order statistic of the random sample $x_1, x_2, \ldots, x_n$ drawn from the distribution. Assume that $F(x)$ has a continuous density function $f(x)$ in the neighbourhood of ξ_p and $f(x) > 0$. Suppose further that $\sqrt{n}\left(\frac{k}{n} - p\right) \to 0$ as $n \to \infty$ and that the convergence is very fast. Then, $\frac{k}{n} = p + \frac{r}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right)$; $n \to \infty$ is said to become $\frac{k}{n} = p + o\left(\frac{1}{\sqrt{n}}\right)$; $n \to \infty$ such that

$$\left|X_{(k)} - \hat{\xi}_{np}\right| \xrightarrow{a.s.} 0 \tag{12}$$

Also, Xu and Miao [2] state that the *k*th order statistic in a sample of size *n* converges in probability to the population *p*th quantile, $\xi_p, p \in (0, 1)$, as $\frac{k}{n}$ tends to p. This statement is true and is in line with Serfling [11] who stated the following theorem:

Theorem (Serfling [11]): Let $0 < p < 1$, if ξ_p is the unique solution of x of $F(x-) \le p \le F(x)$, then for every $\varepsilon > 0$,

$$P\left(\left|\hat{\xi}_{np} - \xi_p\right| \ge \varepsilon\right) \le 2e^{-2n\delta_\varepsilon^2} \tag{13}$$

where $\delta_\varepsilon = \min\left\{F\left(\xi_p + \varepsilon\right) - p;\ p - F\left(\xi_p - \varepsilon\right)\right\}$.

Applying the equivalence between $X_{(k)}$ and $\hat{\xi}_{np}$ which has so far been established, (13) can be written as

$$P\left(\left|X_{(k)} - \xi_p\right| \ge \varepsilon\right) \le 2e^{-2n\delta_\varepsilon^2} \tag{14}$$

Convergence in probability of $X_{(k)}$ to ξ_p implies that for every $\varepsilon > 0$ as $n \to \infty$, $P\left(\left|X_{(k)} - \xi_p\right| \ge \varepsilon\right) = 0$. Hence as $n \to \infty$, $P\left(\left|X_{(k)} - \xi_p\right| \ge \varepsilon\right) \le 2e^{-2n\delta_\varepsilon^2} \to 0$. This is because

$$\lim_{n \to \infty} P\left(\left|X_{(k)} - \xi_p\right| \ge \varepsilon\right) \le \lim_{n \to \infty} 2e^{-2n\delta_\varepsilon^2} = 0 \tag{15}$$

3. Asymptotic Normality of the *k*th Order Statistics

Suppose $x_1, x_2, \ldots, x_n$ is a random sample of size *n* from a distribution $F(x)$ whose density is $f(x)$. If $F(x)$ is continuous at ξ_p, $p \in (0, 1)$ and *n* is large, Bahadur [10], Babu [12], Miao et al [13], and Mood et al [14] have stated that $\sqrt{n}\left(X_{(k)} - \xi_p\right) \sim N\left(0; \frac{p(1-p)}{f^2(\xi_p)}\right)$. This implies that

$$\sqrt{n} f(\xi_p) \frac{\left(X_{(k)} - \xi_p\right)}{\sqrt{p(1-p)}} \sim N\left(0; 1\right) \tag{16}$$

The statistic in (16) has ξ_p as the expected value of the *k*th order statistic, usually denoted by $E(X_{(k)})$. That is, $E(X_{(k)})$ is the inverse distribution function at $p \in (0, 1)$. However, $p \in (0, 1)$ cannot be determined explicitly. David and Nagaraja [15] state that for sufficiently large *n*, an approximation to the expected *k*th order statistic is given as

$$E(X_{(k)}) = F^{-1}\left(\frac{k}{n+1}\right) = \xi_{\left(\frac{k}{n+1}\right)} \tag{17}$$

Since $f(x) > 0$ is continuous, it is well known that its point wise probability approaches zero. However, van der Vaart [16] states that the empirical quantile function is related to the order statistic of a sample through $F_n^{-1}(p) = X_{(k)}$ for $p \in \left(\frac{k-1}{n}, \frac{k}{n}\right]$. The asymptotic distribution in (16) can therefore be obtained with $f(x)$ evaluated in the interval $\left(E(X_{(k-1)}), E(X_{(k)})\right)$. A good number of authors have investigated the asymptotic normality of functions of order statistics. Such authors include Pagurova [17], Dembiraska [18] and Jasinski [19].

