Incorporating risk in an optimization model of reliability engineering

Akio Matsumoto[1], Ferenc Szidarovszky[2], Miklós Szidarovszky[2]

[1]Department of Economics, Chuo University, 742-1, Higashi-Nakano, Hachioji-shi, Tokyo, 192-0393, Japan
[2]ReliaSoft Corporation, 1450 S. Eastside Loop, Tucson, Arizona 85710, USA

Email address:
akiom@tamacc.chuo-u.ac.jp (A. Matsumoto)

Abstract: A non-repairable system is considered and the problem of finding its optimal preventive replacement time is revisited. In addition to minimizing the expected cost per unit time in a cycle, we also consider its variance as the measure of the risk of the optimal decision. A multi-objective optimization problem is then formulated where the two objective functions are the expectation and the variance. A sufficient condition is given for the existence of finite optimum in the case of the weighting method, where either the weight of the variance or the replacement costs are sufficiently small. In applying the ε - constraint method there is always finite optimum if the upper bound for the expectation is close to its minimal value.

Keywords: Certainty Equivalent, Reliability, Risk

1. Introduction

Scheduling preventive maintenance and replacement is one of the most important problems of reliability engineering. There is a huge selection of optimization models known from literature. The survey papers of Beichelt (1993) and Wang (2002) give a good background, in addition there are a large number of books such as Nakagawa (2006, 2008), Elsayed (2012) among others. Most models maximize reliability or minimize expected cost or downtime per unit time without any additional consideration to the risks which can be characterized by the variances of the random outcomes. In the economic literature certainty equivalents are derived and shown to be equivalent to random outcomes (Sargent, 1979). It is a linear combination of the expectation and variance of the random phenomenon. This idea can also be interpreted as the application of a multiobjective solution method when weighting is used for minimizing expected cost or downtime and also minimizing their variances in order to minimize risk. In this paper this idea will be incorporated into a classical optimization model. The single-objective model will be first discussed, and then the multiobjective model will be introduced. There is a large variety of solution concepts and methods (Szidarovszky et al., 1986). In our study the weighting and the ε-constraint methods are selected.

Conditions are given for the existence of finite optimum, and a numerical example illustrates the methodology.

2. Optimum Age Replacement with One-Cycle Criterion

Consider a system which is non-repairable and its time to failure distribution is a Weibull distribution with parameters η and β. The question is to find the optimal time T for its preventive replacement. However, if the system breaks down before it is scheduled, then failure replacement has to be performed. Let c_p and c_f denote the costs of preventive and failure replacements, respectively. It is clear that $c_p < c_f$. In order to formulate the expected cost per unit time, we have to consider the cost per unit time (CPU) as a random variable depending on the time $\tilde{x}$ of the break down and on the decision variable T:

$$CPU(T) = \begin{cases} \frac{c_f}{\tilde{x}} & if\ \tilde{x} < T \\ \frac{c_p}{T} & if\ \tilde{x} \geq T \end{cases} \qquad (1)$$

The CDF of $\tilde{x}$ is given as $F(x) = 1 - e^{-\left(\frac{x}{\eta}\right)^\beta}$, the corresponding pdf is $f(x) = \frac{\beta}{\eta^\beta} x^{\beta-1} e^{-\left(\frac{x}{\eta}\right)^\beta}$ and the reliability function is $R(x) = e^{-\left(\frac{x}{\eta}\right)^\beta}$. The expected value of $CPU(T)$ is given as

$$E\big(CPU(T)\big) = \int_0^T \frac{c_f}{x} f(x) dx + \frac{c_p}{T} R(T) \qquad (2)$$

which is minimized with respect to T in the classical model. If $\beta > 1$, then the first term of this expression is finite, since $\beta - 2 > -1$ and so the function $\frac{f(x)}{x} = \frac{\beta}{\eta^\beta} x^{\beta-2} e^{-\left(\frac{x}{\eta}\right)^\beta}$ has finite integral in any interval $[0, T]$.

The derivative of this objective function has the following form:

$$\frac{c_f}{T} f(T) - \frac{c_p}{T^2} R(T) - \frac{c_p}{T} f(T) = \frac{(c_f - c_p)R(T)}{T}\left(\rho(T) - \frac{c_p}{(c_f - c_p)T}\right) \quad (3)$$

where

$$\rho(T) = \frac{f(T)}{R(T)} = \frac{\beta}{\eta^\beta} T^{\beta-1} \qquad (4)$$

is the failure rate of $\tilde{x}$. Based on this form of $\rho(T)$, expression (3) can be rewritten as

$$\frac{(c_f - c_p)R(T)}{T^2}\left(\frac{\beta}{\eta^\beta} T^\beta - \frac{c_p}{c_f - c_p}\right). \qquad (5)$$

This expression is zero at

$$T^* = \eta\left(\frac{c_p}{\beta(c_f - c_p)}\right)^{\frac{1}{\beta}}, \qquad (6)$$

it is negative for $T < T^*$ and positive for $T > T^*$. Therefore T^* gives the global minimum of $E(CPU(T))$.

