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Notes on the Boussinesq integrable hierarchy

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Abstract: This work is dedicated to some notes on the Moyal momentum algebras applied to the sl_3 Boussinesq integrable hierarchy. Starting from a brief review of the Moyal momentum algebra structures, we establish in detail the Non-commutative Boussinesq hierarchy by using the Lax pair Generating Technique. Then we shows that these equations can be obtained as 3-reduction of Non-commutative KP hierarchy in a similarly form via some conformal realizations.

Keywords: Moyal Momentum Algebra, Moyal KP Hierarchy, Non-Commutative Boussinesq Hierarchy

1. Introduction

The origin of integrable system dates back to the 19th century with the KdV equation, which describe the long solitary wave in the shallow water [1]. Since the study of integrability of nonlinear system, has taken more consideration [2]. For such systems integrability means the existence of an infinite number of conserved quantities in involutions. A definition given by Ward is that such system, more precisely few of them, can be derived from the anti-self-dual Yang Mills equations by reduction with gauge groups [3, 4].

These studies yield exacts solutions in many problem in theoretical high energy physics and mathematics. It appears that the geometry of integrable system is crucial for understanding many aspects of field theories [5]. E. Witten has conjectured that the energy of 2-dimensional gravity coincide with the Tau-function of KdV hierarchy [6].In addition the integrable systems can be linked to the infinite-dimensional conformal algebra and its extensions. From their Poisson bracket structure it turns out that Boussinesq and KP hierarchy are respectively isomorphic to W_3 and $W_{1+\infty}$ algebra. In current days, there are deep interest in the non-commutative aspect of different soliton equations [7, 8, 9], with successful applications to string theories [10]... It appears that the Moyal momentum algebra $\hat{\Sigma}_{\mathrm{m}}^{(r,s)}$ via its $sl_{n-1}-$momentum Lax operators provides an interesting tools for the study of θ $-$deformed integrable systems.

We will study integrable systems of (1 + 1) and (2 + 1) dimensional evolution equation namely the Boussinesq and KP equation respectively. We starts with some basic properties of the Moyal$\star$ Product, introducing the Moyal Momentum Algebra $\hat{\Sigma}_{\mathrm{m}}^{(r,s)}$. Then we adopt the Lax Pair Generating Technique to study the evolution equations of Non-commutative Boussinesq hierarchy. By the way we establish the Non-commutative KP hierarchy before discussing the 3-reduction of NC KP hierarchy and the link with the previous Boussinesq hierarchy.

2. Moyal Product $\star$ and Operators Algebra $\hat{\Sigma}_{\mathbf{m}}^{(r,s)}$

Our formulation will be based on star product $\star$ called the Moyal product. Given a smooth manifold M with $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ coordinates system. This manifold will be endowed with the skew-symmetric bilinear bracket defined on $C^\infty(M)$ by [11, 12]:

$$\{f, g\} = \omega^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} \quad f, g \in C^\infty(M) \qquad (1)$$

$\{,\}$ verifies the Jacobi identity, if ω^{ij} is a non-degenerate skew-symmetric matrix, hence M is symplectic manifold with and even dimension. We consider extend tensorial manifold $F = M \otimes T$ with $\boldsymbol{t} = (t1, t2, \ldots, tn)$ denoting the extra coordinates system of *T*. The Moyal product will not affect **t** and it is given by [2, 13]:

$$f(x,t) \star g(x,t) = \exp\left(\theta\, \omega^{ij} \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\right) f(x,t) g(\tilde{x},t)\Big|_{x=\tilde{x}} \quad (2)$$

Expanding this equation we find:

$$f(x,t) \star g(x,t) = \sum_{s=0}^{\infty} \frac{\theta^s}{s!}\, \omega^{i_1 j_1} \ldots \omega^{i_s j_s} \frac{\partial^s f(x,t)}{\partial x^{i_1} \ldots \partial x^{i_s}} \frac{\partial^s g(x,t)}{\partial x^{j_1} \ldots \partial x^{j_s}} \quad (3)$$

The Moyal bracket is defined as follow:

$$\{f, g\} = \frac{f \star g - g \star f}{2\theta} \quad (4)$$

With $lim_{\theta \to 0} f \star g = fg$.

