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Note on the 3-graded modified classical Yang-Baxter equations and integrable systems

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Abstract: The $6 = 3 \times 2$ huge Lie algebra Ξ of all local and non-local differential operators on a circle is applied to the standard Adler-Kostant-Symes (AKS) R-bracket scheme. It is shown in particular that there exist three additional Lie structures, associated to three graded modified classical Yang-Baxter (GMCYB) equations. As we know from the standard case, these structures can be used to classify in a more consistent way a wide class of integrable systems. Other algebraic properties are also presented.

Keywords: Huge Lie Algebra, Graded Modified Classical Yang-Baxter Equations, Integrable Hamiltonian Systems

1. Introduction

This is an expository account of the basic ideas concerning the application of the huge Lie algebra $\Xi \equiv \mathcal{G}$of arbitrary pseudo-differential operators on the circle [1], to the standard AKS scheme [2]. As pointed out in [3, 4], the AKS theory with a Poisson bracket structure defined in terms of the R-matrix [5] Lie-Poisson bracket is the right setting to study various integrable systems and their gauge equivalence. In [3], it is shown, in a successful analysis, that there exist three KP-integrable systems which are gauge equivalent.

The origin of these three integrable systems is traced to the fact that there exist three decompositions of G into a linear sum of two sub-algebras parameterized by the same index $l$taking values $l = 0, 1, 2$. Focusing on this analysis [3] and referring to our language [1], the different values of the index l describes in some sense a constraint on the degrees quantum numbers and no manifest description of the conformal spin quantum number, of local and non-local differential operators, is specified.

In the present work, we go beyond this analysis by using the more general results obtained in [1] for the huge Lie algebra $\mathcal{G}$, which provides also a consistent way to construct a wide class of KP-integrable systems.

2. Generalities on the Huge Lie Algebra of Pseudo-Differential Operators

We describe in this section, the basic features of nonlinear differential operators on the ring of analytic functions. We show in particular that any such differential operator is completely specified by a spin$s; s \in \mathbb{Z}$, two integers $p$and $q = p + n; n \geq 0$ defining the lowest and highest degrees, respectively, and finally $1 + p - q = n + 1$ analytic fields $u_j(z)$. We also show that the set Ξ of all nonlinear differential operators admits a Lie algebra structure with respect to the commutator of differential operators built out of the Leibnitz product.

Moreover, we find that Ξ splits into $3 \times 2 = 6$ sub-algebras Ξ_s^+ and Ξ_s^-, $s = 0, \pm$ related to each other by two types of conjugations, namely the spin and degrees conjugations. The algebra Ξ_+^+ and Ξ_-^- are of particular interest [1], as they are used in the construction of the Gelfand-Dickey algebra $GD(sl_n)$. For this purpose, we shall proceed as follows: First we introduce the ring $\Xi^{(0,0)}$of analytic fields and W-currents. This is a tensor algebra of analytic functions of arbitrary conformal spin. Then we introduce the space $\Xi_s^{(p,q)}$of pseudo-differential operators of fixed spin $s$and fixed degrees (p, q). The space of pseudo-differential operators of fixed degrees (p, q) but arbitrary spin will be denoted $\Xi^{(p,q)}$. Finally we build our desired space Ξ which is the huge Lie algebra of pseudo-differential operators of arbitrary conformal spin and arbitrary degrees quantum numbers. The convention notations used in the present work closely follows the analysis developed in [1].

2.1. The Ring $\Xi^{(0,0)}$ of Analytic Functions

The two dimensional Euclidean space $R^2 \cong C$ we consider, are parameterized by $z = t + ix$ and $\bar{z} = t - ix$. As a matter of convention, we set $z = z^+$and$\bar{z} = z^+$so that the derivatives $\frac{\partial}{\partial_z} = \partial$ and $\frac{\partial}{\partial_{\bar{z}}} = \bar{\partial}$ are respectively represented by $\partial_+ = \partial$ and $\partial_- = \bar{\partial}$. The $SO(2)$ Lorentz representation fields is given by the set of analytic fields $\phi_k(z)$. These are $SO(2) \cong U(1)$ tensor fields that obey the analyticity condition $\partial_- \phi_k(z) = 0$. In this case the conformal spin k coincides with the conformal dimension Δ. Note that under a $U(1)$ global transformation of parameter θ, the object $z^\pm$, $\phi_k(z)$ transform as:

$$z^{\pm\prime} = e^{(\pm i\theta)} z^\pm, \partial'_\pm = e^{(\pm i\theta)} \partial_\pm, \phi'_k(z) = e^{(ik\theta)} \phi_k(z) \quad (1)$$

so that $dz\partial_z$and $(dz)^k \phi_k(z)$ remain invariant. Notice that in the pure bosonic theory, only integer values of conformal spin k are involved.