The shape of $E(CPU(T))$ as the function of T is shown in Figure 1 with $\eta = 5$, $\beta = 3$, $c_f = 200$, $c_p = 100$, in which case $T^* = 3.47$.
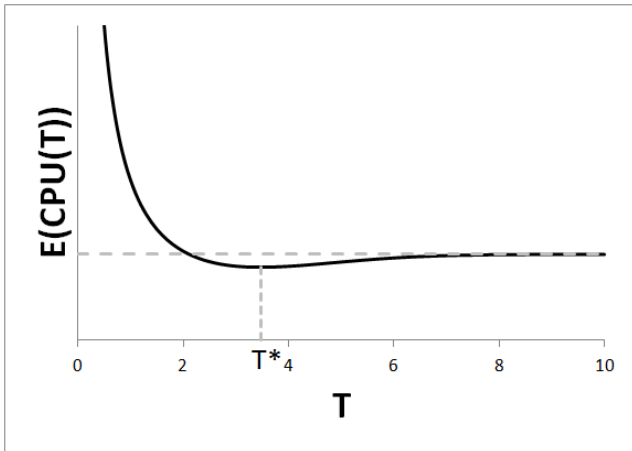

Figure 1. *The shape of* $E(CPU(T))$

This model does not consider the risk which should be a measure of the difference of the actual cost and its expectation. The multiobjective model incorporating risk into this model will be introduced next.

3. Incorporating Risk into the Optimization Model

Notice first that the second moment of $CPU(T)$ is the following:

$$E(CPU(T)^2) = \int_0^T \frac{c_f^2}{x^2} f(x) dx + \frac{c_p^2}{T^2} R(T) \qquad (7)$$

so its variance is given as

$$Var\big(CPU(T)\big) = \int_0^T \frac{c_f^2}{x^2} f(x) dx + \frac{c_p^2}{T^2} R(T) -$$

$$\left\{\int_0^T \frac{c_f}{x} f(x) dx + \frac{c_p}{T} R(T)\right\}^2 \qquad (8)$$

Minimizing expected cost per unit time and minimizing the associated risk result in a multi-objective optimization problem:

$$\text{minimize } \big\{E\big(CPU(T)\big) \, ; \, Var\big(CPU(T)\big)\big\}$$

$$\text{subject to } T \geq 0 \qquad (9)$$

In order to have finite variance we have to assume that $\beta > 2$, since in this case $\beta - 3 > -1$ and so function

$$\frac{f(x)}{x^2} = \frac{\beta}{\eta^\beta} x^{\beta-3} e^{-\left(\frac{x}{\eta}\right)^\beta}$$

has finite integral in any interval $[0, T]$.

We now investigate $Var(CPU(T))$ as function of T. Notice first that

$$\lim_{T \to 0} Var\big(CPU(T)\big)$$
$$= \lim_{T \to 0} \frac{c_p^2}{T^2}(R(T) - R(T)^2)$$
$$= \lim_{T \to 0} \frac{C_p^2\big(-f(T) + 2R(T)f(T)\big)}{2T}$$

$$= \lim_{T \to 0} \frac{c_p^2 f(T)}{2T}(-1 + 2R(T)) = 0$$

since $R(0) = 1$ and $\frac{f(T)}{T} = \frac{\beta}{\eta^\beta} T^{\beta-2} e^{-\left(\frac{x}{\eta}\right)^\beta} \to 0$.

It is also clear that

$$\lim_{T \to \infty} Var(CPU(T)) = \left(\int_0^\infty \frac{f(x)}{x^2} dx - \left(\int_0^\infty \frac{f(x)}{x} dx\right)^2\right) c_f^2 > 0$$

since $\frac{R(T)}{T}$ and $\frac{R(T)}{T^2}$ converge to 0.

The derivative of $Var(CPU(T))$ has the following form:

$$\frac{c_f^2}{T^2}f(T) - \frac{2c_p^2}{T^3}R(T) - \frac{c_p^2}{T^2}f(T)$$
$$- 2\left(\int_0^T \frac{c_f}{x}f(x)dx + \frac{c_p}{T}R(T)\right)\left(\frac{c_f}{T}f(T) - \frac{c_p}{T^2}R(T)\right)$$
$$- \frac{c_p}{T}f(T)\right) =$$