If we consider the 2d-phase space M, with $\boldsymbol{x}(x_1 = x, x_2 = p)$ coordinate, the matrix ω^{ij} becomes:

$$\omega^{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

hence expression (3) can be written as [13]:

$$f(x,p,t) \star g(x,p,t) =$$

$$\sum_{s=0}^{\infty} \sum_{i=0}^{s} \frac{\theta^s}{s!} (-1)^i\, C_s^i \left(\partial_x^i \partial_p^{s-i} f(x,p,t)\right)\left(\partial_x^{s-i} \partial_p^i g(x,p,t)\right) \quad (5)$$

and the Moyal bracket :

$$\{f(x,t), g(x,t)\} = \frac{f \star g - g \star f}{2\theta}$$

$$\{f(x,t), g(x,t)\} = \sum_{s=0}^{\infty} \frac{\theta^{2s}}{(2s+1)!} \sum_{i=0}^{2s+1} (-1)^i \binom{2s+1}{i} \times$$

$$\left(\partial_x^{2s+1-i} \partial_p^i f(x,t)\right)\left(\partial_x^i \partial_p^{2s-i} g(x,t)\right)$$

$$f \text{ and } g \in C^\infty(F) = C^\infty(M \otimes T) \quad (6)$$

The point of introducing the above properties is to define the Moyal momentum algebra. The Moyal momentum was introduced first by authors [14], and systematically studied later with some applications to conformal field theory and θ-deformed integrable models [15]. This algebra is a pseudo momentum operators algebra denoted by $\hat{\Sigma}(\theta)$.

$\hat{\Sigma}(\theta)$ consist of the object of the form $u(x,\tau) \star f(p)$ where $f(p)$ is polynomial in momentum $p, \tau = (t_1, t_2, \ldots, t_n)$. The Moyal momentum algebra is isomorphic to the ordinary pseudo differential operator $L_m = \sum_{i \in Z} u_{m-i} \star \partial^i$. The construction of $\hat{\Sigma}(\theta)$ consist of replacing the ordinary pseudo differential lax operators by the Lax momentum operators:

$$L_m = \sum_{j \in Z} u_{m-j}(x,\tau) \star p^j \quad (7)$$

L_m is a $C^\infty(F)$ function of ordinary spin m living in a non-commutative space parameterized by θ. The conformal dimensions are given as follow:

$$[u_i] = i, [\theta] = 0, [p] = [\partial_x] = -[x] = 1, [\partial_{t_k}] = -[t_k] = k. \quad (8)$$

$\hat{\Sigma}(\theta)$ can be decomposed as:

$$\hat{\Sigma}(\theta) = \oplus_{r \le s} \oplus_{m \in Z} \hat{\Sigma}_m^{(r,s)}(\theta) \quad (9)$$

where $\hat{\Sigma}_m^{(r,s)}$ denotes the space of momentum lax operators of conformal spin m and degrees start from r to s:

$$L_m^{(r,s)} = \sum_{j=r}^{s} u_{m-j} \star p^j \quad (10)$$

$L_m^{(r,s)}$ involving zero value u_{m-k} $(r \le k \le s)$ term belong to the space $\Sigma_m^{(r,s)} / \Sigma_m^{(k,k)}$.

$\Sigma_m^{(0,0)}$ is the space of operators of degree 0 denoting function coefficient of conformal spin m:

$$u_l \star u_m = u_l\, u_m.$$

The Moyal bracket of two operators $\in \Sigma_m^{(r,s)}$ gives rise to an operator $\in \Sigma_m^{(r,2s-1)}$. To perform all the forthcoming calculations, the formulae (3) will be use its more simplified way. This has be done in several papers [8, 15].

We have

$$p^n \star f(x,p) = \sum_{s=0}^{n} \theta^s C_n^s f^{(s)}(x,p)\, p^{n-s}. \quad (11)$$

$$p^{-n} \star f(x,p) = \sum_{s=0}^{\infty} (-1)^s \theta^s C_{n+s-1}^s f^{(s)}(x,p)\, p^{-n-s}. \quad (12)$$

$$\{p^n, f\}_\theta = \sum_{k=0} \theta^{2k} C_n^{2k+1} f^{(2k+1)} p^{n-2k-1}. \quad (13)$$

$$\{p^{-n}, f\}_\theta = -\sum_{k=0} \theta^{2k} C_{2k+n}^{2k+1} f^{(2k+1)} p^{-n-2k-1}. \quad (14)$$