Let $\Xi^{(0,0)}$ be the tensor algebra of analytic fields of arbitrary conformal spin. This is a completely reducible infinite dimensional $SO(2)$Lorentz representation that can be written as:

$$\Xi^{(0,0)} = \oplus_{k \in \mathbb{Z}} \Xi_k^{(0,0)} \quad (2)$$

where the $\Xi_k^{(0,0)}$ are one dimensional $SO(2)$ spin k irreducible modules. The upper indices $(0,0)$ carried by the spaces figuring in Eq.(2) are special values of general indices (p, q) to be introduced later on. The generators of these spaces are given by the spin k-analytic fields $u_k(z)$. They may be viewed as analytic maps u_k which associate to each point z, on the unit circle S^1, the fields $u_k(z)$.

For $k \geq 2$, these u_k fields can be thought of as the higher spin currents involved in the construction of the W-algebras. As an example, the following fields:

$$W_2 = u_2(z), W_3 = u_3(z) - \tfrac{1}{2}\partial_z u_2(z) \quad (3)$$

are the well-known spin-2 and spin-3 conserved currents of the Zamolodchikov W_3 -algebra [6]. As in infinite dimensional spaces, elements ϕ of the spin tensor algebra $\Xi^{(0,0)}$ in Eq.(2) are built from the vector basis $\{u_k, k \in \mathbb{Z}\}$ as follows:

$$\phi = \sum_{k \in \mathbb{Z}} c(k) u_k \quad (4)$$

where only a finite number of the decomposition coefficients $c(k)$is non vanishing.

Introducing the following scalar product$\langle , \rangle$ in the tensor algebra $\Xi^{(0,0)}$

$$\langle u_l, u_k \rangle = \delta_{k+l,0} \int u_{1-k}(z) u_k(z) dz \quad (5)$$

where $\delta_{k,l}$ is the Kronecker index, it is not difficult to see that the one dimensional subspaces $\Xi_k^{(0,0)}$and $\Xi_{1-k}^{(0,0)}$ are dual to each other. As a consequence the tensor algebra $\Xi^{(0,0)}$ splits into two semi-infinite tensor algebras $\sum_+^{(0,0)}$ and $\sum_-^{(0,0)}$, respectively, characterized by positive and negative conformal spins as shown here below:

$$\textstyle\sum_+^{(0,0)} = \oplus_{k>0} \Xi_k^{(0,0)} \quad (6)$$

$$\textstyle\sum_-^{(0,0)} = \oplus_{k>0} \Xi_{1-k}^{(0,0)} \quad (7)$$

From these equations we read in particular that $\Xi_0^{(0,0)}$ is the dual of $\Xi_1^{(0,0)}$ and if half integers were allowed, $\Xi_{1/2}^{(0,0)}$would be self dual with respect to the form Eq.(5). Note that the product Eq.(5) carries a conformal spin structure since from dimensional arguments, it behaves as an object of conformal weight $\Delta[\langle , \rangle] = -1$. Latter on we will introduce a combined scalar product $\langle\langle , \rangle\rangle$ built out of Eq.(5) and a pairing product $(,)$ see Eq.(20), of conformal weight Δ = 1 so that we get Δ[<<;>>] = 0. Note moreover that, the infinite tensor algebra $\sum_+^{(0,0)}$ of Eq.(6) contains, in addition to the spin-1 current, all the $W_n$currents $n \geq 2$. These fields are used in the construction of higher spin local differential operators as it explained in detail in [1]. Analytic fields with negative conformal spins, Eq.(7) are involved in the building of non-local pseudo differential operators. Both these local and non-local operators are very useful in the derivation of classical W_n-algebras from the GD algebra of sl_n [1, 7].

2.2. The Algebra of Higher Spin Differential Operators

2.2.1. The $\Xi_k^{(0,0)}$ space

We define this space as the set of pseudo differential operators whose elements $d_s^{(p,q)}$ are the generalization of the well-known differential Lax operators involved in the analysis of the so-called KdV hierarchies and in Toda theories [8]. The simplest example is given by the Hill operator:

$$L = \partial^2 + u(z) \quad (8)$$

which plays an important role in the study of the Liouville theory and in the KdV equation. A natural generalization of the above relation is given by:

$$d_s^{(p,q)} = \sum_{i=p}^{q} u_{s-i}(z) \partial^i \quad (9)$$

where the $u_{s-i}(z)$`s are analytic fields of spin $(s - i)$. p and q, with $p \geq q$areintegers that we suppose positive for the moment. We shall refer hereafter to p as the lowest degree of $d_s^{(p,q)}$and to q as the highest degree. We combine these two features of Eq.(9) by setting:

$$Deg\left(d_s^{(p,q)}\right) = (p, q) \quad (10)$$

As noted above, $s$is the conformal spin of the $(1 + q - p)$ monomials of the r.h.s of Eq.(9) and then of $d_s^{(p,q)}$ itself.