$$\frac{c_f^2}{T^2}f(T) - \frac{2c_p^2}{T^3}R(T) - \frac{c_p^2}{T^2}f(T) - 2\frac{c_f^2}{T}f(T)\int_0^T \frac{f(x)}{x}dx$$
$$+ 2\frac{c_p c_f}{T^2}R(T)\int_0^T \frac{f(x)}{x}dx + 2\frac{c_p c_f}{T}f(T)\int_0^T \frac{f(x)}{x}dx$$
$$- 2\frac{c_p c_f}{T^2}f(T)R(T) + 2\frac{c_p^2}{T^3}R(T)^2 + 2\frac{c_p^2}{T^2}\frac{R(T)f(T)}{T^2} =$$

$$\frac{(c_f^2 - c_p^2)f(T)}{T^2} - \frac{2c_p^2 R(T)(1-R(T))}{T^3} - \frac{2c_p(c_f - c_p)R(T)f(T)}{T^2} +$$
$$\int_0^T \frac{f(x)}{x}dx\left\{-\frac{2(c_f - c_p)c_f f(T)}{T} + \frac{2c_p c_f R(T)}{T^2}\right\} \quad (10)$$

This can be rewritten as

$$\frac{f(T)}{T}\left\{\frac{c_f^2 - c_p^2}{T} - \frac{2c_p^2(1-R(T))}{T^2\rho(T)} - \frac{2c_p(c_f - c_p)R(T)}{T} + \int_0^T \frac{f(x)}{x}dx\left(-2(c_f - c_p)c_f + 2\frac{c_p c_f}{T\rho(T)}\right)\right\} \quad (11)$$

As $T \to \infty$, the multiplier of $\frac{f(T)}{T}$ converges to

$$-2\int_0^\infty \frac{f(x)}{x}dx\,(c_f - c_p)c_f < 0,$$

so as $T \to \infty$ the derivative of $Var(CPU(T))$ converges to 0 through negative numbers. The shape of this function is shown in Figure 2 with the same parameter selection as in the case of Figure 1.
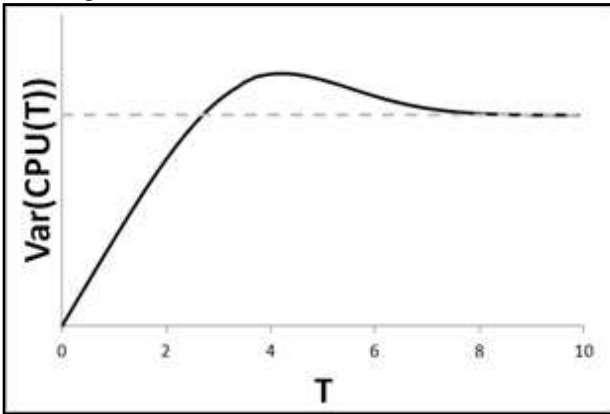

Figure 2. The shape of $Var(CPU(T))$

If problem (9) is solved by the weighting method, then with some $\alpha > 0$ function

$$Q(T) = E\big(CPU(T)\big) + \alpha Var\big(CPU(T)\big) \quad (12)$$

is minimized. Observe first that $Q(0) = \infty$ and

$$\lim_{T\to\infty} Q(T) = \int_0^\infty \frac{c_f}{x}f(x)dx$$
$$+ \alpha\left(\int_0^\infty \frac{c_f^2}{x^2}f(x)dx - \left(\int_0^\infty \frac{c_f}{x}f(x)dx\right)^2\right)$$

which is a positive constant. Using (3) and (10),

$$Q'(T) = \frac{f(T)}{T}\left\{c_f - \frac{c_p}{T\rho(T)} - c_p\right.$$
$$+ \alpha\left(\frac{c_f^2 - c_p^2}{T} - \frac{2c_p^2(1 - R(T))}{T^2\rho(T)}\right.$$
$$\left. - \frac{2c_p(c_f - c_p)R(T)}{T} +\right.$$
$$\left.\int_0^T \frac{f(x)}{x}dx\left(-2(c_f - c_p)c_f + 2\frac{c_p c_f}{T\rho(T)}\right)\right)\right\} \quad (13)$$

If $T \to \infty$, then the multiplier of $\frac{f(T)}{T}$ converges to

$$(c_f - c_p)\left[1 - 2\alpha c_f \int_0^\infty \frac{f(x)}{x}dx\right]$$

from which we see that $Q'(T) \to 0$ as $T \to \infty$ through positive numbers if

$$\alpha c_f < \frac{1}{2\int_0^\infty \frac{f(x)}{x}dx} = \frac{1}{2E\left(\frac{1}{x}\right)} \quad (14)$$

and through negative numbers if

$$\alpha c_f > \frac{1}{2\int_0^\infty \frac{f(x)}{x}dx} = \frac{1}{2E\left(\frac{1}{x}\right)} \quad (15)$$

In the first case there is a finite minimum of $Q(T)$, which can be obtained either by an optimization software or by solving the first order condition. Since the problem is single dimensional, it is easy to find the optimum. In the second case however there is no theoretical guarantee for the existence of finite optimum, it depends on the particular model parameters.