3. Moyal Boussinesq Hierarchy

The sl_n − moyal hierarchy is defined by the lax equation [16]

$$\frac{\partial L}{\partial t_k} = \left\{\left(L^{1/n}\right)_+^k, L\right\}_\theta \quad (15)$$

Where

$$\left(L^{1/n}\right)_+^k = \underbrace{\left(L^{\frac{1}{n}} \star L^{\frac{1}{n}} \star \ldots \star L^{1/n}\right)_+}_{k}$$

It follows that the coefficient in order $n - 1$ vanishes, we have the special form of $L \in \Sigma_n^{(0,n)} / \Sigma_n^{(n-1,n-1)}$ called $L-$ hierarchy.

$$L = L_n = p^n + \sum_{i=0}^{n-2} u_{n-i} \star p^i \tag{16}$$

and

$$L^{1/n} = p + \sum_{i=1} \omega_{i+1} \star p^{-i} \tag{17}$$

is the n^{th} root of L. Thus the sl_3 -Boussinesq moyal momentum Lax operator we will deal with is :

$$L_3 = p^3 + u_2 \star p + u_3 \tag{18}$$

The explicit expression of $L^{1/3}$ and the straightforward calculations gives the Boussinesq hierrarchy. This has been done by many authors [8, 15].

Instead of the above approach in this section, we will adopt the Lax pair generating technique to determine the Non-commutative Boussinesq hierarchy [4]. Briery, the Lax pair generating technique consist of finding for a given L_m, the operator T such that:

$$\{L_m, T + \partial_t\}_\theta = 0 \tag{19}$$

The equation (19) is equivalent to (15) where T is the analogue of $\left(L^{k/m}\right)_+$ [4, 8]. This technique is based on the following ansatz:

$$T = p^\alpha \star L^\beta + \tilde{T} \tag{20}$$

with α and $\beta \in Z$.
where $p^\alpha$is a monome of momentum operator. Actually the clue of the problem is to determine the expression of the operator $\tilde{T}$; keeping in mind that T and $\tilde{T}$ have the same degree.

- The t_1 flow $\partial_{t_1} u = \dot{u}$:

$$L = p^3 + u_2 \star p + u_3 = p^3 + u_2 p + u_3 - \theta u_2'$$

$$T = p^{-2} \star L + \tilde{T} \tag{21}$$

The equation (19) leads to trivial equations with $\tilde{T} = A \in C$:

$$\frac{\dot{u}_2}{2\theta} = -u_2' , \tag{22}$$

$$\frac{\dot{u}_3}{2\theta} = -u_3' , \tag{23}$$

where we denote by $\frac{\partial u}{\partial x} = u'$ and $\frac{\partial u}{\partial t} = \dot{u}$ We can obtain the ordinary form of the Boussinesq hierarchy via the correspondence $\frac{1}{2\theta}\frac{\partial}{\partial t} \longrightarrow -\frac{\partial}{\partial t}$.

- The t_2 flow $\partial_{t_2} u = \dot{u}$:

We consider the ansatz :

$$T = p^{-1} \star L + \tilde{T}, \tag{24}$$

where $\tilde{T} = A \in \hat{\Sigma}_2^{(0,0)}$.

Considering the differential part of $p^{-1} \star L$, we get:

$$\{L, (p^{-1} \star L)_+ + \partial_t\} = (3u_2' - 2u_2')p^2 - \left(2(u_3' - \theta u_2'') + \frac{\dot{u}_2}{2\theta}\right)p + u_2 u_2' - \frac{\dot{u}_3}{2\theta} + \frac{\dot{u}_2'}{2} + \theta^2 u_2'''$$

By identifying with:

$$-\{L, \tilde{T}\} = -3 A' p^2 - u_2 A' - \theta A''' ,$$

one gets :

$$A = -\tfrac{1}{3} u_2 + a,$$

taking $a = 0$, then:

$$A = -\tfrac{1}{3} u_2. \tag{25}$$

Therefore:

$$T = p^2 + \tfrac{2}{3} u_2 \tag{26}$$

$$\frac{\dot{u}_2}{2\theta} = -2u_3 + 2\theta u_2''. \tag{27}$$

Substitute A in:

$$u_2 u_2' - \frac{\dot{u}_3}{2\theta} + \frac{\dot{u}_2'}{2} + \theta^2 u_2''' = -u_2 A' - \theta A''',$$

we get:

$$\frac{\dot{u}_3}{2\theta} = \tfrac{2}{3} u_2 u_2' + \tfrac{8}{3}\theta^2 u_2''' - 2\theta u_3''. \tag{28}$$

We recognize the pair of equations (27) and (28) is nothing but the non-commutative Boussinesq equation.