As for the above relation, we set:

$$\Delta\left(d_s^{(p,q)}\right) = s \tag{11}$$

Notice that the KdV operator Eq.(8) is discovered from Eq.(9) as a special case by setting $s = 2, p = 0$and $q = 2$ together with the special choices $u_0(z) = 1$and $u_1(z) = 0$. Moreover, Eq.(9) which is well defined for $q \geq p \geq 0$, may be extended to negative integers by introducing pseudo-differential operators of type $\partial^{-k}, k \geq 1$, whose action on the fields $u_s(z)$ is defined as:

$$\partial^{-k} u_s(z) = \sum_{l=0}^{\infty}(-1)^l c_{k+l-1}^l u_s^{(l)}(z)\,\partial^{-k-l} \tag{12}$$

where $u_s^{(l)}(z)$ is the l-th derivative of u_s. As can be checked by using the Leibnitz rule, Eq.(12) obeys the expected property:

$$\partial^k \partial^{-k} u_s(z) = u_s(z) \tag{13}$$

Note that a natural representation basis of pseudo-differential operators of arbitrary conformal spin s but negative degrees (p, q) is given by:

$$\delta_s^{(p,q)}(u) = \sum_{i=p}^{q} u_{s-i}(z)\partial^i \tag{14}$$

which is a direct extension of Eq.(9). A more convenient way to write a pseudo-differential operator of arbitrary conformal spin s but negative degrees(p, q) is to use the Volterra basis [1, 7] given by:

$$V_s^{(p,q)}(u) = \sum_{i=p}^{q} \partial^i u_{s-i}(z) \tag{15}$$

As shown in [1], this Volterra configuration, is very useful in the study of the algebraic structure of the spaces $\Xi_s^{(p,q)}$ and $\Xi^{(p,q)}$ and in the derivation of the second Hamiltonian structure of higher conformal spin integrable hierarchies. Now let $\Xi_s^{(p,q)}$; s, p and q arbitrary integers with $p \leq q$; be the space of pseudo-differential operators of spin $s$and degrees(p, q). With respect to the usual addition and multiplication by c-numbers, $\Xi_s^{(p,q)}$ behaves as $(1 + q - p)$ dimensional space generated by the vector basis:

$$\left\{\mathrm{D}_s^{(p,q)}(u) = \sum_{i=p}^{q} u_{s-i}(z)\partial^i, p \leq i \leq q\right\} \tag{16}$$

Thus the space decomposition of $\Xi_s^{(p,q)}$ reads as:

$$\Xi_s^{(p,q)} = \oplus_{i=p}^{q}\, \Xi_s^{(i,i)} \tag{17}$$

where the $\Xi_s^{(i,i)}$ are one dimensional spaces given by:

$$\Xi_s^{(i,i)} = \Xi_{s-i}^{(0,0)} \otimes \partial^i \tag{18}$$

Setting $i = 0$, one discovers the ring $\Xi_s^{(0,0)}$ described previously. Note that the dimension of the space $\Xi_s^{(p,q)}$, is fixed by the number of independent analytic fields $u_j(z)$involved in the expression of Lax operators. In [1], we showed that among all the spaces $\Xi_s^{(p,q)}$, only the sets $\Xi_0^{(p,q)}$ with $p < q < 1$ which admit a Lie algebra structure with respect to the bracket $[D_1; D_2] = D_1 \cdot D_2 - D_2 \cdot D_1$ constructed out of the Leibnitz product. For more details we refer the reader to this reference.

2.2.2. The huge Lie Algebra $\Xi_0^{(p,q)}$ of Differential Operators

Having defined the space $\Xi_s^{(p,q)}$, of pseudo-differential operators of fixed spin $s$and fixed degrees (p, q), we are now in a position to introduce the algebra of differential operators of arbitrary spins and arbitrary degrees. This algebra, which we denote by Ξ is simply obtained by summing over all allowed spins and degrees of the space $\Xi_s^{(p,q)}$. We have:

$$\Xi = \oplus_{p \leq q} \oplus_{s \in \mathbb{Z}} \Xi_s^{(p,q)} \tag{19}$$

A remarkable property of Ξ is that is splits into six infinite sub-algebras $\sum_s^+$ and $\sum_s^-$, $s = 0, \pm$, related to each others by conjugation of the spin and degrees.

Indeed given two integers $q \geq p$, it is not difficult to see that the subspaces $\Xi^{(p,q)}$ and $\Xi^{(-1-p,-1-q)}$are dual to each others with respect to the following pairing product $(,)$ defined as:

$$\left(D^{(r,s)}, D^{(p,q)}\right) = \delta_{1+r+q,0}\delta_{1+s+p,0}\, res\left[D^{(r,s)} \cdot D^{(p,q)}\right] \tag{20}$$

where the symbol (res) stands for the Adler- residue operation defined by:

$$res(\partial^i) = \delta_{i+1,0} \tag{21}$$

As shown also in [1], remark that the operation res. carries a conformal weight $\Delta = 1$ and then the residue of any operator $D_s^{(p,q)}$ is:

$$res\left(\sum_{i=p}^{q} u_{s-i}(z)\partial^i\right) = u_{s+1}(z) \tag{22}$$

if $p \leq -1$ and $q \geq -1$ and zero elsewhere. Furthermore, using the previous degree pairing Eq.(20), we can decompose the space Ξ as:

$$\Xi = \Xi^+ \oplus \Xi^-, \tag{23}$$

with:

$$\Xi^+ = \oplus_{p \geq 0}\left[\oplus_{n \geq 0}\, \Xi^{(p,p+n)}\right] \tag{24}$$

$$\Xi^- = \oplus_{p \geq 0}\left[\oplus_{n \geq 0}\, \Xi^{(-1-p-n,-1-p+)}\right] \tag{25}$$