3.1. Empirical Studies

In this subsection, the asymptotic normality of the *k*th order statistic through an extensive simulation studies shall be investigate empirically. Here, we simulated 10000 samples of the same size, in each trial, from some randomly selected continuous distributions. The distributions selected for this investigation include the standard normal distribution, the beta distribution (first kind) with parameters 0.5 and 1, the beta distribution (first kind) with parameters 2 and 1, the chi square distribution with 2 degrees of freedom, the student's t distribution with 20 degrees of freedom and the uniform distribution in the interval (0, 1). The parameters of these distributions were arbitrarily chosen and each distribution was studied at sample sizes 10, 20, 30, 50 and 100. In each case, the *k*th order statistic, $k = 1, 2, \ldots, n$, is obtained for each sample together with their corresponding approximate expected order statistics. The statistic in (16) is evaluated for each *k*th order statistic with $p = \frac{k - 0.5}{n}$, which may be seen as the mean value of the interval $p \in \left(\frac{k-1}{n}, \frac{k}{n}\right]$. The result of this evaluation on each kth order statistic obtained from each sample gives rise to a transformation of the ordered set of data in each sample to a supposed standard normal order statistics if the null distribution is true. From the standardized set of data in each sample, we selected five data points corresponding to the first order statistic (the minimum), the sample 0.25th quantile (the first quartile), 0.5th quantile (the median), 0.75th quantile (the third quartile) and the *n*th order statistic (the maximum). The sample quantiles were obtained from the transformed data set by taking the appropriate weighted averages since, for instance, all the sample sizes considered were even numbers and as a result, no specific observation in the data could be said to be a median. They are denoted in this work as Min, $Q_{0.25}$, Median, $Q_{0.75}$ and Max respectively.

Table 1. *p-values of the Tests for Normality of Some Standardized Order Statistics Obtained from 10000 Simulated Samples from Some Selected Continuous Distributions at Sample Sizes n = 10, 20, 30, 50 and 100.*

Sample size	Selected order statistics	B(0.5, 1)	B(2, 1)	Chi-square(2)	N(0, 1)	$t_{(20)}$	U(0, 1)
	Min	<0.005	<0.005	<0.005	<0.005	<0.005	<0.005
	$Q_{0.25}$	<0.005	<0.005	<0.005	<0.005	<0.005	<0.005
10	Median	<0.005	<0.005	<0.005	0.2310*	0.5130*	<0.005
	$Q_{0.75}$	<0.005	<0.005	<0.005	<0.005	<0.005	<0.005
	Max	<0.005	<0.005	<0.005	<0.005	<0.005	<0.005
	Min	<0.005	<0.005	<0.005	<0.005	<0.005	<0.005
	$Q_{0.25}$	<0.005	<0.005	<0.005	0.0050	<0.005	<0.005
20	Median	<0.005	<0.005	<0.005	0.8520*	0.3030*	0.0500*
	$Q_{0.75}$	<0.005	<0.005	<0.005	<0.005	<0.005	<0.005
	Max	<0.005	<0.005	<0.005	<0.005	<0.005	<0.005
	Min	<0.005	<0.005	<0.005	<0.005	<0.005	<0.005
	$Q_{0.25}$	<0.005	0.0130*	<0.005	<0.005	<0.005	<0.005
30	Median	<0.005	<0.005	<0.005	0.0440*	0.8210*	0.2680*
	$Q_{0.75}$	<0.005	<0.005	<0.005	<0.005	<0.005	<0.005
	Max	<0.005	<0.005	<0.005	<0.005	<0.005	<0.005
	Min	<0.005	<0.005	<0.005	<0.005	<0.005	<0.005
	$Q_{0.25}$	<0.005	0.1580*	<0.005	<0.005	<0.005	<0.005
50	Median	<0.005	<0.005	<0.005	0.2460*	0.7940*	0.6770*
	$Q_{0.75}$	<0.005	<0.005	<0.005	0.1890*	<0.005	<0.005
	Max	<0.005	<0.005	<0.005	<0.005	<0.005	<0.005
	Min	<0.005	<0.005	<0.005	<0.005	<0.005	<0.005
	$Q_{0.25}$	<0.005	0.2060*	<0.005	0.2120*	0.2090*	<0.005
100	Median	<0.005	<0.005	<0.005	0.9260*	0.5190*	0.1990*
	$Q_{0.75}$	0.1130*	<0.005	<0.005	0.1610*	0.0850*	<0.005
	Max	<0.005	<0.005	<0.005	<0.005	<0.005	<0.005

Note: * implies normal at 1% level of significance.

Table 2. *Computed Values of the Test statistic from the Tests for Normality of Some Standardized Order Statistics Obtained from 10000 Simulated Samples from Some Selected Continuous Distributions at Sample Sizes n = 10, 20, 30, 50 and 100.*