- The t_4 ow $\partial_{t_4} u = \dot{u}$

Here we consider the following ansatz:

$$T = p \star L + \tilde{T} \tag{29}$$

with:

$$\tilde{T} = a \star p^2 + b \star p + c$$

$$\tilde{T} = Ap^2 + Bp + C \tag{30}$$

where coefficients of polynome in p belong to $\hat{\Sigma}^{(0,0)}$. To find the Lax pair of equation

$$\{L, T + \partial_t\}_\theta = 0,$$

we start by calculating the following terms:$\{L, p \star L\}_\theta$ and $\{L, \tilde{T}\}_\theta$:

$$\begin{aligned}\{L, p \star L\}_\theta = & -u_2'\, p^4 + (-u_3' + 4\theta u_2'')p^3 \\ & + (3\theta u_3'' - 6\theta^2 u_2''' - u_2 u_2')p^2 \\ & + \left[-3\theta^2 u_3''' - (u_2u_3)' + 2\theta u_2 u_2'' \right. \\ & \left. + 4\theta^3 u_2^{(4)}\right]p + \theta^3\left(u_3^{(4)} - \theta u_2^{(5)}\right) \\ & + \theta^2 u_2' u_2'' + \theta u_2(u_3'' - \theta u_2''') \\ & - u_3(u_3' - \theta u_2'').\end{aligned}$$

$$\begin{aligned}-\{L, \tilde{T}\}_\theta = & -3A'p^4 - 3B'p^3 \\ & - (\theta^2 A''' + 3C' + u_2A' - 2u_2'A)p^2 \\ & - [\theta^2 B''' + u_2 B' - u_2' B \\ & - 2A(u_3' - \theta u_2'')]p - \theta^2 C''' - u_2 C' \\ & - \theta^2 u_2'' A' + B(u_3' - \theta u_2'').\end{aligned}$$

Then by identifying the order 4, 3, 2 in p, we obtain:

$$A = \frac{1}{3}\, u_2\,, \tag{31}$$

$$B = \frac{1}{3}(u_3 - 4\theta u_2'), \tag{32}$$

$$C = \frac{17}{9}\theta^2 u_2'' - \theta u_3' + \frac{2}{9}(u_2)^2, \tag{33}$$

With these values, the identification in order 1 leads to:

$$\frac{\dot{u}_2}{2\theta} = \frac{4}{3}\left[\theta(u_2u_2'' + (u_2')^2) - (u_2u_3)' - 2\theta^2 u_3''' + 2\theta^3 u_2^{(4)}\right] \tag{34}$$

Finally, the term of the order 0 $\in \hat{\Sigma}^{(0,0)}$ fields:

$$\frac{\dot{u}_3 - \theta \dot{u}_2'}{2\theta} = \frac{4}{3}\Big[-u_3u_3' + \frac{1}{3}u_2^2 u_2' + \frac{2}{3}\theta^4 u_2^{(5)} + \theta\,(u_3 u_2'' + u_2' u_3') + \theta^2\, u_2' u_2'' + u_2 u_2'''. \tag{35}$$

Hence:

$$T = p \star L + \tilde{T},$$

$$T = p^4 + \frac{4}{3}\, u_2\, p^2 + \frac{4}{3}(u_3 - \theta u_2')p + \frac{8}{9}\theta^2 u_2'' + \frac{2}{9}(u_2)^2. \tag{36}$$

Equations (34) and (35) correspond to the t_4 evolution equations of the non-commutative Boussinesq hierarchy.

4. Moyal KP Hierarchy

In this section, we drop the $-\frac{1}{2\theta}\frac{\partial}{\partial t_k}$. time derivation we start with a more familiar notations similar to Lax representation for a hierarchy in Sato's framework. We consider the KP Lax operator:

$$L_{KP} = L = p + \sum_{i=1}^{\infty} u_{i+1} \star p^{-i} \in \hat{\Sigma}_1^{(-\infty,1)} \Big/ \hat{\Sigma}_1^{(0,0)}$$

$$L = p + \textstyle\sum_{i=1}^{\infty} v_{i+1}\, p^{-i}. \tag{37}$$

Then the non-commutative KP evolution equations take the lax form:

$$\frac{\partial L}{\partial t_k} = \{B_k, L\}_\theta = \{(L^k)_+, L\}_\theta \tag{38}$$

We use the later to determine KP hierarchy in a simpler way by using the Moyal $\star$ product and recover the hierarchy similar to the one found by using the supershmidt-Manin $\star$ product, just by a conformal realization of fields v_i. It turns out that the KP hierarchy consists of an infinite set of differential equation for each time t_k [13, 17].