The $+$ and $-$ upper-stairs indices carried by Ξ^+ and Ξ^-refer respectively to the positive and negative degrees. $\Xi^{(p,p+n)}$ as noted above is just the space of arbitrary conformal spin but fixed degrees $(p, p + n)$. This space can be decomposed as:

$$\Xi^{(p,p+n)} = \Sigma_+^{(p,p+n)} \oplus \Sigma_0^{(p,p+n)} \oplus \Sigma_-^{(p,p+n)}, \tag{26}$$

where $\Sigma_+^{(p,p+n)}$ and $\Sigma_-^{(p,p+n)}$ denote the spaces of differential operators of negative and positive definite spin. $\Sigma_0^{(p,p+n)}$ is however the space of Lorentz scalar differential operators. All these spaces can be written as:

$$\sum_{+}^{(p,p+n)} = \oplus_{s>0} \Xi_s^{(p,p+n)}, \tag{27}$$

$$\sum_{0}^{(p,p+n)} = \oplus_{s=0} \Xi_s^{(p,p+n)}, \tag{28}$$

$$\sum_{-}^{(p,p+n)} = \oplus_{s<0} \Xi_s^{(p,p+n)}, \tag{29}$$

Injecting these decompositions into the expression of the space Ξ Eq.(23), one find then that Ξ decomposes into $6 = 3 \times 2$ sub-algebras as:

$$\Xi = \oplus_{s=\pm 1,0} [\textstyle\sum_s^+ \oplus \sum_s^-], \tag{30}$$

where:

$$\textstyle\sum_s^+ = \oplus_{p\geq 0} \left[\oplus_{n\geq 0} \sum_s^{p,p+n} \right], s = 0, \pm 1 \tag{31}$$

$$\textstyle\sum_s^- = \oplus_{p\geq 0} \left[\oplus_{n\geq 0} \sum_s^{-1-p-n,-1-p} \right], s = 0, \pm 1 \tag{32}$$

Introducing the combined scalar product $\langle\langle , \rangle\rangle$, built out of the product Eq.(5) and the pairing Eq.(20), namely

$$\langle\langle D_m^{(r,s)}, D_n^{(p,q)} \rangle\rangle = \delta_{n+m,0}\delta_{1+r+q,0}\delta_{1+s+p,0} \int res\big[D_m^{(r,s)} \cdot D1-m-1-s,-1-rdz, \tag{33}$$

one sees that the Lie algebra $\sum_+^+$, $\sum_0^+$ and $\sum_-^+$ are duals of $\sum_-^-$, $\sum_0^-$ and $\sum_+^-$ respectively.

We conclude this section by noting that this formulation is very important in the sense that it leads in one hand to specify the Lax pseudo-differential operators with respect to their quantum numbers which are the spin and the degrees. On the other hand, knowing that most of the important integrable Hamiltonian systems (KP-Hierarchies) are formulated using Lax operators, one can therefore classify, in consistent way, all these systems if only the quantum numbers of their Lax operators are specified. Note also that in the super-symmetric case, from which emerge a lot of new features, one have another quantum number given by the statistics of the Lax operators (the Z_2-grading), see for instance [9].

We present in the next section of this work an application of our formulation to the well-known Adler-Kostant-Symes R-bracket mechanism.

3. The Generalized Adler-Kostant-Symes Scheme and the 3-Classes of MCYB Equations

We start this section by recalling the content of the standard AKS scheme and apply the previous huge Lie algebraic construction to this scheme. The important thing in this formulation is that one can split the standard MCYB equation into three different classes of equations associated to different realizations of the additional Lie structure on the Lie algebra Ξ. These three classes of integrable equations are shown to play an important role in the classification of a wide class of integrable models. Notice that some parts of the next subsections are extended results inspired from the work of [10].