Sample size	Selected order statistics	B(0.5, 1)	B(2, 1)	Chi-Square(2)	N(0, 1)	$t_{(20)}$	U(0, 1)
	Min	1301.303	30.293	474.143	17.597	48.129	369.537
	$Q_{0.25}$	382.329	2.083	158.067	3.741	4.230	57.664
10	Median	63.659	18.455	85.594	0.482	0.331	5.538
	$Q_{0.75}$	18.193	88.616	72.273	3.108	6.740	58.314
	Max	290.283	404.42	141.526	20.030	50.749	364.622
	Min	1432.735	44.895	443.424	32.140	60.946	424.700
	$Q_{0.25}$	251.245	1.203	87.266	1.145	1.797	29.793
20	Median	38.718	13.324	50.248	0.214	0.433	0.753
	$Q_{0.75}$	9.488	53.100	38.199	1.684	2.174	31.257
	Max	366.792	429.990	131.717	24.621	61.918	414.228
	Min	1490.988	43.773	456.014	25.510	78.864	432.428
	$Q_{0.25}$	189.137	0.993	58.097	1.599	1.794	17.658
30	Median	36.978	11.715	31.193	0.774	0.225	0.456
	$Q_{0.75}$	3.431	38.209	20.102	1.558	3.077	20.366
	Max	388.917	419.722	126.663	35.381	69.508	413.096
	Min	1527.952	55.830	475.230	45.251	75.240	435.736
	$Q_{0.25}$	115.018	0.548	30.593	1.273	1.164	11.944
50	Median	19.761	3.197	19.250	0.471	0.234	0.270
	$Q_{0.75}$	1.858	20.618	18.211	0.518	1.166	11.956
	Max	411.105	442.653	143.623	42.951	81.141	426.268
	Min	1585.250	60.897	473.796	41.634	89.031	455.694
	$Q_{0.25}$	58.018	0.503	19.706	0.497	0.500	7.015
100	Median	10.622	2.784	9.860	0.174	0.327	0.508
	$Q_{0.75}$	0.609	11.088	10.838	0.545	0.660	4.968
	Max	443.556	494.474	132.899	46.146	82.358	471.731

From the foregoing, it is expected that each of these standardized order statistics in a sample is a standard normal observation. We tested for normality of the 10000 standardized observations for each selected sample quantile per sample size. There are several powerful tests for univariate normality in the literature. Our study here is based on one of them, known as the Anderson - Darling test. It is denoted here by AD and its statistic is given as

$$AD = -n - \frac{\sum_{k=1}^{n} (2k-1)\left\{\ln(Z_k) + \ln(1 - Z_{n-k+1})\right\}}{n} \quad (18)$$

where $Z_k = \Phi\left(\frac{X_{(k)} - \overline{X}}{s_{\overline{X}}}\right)$. We therefore test this standard normality for the 5 selected quantiles for sample sizes, n = 10, 20, 30, 50 and n = 100. For each sample size, 10000 samples were used to test the normality of the standardized sample quantiles. This was done for the selected distribution of the original sample data. The results of the normality tests are presented in Tables 1 and 2.

3.2. Discussion of Results

The normality tests conducted on the selected order statistics show some interesting results. Firstly from Table 1, the chi square distribution has all the selected order statistics (both extreme and central order statistics) in all the considered sample sizes as non-normal at 0.01 level of significance. This is because all the p-values obtained for the distribution were significantly less than 0.01. In the case of the beta distribution with parameters 0.5 and 1, only the sample 0.75th quantile was normal at only the sample size of 100. Also, the beta (2, 1) distribution has only the sample 0.25th quantile to be normal only at the sample sizes 30, 50 and 100. In the case of the standard normal distribution, only the median was normal for all the sample sizes at 1% level of significance. In addition, the sample 0.75th quantile was normal for a sample size of 50 while the sample 0.25th quantile was normal for only sample size of 100. This means that all the central order statistics considered were normal when the sample size was 100. The student's t distribution has the median to be normal in all the sample sizes considered with the sample 0.25th and 0.75th quantiles in addition only in the sample size 100. The uniform distribution showed that only the median in sample sizes 20, 30, 50 and 100 was normally distributed.

Secondly, the Anderson-Darling statistic which we employed in this study lacks evidence for rejection of normality of a set of data when the computed value of the statistic is very small. This can be seen clearly when the computed values of the statistic presented in Table 2 are compared with their corresponding p-values as contained in Table 1. From the two tables, the higher the value of the test statistic (Table 2), the closer p-value is to zero (Table 1). Having this in mind, Table 2 shows that the values of the test statistic in (18) for all the central order statistics considered in this study generally decreased progressively with increasing sample size among all the considered distributions. The import of this development is that as the sample size increased beyond 100, all the central order statistics in all the various distributions, at different points, would be expected to tend to normal normal. This is not so with the values of the test statistic obtained for the extreme order statistics (minimum and maximum). These values for all the sample sizes considered among all the distributions were very high leading to the rejection of their normality. Most importantly, there is an observed general increase in the values of the test statistic with increase in the sample size among all the distributions considered here. This clearly suggests an unlikelihood of any of the extreme order statistics becoming normal at any sample size beyond 100.

4. Conclusion

The equivalence of the sample pth quantile and the kth order statistic of a random sample from a distribution has been reviewed in this paper. Although they represent two different concepts in statistics, they can be used interchangeably especially in their applications such as in probability plots tests for multinormality, provided $\frac{k}{n} \to p$. As a result of the equivalence property which holds for the sample pth quantile and the kth order statistic, asymptotic normality of kth order statistic implies asymptotic normality of the sample pth quantile provided $\frac{k}{n} \to p$. Also, the asymptotic normality of the kth order statistic holds only for the central order statistics (i.e. non extreme order statistics). From our empirical study, we also conclude that the sample size from which the asymptotic normal distribution of the central order statistics holds depends on the centeredness of p as well as the parent continuous distribution.

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