** The $t_1$flow $\partial_{t_1} u = \dot{u}$:

$$\dot{v}_{i+1} = v_{i+1}', \tag{39}$$

** The $t_2$flow $\partial_{t_2} u = \dot{u}$:

$$\begin{aligned}\{B^2, L\}_\theta &= \{p^2 + 2v_2, p + \textstyle\sum_{i=1}^{\infty} v_{i+1}\, p^{-i}\}_\theta \\ &= \textstyle\sum_{i=1}^{\infty} 2v_{i+1}'\, p^{-i+1} - 2v_2' - 2\sum_{i=1}^{\infty} v_{i+1}\, \{p^{-i}, v_2\},\end{aligned}$$

we keep the terms in p^{-1}, p^{-2} and p^{-3} then we get the following equations:

$$\begin{aligned}\dot{v}_2 &= v_3' \\ \dot{v}_3 &= 2v_4' + 2v_2v_2' \\ \dot{v}_4 &= 2v_5' + 4v_2'v_3 \\ &\vdots\end{aligned} \tag{40}$$

by a conformal realization the field v_i is expressed in term of u_i:

$$\begin{aligned} v_2 &= u_2\,, v_3 = u_3 + \theta\, u_2' \\ v_4 &= u_4 + 2\theta\, u_3' + \theta^2 u_2'', \\ v_5 &= u_5 + 3\theta\, u_4' + 3\theta^2 u_3'' + \theta^3 u_2''',\end{aligned} \tag{41}$$

we find the previous hierarchy in the following form:

$$\begin{aligned}\dot{u}_2 &= 2(u_3' + \theta\, u_2')\,, \\ \dot{u}_3 &= 2u_4' + 2\theta\, u_3'' + 2u_2u_2' \\ \dot{u}_4 &= 2u_5' + 2\theta\, u_4'' + 4u_3u_2' - 4\theta u_2 u_2'', \\ &\vdots\end{aligned} \tag{42}$$

** The $t_3$flow $\partial_{t_3} u = \dot{u}$:

$$\{B^3, L\}_\theta = \{p^3 + 3v_2 p + 3v_3, p + \textstyle\sum_{i=1}^{\infty} v_{i+1}\, p^{-i}\}_\theta$$

keeping the term up to p^{-2}, we find:

$$\begin{aligned}\dot{v}_2 &= 6v_2v_2' + 3v_4' + \theta^2 v_2''' \\ \dot{v}_3 &= 6(v_2v_3)' + 3v_5' + \theta^2 v_3'''\end{aligned} \tag{43}$$

using the conformal realization (41), we get:

$$\begin{aligned}\dot{u}_2 &= 6u_2u_2' + 4\theta^2 u_2''' + 3u_4' + 6\theta\, u_3''\,, \\ \dot{u}_3 &= 6(u_2u_3)' + 4\theta^2 u_3''' + 3u_5' + 6\theta\, u_4'' \\ &\vdots\end{aligned} \tag{44}$$

It appears that if one takes the first two equations of (42) and eliminating u_3 and u_4 in the fisrt equation of (44) we get the non-commutative KP equation where $t_2 \equiv y$ and $t_3 \equiv t$.

5. Boussinesq Hierarchy as 3-Reduction of Moyal KP Hierarchy

This approach pictures the link between the KP Lax operator and others integrables models. Let's rewrite the KP Lax operator $L = p + \sum_{i=1}^{\infty} u_{i+1} \star p^{-i}$ or in the form $L = p + \sum_{i=1}^{\infty} v_{i+1}\, p^{-i}$. For Boussinesq equation, we denote the Lax operators by $\mathcal{L} = p^3 + u_2 \star p + u_3$ or $\mathcal{L} = p^3 + v_2\, p + v_3$ .Then the Boussinesq hierarchy obtained by 3-reduction is given by the following Lax equation.

$$\frac{\partial \mathcal{L}}{\partial t_k} = [B_k, L^3] \tag{45}$$

Where $B_k = (L^k)_+$with the contrain $L^3 = B_3$. The t_1 flows are trivial.