3.1. The Standard AKS Scheme

It is well known that a very wide class of non-linear integrable systems can be constructed by using the AKS method having roots in the co-adjoint orbit formulation. We will show later that the basic structure of this method is the R-matrix which defines an operator approach to integrable systems [2, 5]. Before going to the description of the R-operator approach, we will recall _rst what is the co-adjoint orbit formulation. Denoting by G a Lie group and $\mathcal{G}$ its associated Lie algebra. The action of this group on its Lie algebra is given by the co-adjoint action:

$$Ad(g)X = gXg^{-1}, \tag{34}$$

with $g \in G$ and $X \in G$. Denoting also by $\mathcal{G}^*$the dual space of $\mathcal{G}$with respect to a non-degenerate bilinear form $\langle , \rangle$ on $\mathcal{G}^* \times \mathcal{G}$. The corresponding coadjoint action of the Lie group G on $\mathcal{G}^*$is defined through the duality of $\langle , \rangle$ as:

$$\langle Ad^*(g)U|X\rangle = \langle U|Ad(g^{-1})X\rangle, \tag{35}$$

where $U \in \mathcal{G}^*$ and $X \in \mathcal{G}$. In the infinitesimal case, the adjoint and co-adjoint transformations $Ad(g)$ and $Ad^*(g)$ reduce to $Ad(X)$ and $Ad^*(X)$ respectively, with $g = e^X$. On the space $C^\infty(\mathcal{G}^*, \mathrm{R})$, of smooth real-valued functions on $\mathcal{G}^*$, one can introduce a natural Poisson (LP) bracket:

$$\{F, G\}(U) = -\langle U|[\nabla F(U), \nabla H(U)]\rangle, \tag{36}$$

with $F, H \in C^\infty(\mathcal{G}^*, R)$ where ∇F is the gradient operator defined by the usual formula:

$$\nabla F : \mathcal{G}^* \to \mathcal{G}$$

$$\frac{dF}{dt}(U + tV)|_{t=0} = \langle V|\nabla F(U)\rangle \tag{37}$$

and where $[,]$ in Eq.(36) is the usual Lie bracket of $\mathcal{G}$. Note that the properties of anti-symmetry and Jacobi identity of the LP bracket, l.h.s of Eq.(36) are naturally deduced from the usual Lie bracket $[,]$, r.h.s of Eq.(36).

The important point in the AKS construction is that one can define an additional Lie structure on $\mathcal{G}$. The idea consists first in introducing a generalized R-matrix (R-operator) as a linear map from the Lie algebra $\mathcal{G}$ to it self such that the bracket:

$$[X, Y]_R = \frac{1}{2}([RX, Y] + [X, RY]), \tag{38}$$

defines a second Lie structure on $\mathcal{G}$. In order to ensure the Jacobi identity for this additional Lie structure, the R-operator must satisfy an algebraic relation, namely the modified classical Yang-Baxter equation [5]:

$$[RX, RY] - R([RX, Y] + [X, RY]) = -[X, Y], X, Y \in \mathcal{G} \tag{39}$$

Note that one can furthermore introduce a LP bracket $\{ , \}_R$ induced in Eq.(36) from the substitution of the usual bracket $[,]$ by the R-Lie bracket $[,]_R$ Eq.(38) (see for instance[3]). We have:

$$\{F,H\}_R(U) = -\langle U|[\nabla F(U), \nabla H(U)]_R\rangle, \tag{40}$$

which correspond to a dynamical system with the equation of motion:

$$\frac{dF}{dt} = \{H,F\}_R. \tag{41}$$

Later on, we will show that the previous $6 = 3 \times 2$ decomposition of the Lie algebra $\mathcal{G} \equiv \Xi$ induces three copies of the additional Lie structure Eq.(38). These three copies are associated to three classes of R-matrix

$$R_{(s)} = R_s^+ - R_s^-, \tag{42}$$

where $s = 0, \pm$ is the conformal spin quantum number introduced in previously and where the graded operators $R_s^\pm$ are projections on the space $\Xi_s^\pm$ of local (respectively non local) differential operators carrying a conformal spin s.

3.2. Three Classes of Integrable Hamiltonian Systems

We showed previously, that the huge Lie algebra Ξ of pseudo-differential operators, splits into 3×2 Lie algebras given by [1]:

$$\Xi = \Xi^+ \oplus \Xi^-, \tag{43}$$

$$\Xi^\pm = \textstyle\sum_+^\pm \oplus \sum_0^\pm \oplus \sum_-^\pm . \tag{44}$$

$\sum_+^+$ denote the algebra of local differential operators of positive spin and $\sum_-^-$,its dual with respect to the combined scalar product Eq.(33), the algebra of non-local differential operators of negative spin. $\sum_0^+$ is the algebra of Lorentz scalar local differential operators and $\sum_0^-$ its dual, the algebra of scalar non local operators. Finally, $\sum_-^+$ is the algebra of local differential operators of negative conformal spin and $\sum_+^-$, its dual, the algebra of non-local operators with positive spin. As shown in [1], one can define three classes of integrable systems for which one can introduce separately the first and the second Hamiltonian structures. The existence of these three classes of integrable systems has origin in fundamental algebraic properties of the algebra Ξ of pseudo-differential operators on a circle. More precisely, this existence can be traced to the fact that there exist three self-dual algebras:

$$\mathcal{G}_1 = \textstyle\sum_+^+ \oplus \sum_-^- = \mathcal{G}_1^* \tag{45}$$

$$\mathcal{G}_0 = \textstyle\sum_0^+ \oplus \sum_0^- = \mathcal{G}_0^* \tag{46}$$

$$\mathcal{G}_2 = \textstyle\sum_-^+ \oplus \sum_+^- = \mathcal{G}_2^* \tag{47}$$

Note by the way that $\mathcal{G}_1$ is nothing but the self-dual algebra used in the construction of the GD Poisson bracket [1, 7]. By analogy with $\mathcal{G}_1$, the algebras $\mathcal{G}_0$ and $\mathcal{G}_2$ are suspected to play an important role in the description of other integrable systems.