We can explain the condition (14). If α is small, then very small weight is given to variance, so the first term of (12) dominates, which has a finite optimum. If c_f is small, then $c_p < c_f$ is also small, so the first term of (12) dominates again, since it is linear in c_p and c_f, while the second term is quadratic (every term includes either c_p^2, c_f^2, or $c_p c_f$).

Consider next the application of the ε-constraint method, when the variance is minimized subject to an upper bound for the cost. The constraint has the form

$$\int_0^T \frac{c_f}{x}f(x)dx + \frac{c_p}{T}R(T) \le \varepsilon \quad (16)$$

Let E_{min} denote the minimum value of the expression and let E_∞ be its limit at infinity. Figure 1 shows the shape of

$E(CPU(T))$.

Assume first that $\varepsilon \geq E_\infty$. Then there is a unique $\bar{T}$ such that $E(CPU(\bar{T})) = \varepsilon$, and (16) holds for $T \geq \bar{T}$.

If $E_{min} < \varepsilon < E_\infty$, then there are values $\bar{T}_1 < \bar{T}_2$ such that $E(CPU(\bar{T}_1)) = E(CPU(\bar{T}_2)) = \varepsilon$ and (16) holds for $\bar{T}_1 \leq T \leq \bar{T}_2$.

If $\varepsilon = E_{min}$, then only $T = T^*$ is feasible, and if $\varepsilon < E_{min}$, then there is no feasible solution.

In the case of $E_{min} < \varepsilon < E_\infty$ the feasible region for T is a closed interval, and since $Var(CPU(T))$ is continuous, there is finite optimum. If $\varepsilon \geq E_\infty$, then there is no guarantee for the existence of finite optimum.

4. Numerical Example

The same parameter values are selected as before, $\eta = 5$, $\beta = 3$, $c_f = 200$ and $c_p = 100$. We selected the weighting method with several values of α. The optimal solutions are shown in Table 1.

It is clear that a larger value of α, that is, a higher weight for risk results in a smaller optimal solution. In other words, when more importance is given to risk, then the optimal solution becomes more cautious by replacing the system more often.

Table 1. *Optimal Solutions*

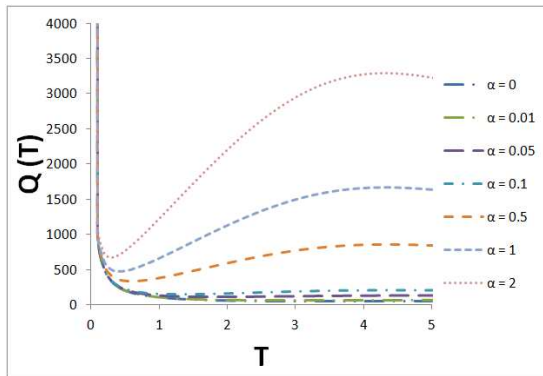| α	Optimal Solution (T)
0	3.47
0.001	3.43
0.01	3.06
0.05	1.83
0.1	1.32
0.5	0.60
1	0.42
2	0.30

Figure 3. *The shape of Q(T) for various α*

5. Conclusions

A multiobjective programming model was formulated for the simultaneous minimization of the expectation and the variance of the cost per unit time in a cycle of a non-repairable system.

In applying the weighting method sufficient condition was given for the existence of finite optimum. This condition requires that either the weight of the variance and/or the failure replacement cost are sufficiently small. In applying the ε-constraint method there is always finite optimum when the upper bound of the expected cost is close enough to its minimal value.

Since the problem is single dimensional, simple line search algorithm or an equation solver can be used to find the optimal solution.

This model can be easily modified to minimize expected downtime per unit time, in which case c_p and c_f are replaced by T_p and T_f, respectively.

Acknowledgements

References

[1] Beichelt, F (1993) A Unifying Treatment of Replacement Policies with Minimal Repair. *Naval Research Logistics, 40*: 51-67.

[2] Wang, H.Z. (2002) A Survey of Maintenance Policies of Deteriorating Systems. *European Journal of Operational Research*, *139 (3):* 469-489.

[3] Nakagawa, T. (2006) *Maintenance Theory and Reliability*. Springer-Verlag, Berlin/Tokyo.

[4] Nakagawa, T. (2008) *Advanced Reliability Models and Maintenance Policies*. Springer-Verlag, Berlin/Tokyo.

[5] Elsayed, E.A. (2012) *Reliability Engineering*. John Wiley & Sons, Hoboken, New Jersey.

[6] Sargent, T.J. (1979) *Macroeconomic Theory*. Academic Press, New York.

[7] Szidarovszky, F., Gershon M.E., Duckstein, L. (1986) *Techniques of Multiobjective Decision Making in Systems Management*. Elsevier, Amsterdam.