For The t_2 flows we have:

$$B_2 = (L^2)_+ = p^2 + 2v_2 \tag{46}$$

$$\frac{\partial \mathcal{L}}{\partial t_2} = [B_2, L^3]_\theta. \tag{47}$$

With:

$$L^3 = p^3 + 3v_2\, p + 3v_3 + (3v_2^2 + 3v_4 + \theta^2 v_2'')p^{-1} + \cdots \tag{48}$$

and the contrain $L^3 = B_3 \Rightarrow Res(L^3) = 0$, we find:

The term$[\mathrm{B}_2, \mathrm{L}^3]_\theta$ yields:

$$v_4 = -v_2^2 - \frac{\theta^2}{3} v_2'' \tag{49}$$

$$[B_2, L^3]_\theta = 6v_3' p - 2(\theta^2 v_2''' + 3v_2 v_2').$$

Finally with the lax equation (47) we obtain:

$$\dot{v}_2 = 6v_3' \tag{50}$$

$$\dot{v}_3 = -2\,(\theta^2 v_2''' + 3\, v_2 v_2') \tag{51}$$

it turns out that the ∂_{t_2}time derivation of equation (50) yields:

$$\ddot{v}_2 = 6\dot{v}_3' = 6\big(-2\,(\theta^2 v_2''' + 3\, v_2 v_2')\big)',$$

Therfore we get the Non-commutative Boussinesq equation:

$$\ddot{v}_2 = -12(3\, v_2 v_2' + \theta^2 v_2''')'. \tag{52}$$

Taking classical limit $\theta = \frac{1}{2}$ we obtain the Boussinesq equation in the ordinary form [18]. Notice that the map (41) doesn't change equation (52).

The t_4 flows are given as follow: we start by calculating:

$$L^4 = \mathrm{B}_4 = p^4 + 4v_2\, p^2 + 4v_3 p + 4\theta^2 v_2'' + 6v_2^2 + 4v_4.$$

The condition $Res(L^3) = 0$ yields :

$$L^4 = \mathrm{B}_4 = p^4 + 4v_2\, p^2 + 4v_3 p + 2v_2^2 + \frac{8}{3}\theta^2 v_2''. \tag{53}$$

then the equation :

$$\frac{\partial \mathcal{L}}{\partial t_4} = [B_4, L^3]_\theta \tag{54}$$

gives rise to:

$$\dot{v}_2 = 4(\,3(v_2 v_3)' + 2\theta^2 v_3'''), \tag{55}$$

$$\dot{v}_3 = 4\left(\,3v_3 v_3' - 3v_2^2 v_2' - 6\theta^2 v_2' v_2'' - 3\theta^2 v_2 v_2''' - \frac{2}{3}\theta^4 v_2^{(5)}\right). \tag{56}$$

Notice that the presence of the term $-6\theta^2 v_2' v_2''$ in the last equation doesn't matter, since when applying the conformal map (41) to the terms v_i that coming from the KP Lax operator L we recover the following equations.

$$\ddot{u}_2 = 4\left(\,3(u_2 u_3)' + 3\theta\big(u_2 u_2'' + {u_2'}^2\big) + 2\theta^2\left(u_3''' + \theta u_2^{(4)}\right)\right), \tag{57}$$

$$\dot{u}_3 - \theta \dot{u}_2' = 12\left(\,u_3 u_3' - u_2^2 u_2' + \theta(u_3 u_2'' + u_2' u_3') - \theta^2 (u_2' u_2'' + u2u2''' - 29\theta4u25. \right. \tag{58}$$

6. Conclusions

We have presented by two different methods how to obtain the deformed Boussinesq hierarchy. Of course there are several versions of theory and each has its advantages and flaws. In this work, the results found in the first method show the consistency of Lax pair generating technique. Where by rescaling time derivation we recover the ordinary form of Boussinesq hierarchy. We also got a look to the KP hierarchy which has been simplified by using a conformal realization that shows the equivalence between the Moyal $\star$product and the Kupershmidt-Manin $\star$ product. We have also shown that the Boussinesq hierarchy obtained by the 3-reduction of KP hierarchy using the same conformal map gives rise to equations similar to that obtained by Lax Pair Generating Technique. We hope our discussion will make the Moyal momentum be more accessible in the study of some integrable models.

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