3.3. The R_s Approach of the AKS Construction

Knowing that $\mathcal{G} = \Xi$ decomposes as:

$$\Xi = \Xi^+ \oplus \Xi^-, \tag{48}$$

with respect to the degrees quantum numbers, one can introduce a particular decomposition with respect to the conformal spin quantum number. This decomposition is given by:

$$\Xi \equiv \mathcal{G} = \mathcal{G}_+ \oplus \mathcal{G}_0 \oplus \mathcal{G}_-. \tag{49}$$

$$\mathcal{G}_+ = \textstyle\sum_+^+ \oplus \sum_+^- \tag{50}$$

$$\mathcal{G}_0 = \textstyle\sum_0^+ \oplus \sum_0^- \tag{51}$$

$$\mathcal{G}_- = \textstyle\sum_-^- \oplus \sum_-^+ \tag{52}$$

where $\sum_\pm^\pm$ and $\sum_0^\pm$ are subspaces of $\mathcal{G}$ realized in terms of pseudo-differential operators as:

$$\textstyle\sum_\pm^+ = \left\{L_{\pm n}(z) = \sum_{i\geq 0} u_{\pm n-i}(z)\partial^i, \pm n \geq 0\right\}, \tag{53}$$

$$\textstyle\sum_0^+ = \left\{L_0(z) = \sum_{i\geq 0} u_{-i}(z)\partial^i\right\}, \tag{54}$$

$$\textstyle\sum_\pm^- = \left\{V_{\pm n}(z) = \sum_{j\geq 1} \partial^{-j} v_{j\pm n}(z), \pm n > 0\right\}, \tag{55}$$

$$\textstyle\sum_0^- = \left\{V_0(z) = \sum_{j\geq 1} \partial^{-j} v_j(z)\right\}. \tag{56}$$

Note that the local and non-local differential operators are given respectively by the Lax operators $L_{0,\pm}(z)$ and their dual $V_{0,\pm}(z)$ in the Volterra representation. The indices carried by these Lax operators are the conformal spin quantum numbers. The sub-algebras $\mathcal{G}_\pm$ and $\mathcal{G}_0$ given in Eqs(50-52) satisfy the following duality relations with respect to the combined scalar product Eq.(33) :

$$\mathcal{G}_+^* = \mathcal{G}_-, \tag{57}$$

$$\mathcal{G}_0^* = \mathcal{G}_0, \tag{58}$$

As shown in Section.2, this duality transformation imposes constraints on the degrees quantum numbers, namely, if (p,q) are the lowest and the highest degrees of some algebra, the degrees of the corresponding dual algebra are $(-q-1; -p-1)$ i.e.:

$$L_n(z) = \textstyle\sum_{i=p}^{q} u_{n-i}(z)\partial^i \leftrightarrow X_n(z) = \sum_{j=p+1}^{q+1} \partial^{-j} u_{j+n}(z), \tag{59}$$

the sub-algebras $\mathcal{G}^\pm$ and $\mathcal{G}^0$ are shown to correspond to eigen-spaces of a particular generalized R_s -matrix exhibiting a conformal spin quantum number $s = +, -$ or 0 and which we define as:

$$R_s = R_s^+ - R_s^-, \tag{60}$$

where $R_s^\pm$ are the projections into the subspaces $\sum_s^\pm$:

$$R_s^\pm = Proj_{\sum_s^\pm}, \tag{61}$$

These spin-graded R_s-matrix define an endomorphism $End(\sum_s)$ satisfying the following graded-modified classical Yang-Baxter (GMCYB) equations:

$$[R_sX, R_sY] - R_s([R_sX, Y] + [X, R_sY]) = -[X,Y], \tag{62}$$

for which X, Y are elements of $\sum_s = \sum_s^+ \oplus \sum_s^-$. The action of the graded R_s -operator on the huge Lie algebra $\mathcal{G} = \oplus_{s=0,\pm} \mathcal{G}_s$ is simply given by:

$$\mathcal{G}_s = \{X/R_sX = \pm X, X \in \textstyle\sum_s^\pm \}, \tag{63}$$

where $\sum_s^\pm$ are the $6 = 3 \times 2$ Lie sub-algebras defined previously. The expressions of the projections operators $R_s^\pm$ in terms of R_s are given by:

$$R_s^+ = \tfrac{1}{2}(R_s + 1), \tag{64}$$

$$R_s^- = -\tfrac{1}{2}(R_s - 1), \tag{65}$$

where R_s is the difference of the projections. Notice that $R_s^\pm X = X$ if $X \in \sum_s^\pm$ for $s = 0, \pm$ and that $R_s^\pm X = 0$ if $X \in \sum_s^\mp$. On sees that there exist three additional Lie structures:

$$[X, Y]_{R_s} = \tfrac{1}{2}([R_sX, Y] + [X, R_sY]), X, Y \in \textstyle\sum_s . \tag{66}$$

Using the self-duality conditions with respect to the combined scalar product introduced in previously, one can suspect from the AKS scheme that the KP-integrable systems associated to the additional structures $[,]_{R_\pm}$ are dual to each others. Fact which means also the possibility to connect their second Hamiltonian structures. The integrable systems associated to the spin-zero additional structure $[,]_{R_0}$is self-dual. Later on, we will denote these three $R_s$systems simply by $(2 + 1)$-integrable systems. To derive these $(2 + 1)$ integrable systems, one have to develop a technical analysis starting from the following R_s– bracket:

$$\frac{dF}{dt} = \{F, H\}_{R_s}(U) = -\langle U\big|[\nabla F(U), \nabla H(U)]_{R_s}\rangle, \tag{67}$$

generalizing the well known one to our huge Lie algebra $\mathcal{G} \equiv \Xi$.

We present here below some algebraic results concerning our decomposition Eqs.(50-52) and its connection with the GMCYB equations.

Proposition 1: The eigen spaces $\mathcal{G}_\pm$ and $\mathcal{G}_0$ of pseudo-differential operators of positive (resp. negative) and vanishing conformal spin are sub-algebras of $\mathcal{G} \equiv \Xi$.

The proof of this proposition follows straightforwardly by starting from the decomposition Eqs.(50-52) of the huge Lie algebra Ξ and proceeding step by step by using the definition of our subspaces $\sum_\pm^\pm$, $\sum_0^\pm$ and the formulas for the graded modified classical Yang-Baxter (GMCYB) equation.

i) $X, Y \in \mathcal{G}_+ = \sum_+^+ \oplus \sum_+^-$

$\mathcal{G}_+$ is a subspace of $\mathcal{G}$ generated by local and non-local operators of positive conformal spin. $\mathcal{G}_+$ is therefore invariant under the action of R_+:

$$R_+(\mathcal{G}_+) = \mathcal{G}_+$$

$$R_0(\mathcal{G}_+) = 0$$

$$R_-(\mathcal{G}_+) = 0$$

- if $X, Y \in \sum_+^+$ or $X, Y \in \sum_+^-$, it is easy to see that:

$$[R_+X, R_+Y] = R_+[X, Y]_{R_+}$$

with $[X, Y]_{R_+} = \frac{1}{2}([R_+X, Y] + [X, R_+Y])$.

Suppose $X, Y \in \sum_+^+$, we have $R_+X = X$, $R_+Y = Y$ and then:

$$[R_+X, R_+Y] = [X, Y] = R_+[X, Y],$$

Since $[X, Y]_{R_+} = [X, Y]$. We conclude then that $[X, Y] \in \sum_+^+$.

- If $X, Y \in \sum_+^-$, we have $R_+X = -X$, $R_-Y = -Y$ and then

$$[R_+X, R_+Y] = [X, Y] = -R_+[X, Y]$$

Since $[X, Y]_{R_+} = -[X, Y]$. Then we conclude that $[X, Y] \in \sum_+^-$.

- If $X \in \sum_+^+$ and $Y \in \sum_+^-$, it is easy to check that the additional structure $[X, Y]_{R_+}$ is zero and the GMCYB equation simply means$[R_+X, R_+Y] = -[X, Y]$.

We conclude finally that $\mathcal{G}_+$ is a sub-algebra of Ξ, ie.

$$[\mathcal{G}_+, \mathcal{G}_+] \subset \mathcal{G}_+.$$

(i) It is easy also to show that $\mathcal{G}_-$ is a sub-algebra of $\mathcal{G}$ in the same way,

$$[\mathcal{G}_-, \mathcal{G}_-] \subset \mathcal{G}_-.$$

(ii) Now suppose $X, Y \in \mathcal{G}_0$ which is invariant under the action of R_0 . The pseudo-differential operators belonging to this subspace are Lorentz scalar (spin zero) operators. Let first consider $X, Y \in \sum_0^+$ and proceeding as previously, we find$R_0[X, Y] = [X, Y]$ since $[X, Y]_{R_0} = [X, Y]$ and therefore $[X, Y] \in \sum_0^+$.

The same thing holds for $, Y \in \sum_0^-$. Indeed, $[X, Y]_{R_0} = -[X, Y]$and then $R_0[X, Y] = -[X, Y]$ showing that $[X, Y] \in \sum_0^+$.

If X and Y are elements of $\mathcal{G}_0$ with opposite degrees quantum numbers, say $X \in \sum_0^+$ and $Y \in \sum_0^-$ (or the inverse), one can check easily that $[X, Y]_{R_0} = 0$ and the GMYBE simply means$[R_0X, R_0Y] = -[X, Y]$. One have then $[\mathcal{G}_0, \mathcal{G}_0] \subset \mathcal{G}_0$.

Proposition 2: $[\mathcal{G}_\pm, \mathcal{G}_0] \subset \mathcal{G}_\pm$

Proof:

(i) Suppose, $X \in \mathcal{G}_+$ and $Y \in \mathcal{G}_0$, from the GMCYB equation, one have $2R_+[X, Y]_{R_+} = R_+[R_+X, Y]$ since $R_+Y = 0$. If $X \in \sum_+^+$:

$$R_+[X, Y] = [X, Y] \; i{:}e \; [X, Y] \in \sum\nolimits_+^+$$

If $X \in \sum_+^-$:

$$R_+[X, Y] = -[X, Y] \; i{:}e \; [X, Y] \in \sum\nolimits_+^-$$

We conclude then that, if$X \in \mathcal{G}_+$and $Y \in \mathcal{G}_0$then $[X,Y] \in \mathcal{G}_+$.

(ii) Similarly, we have for $\mathcal{G}_-$:

$$X \in \mathcal{G}_-; Y \in \mathcal{G}_0 \text{ then } [X,Y] \in \mathcal{G}_- .$$

Before closing this section, we give here below some important remarks.

1 The derived three additional structures $[,]R_s$; $s = 0, \pm$ corresponding to three GMCYB equations are now used to construct three R_s-Lie Poisson brackets generalizing Eq.(40) to:

$$\{F,H\}_{R_s}(U) = -\left\langle U \middle| [\nabla F(U), \nabla U(U)]_{R\,s} \right\rangle,$$

Where $F, H \in C^\infty(\mathcal{G}^*, R)$. As pointed before, these Poisson-brackets should describe three dynamical systems given by:

$$\frac{dF}{dt} = \{H,F\}_{R_s}.$$

2 The combined scalar product, satisfy the following properties

(i) Its conformal weight $\Delta[<<,>>] = 0$

(ii) The Lie sub-algebras $\sum_+^+$, $\sum_0^+$ and $\sum_-^+$ are dual to $\sum_-^-$, $\sum_0^-$ and $\sum_+^-$ with respect to this combined scalar product in the following sense:

$$\langle\langle \sum\nolimits_+^+ , \sum\nolimits_-^- \rangle\rangle = \langle\langle R_s \sum\nolimits_+^+ , \sum\nolimits_-^- \rangle\rangle = -\langle\langle \sum\nolimits_+^+ , R_s \sum\nolimits_-^- \rangle\rangle$$

$$\langle\langle \sum\nolimits_0^+ , \sum\nolimits_0^- \rangle\rangle = \langle\langle R_s \sum\nolimits_0^+ , \sum\nolimits_0^- \rangle\rangle = -\langle\langle \sum\nolimits_0^+ , R_s \sum\nolimits_0^- \rangle\rangle$$

$$\langle\langle \sum\nolimits_-^+ , \sum\nolimits_+^- \rangle\rangle = \langle\langle R_s \sum\nolimits_-^+ , \sum\nolimits_+^- \rangle\rangle = -\langle\langle \sum\nolimits_-^+ , R_s \sum\nolimits_+^- \rangle\rangle$$

showing that R_s is skew symmetric with respect to the combined scalar product$\langle\langle , \rangle\rangle$.

Each $\mathcal{G}_+ = \sum_+^+ \oplus \sum_+^-$ and $\mathcal{G}_- = \sum_-^+ \oplus \sum_-^-$ are isotropic with respect tothe combined scalar product $\langle\langle , \rangle\rangle$. Moreover $\mathcal{G}_0 = \sum_0^+ \oplus \sum_0^-$ is orthogonal to $\mathcal{G}_\pm$.

These properties follow naturally from the definition of the product $\langle\langle , \rangle\rangle$. Indeed for any $X, Y \in \mathcal{G}_+ (or\ X, Y \in \mathcal{G}_-)$ we have:

$$\langle\langle X, Y \rangle\rangle = 0.$$

Remark finally that this result can be also obtained from the skew symmetry of the $R_s$with respect to $\langle\langle , \rangle\rangle$. We deduce also that $\mathcal{G}_0$ is orthogonal to $\mathcal{G}_\pm$ since:

$$\langle\langle X, Y \rangle\rangle = 0$$

for all $X \in \mathcal{G}_\pm$ and $Y \in \mathcal{G}_0$.

4 Conclusion

We have proposed a consistent approach for the AKS construction of integrable systems. This approach is based on the conformal spin decomposition of the huge Lie algebra $\mathcal{G}$ of pseudo-differential operators Eqs.(50-52). We used this decomposition to derive three additional Lie brackets associated to three GMCYB equations. The method exposed here allows, furthermore to draw a consistent classification principle of integrable systems. Future works, will focus to apply the present formulation to KP-integrable hierarchies which proved to be relevant for a variety of physical problems both in the bosonic and super symmetric cases.

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