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Hosaya Polynomial and Weiner Index of Abid-Waheed Graph $(AW)_p^6$

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Abstract: Graph theory is an area of mathematics and computer science that deals with graphs, or diagrams containing points and lines that represent mathematical truths pictorially. It has a broad scope of applications. The use of graph theory has exponentially increased. It is effective to understand the flow of computation, networks of communication, data organization, and Google maps in computers. Graphs have great importance in electrical engineering (design of electrical connections), linguistics (parsing of language trees, grammar of a language tree, phonology, and morphology), chemistry, physics, mathematics, and biology. Graph theory plays an important role in the development of theoretical chemistry. A special type of graph invariant called a topological index is a real number associated with the structure of a connected graph. In this paper, we calculate the Wiener index (WI) and Hosoya polynomial of newly defined "Abid Waheed graph $(AW)_p^6$".

Keywords: Graph Theory, Topological Index, Distance, Hosoya Polynomial, Weiner Index, Abid-Waheed Graph

1. Introduction

The graph is made up of points and lines. Let G be a planar, simple, and connected graph. The graph G has a set of nodes and an edge set, which are respectively represented by V(G) and E(G). The degree of a vertex $\upsilon \varepsilon V(G)$ is the count of edges joining to υ and represented as $\Phi(\upsilon)$. It was Leonard Euler who first introduced graph theory with his solution to the famous problem named as, "Seven Bridges of Königsberg problem" in 1736. As a result, he is known as "The Father of Graph Theory" because he was the first to convert a real-life problem into a mathematical problem. Due to its great significance, it has numerous applications in computer science, linguistics, physics, chemistry, social sciences, biology, engineering, and mathematics [1-10].

A branch of mathematical chemistry called chemical graph theory deals with mathematically solving problems of chemical structure. Atoms and bonds are symbolically represented as nodes and lines (edges) of a chemical graph. It is similar to the degree of a vertex in graph theory to think of valence electrons in chemistry. The chemical graph theory is more important because of isomorphism in chemical structures. A number of applications can be found in medicine and food production of chemical graph theory [11-15].

Topological indices (TIs) of a simple and connected graph are numerical descriptors that are derived from graphs of chemical compounds [16-20]. Topological indices have different classifications as counting-based, degree-based, distance-based, and eccentricity-based. Such indices based on the distances in graphs are widely used in QSAR (quantitative structure activity relationship) for establishing relationships between the structure of molecular graphs and their physicochemical properties. Usage of TIs in biology and chemistry began in 1947 when chemist Harold Wiener proposed the W-index while estimating the boiling point of paraffin [21]. He demonstrated the correlations between the physicochemical properties of organic compounds and the index of their molecular graphs. Wiener applied the index to trees and studied its use for correlations of physicochemical properties of alkanes, alcohols, amines, and their analogous compounds. The W-index for a graph G is mathematically stated as:-

$$W(G) = \frac{1}{2}\sum_{\tau,\upsilon\varepsilon V(G)} \Phi(\tau,\upsilon) \quad (1)$$

For more information about the W-index, see [22-25]. In 1988, Hosoya determined the Hosoya polynomial [26-30]. It is mathematically stated as:-

$$H(G,k) = \sum_{\tau,\upsilon\varepsilon V(G)} \Phi(\tau,\upsilon)k^{\Phi(\tau,\upsilon)} \quad (2)$$

where $\Phi(\tau,\upsilon)$ represents the distance between τ and υ. The W-index can be calculated by taking the first derivative of Hosoya polynomial at k=1. It is an easy and short method to find the W-index when H-polynomial is given.

$$W(G) = \frac{\partial H(G,k)}{\partial k}|_{k=1} \quad (3)$$

The topological diameter D(G) of a graph is the greatest distance between nodes, expressed as:-

$$D(G) = \max_{\upsilon\varepsilon V(G)}\{\Phi(\tau,\upsilon)|\forall\upsilon\varepsilon V(G)\} \quad (4)$$

2. Construction of Abid Waheed $(AW)_p^q$ Graph for $q \geq 3$ and $p \geq 1$

This section deals with the construction and methodology of a new notion of a connected graph known as the Abid Waheed (AW-graph) and defined as;
Abid Waheed graph AW_p^q consists of pq+1 vertices and p(q+1) edges for all $q \geq 3$ and $p \geq 1$, i.e. a graph generated by p-cycles (with each of order q), meeting at an external vertex of degree p. It is denoted by $(AW)_p^q$ for $q \geq 3$ and $p \geq 1$. Representation of Abid-Waheed Graph Shown in Figure 1.

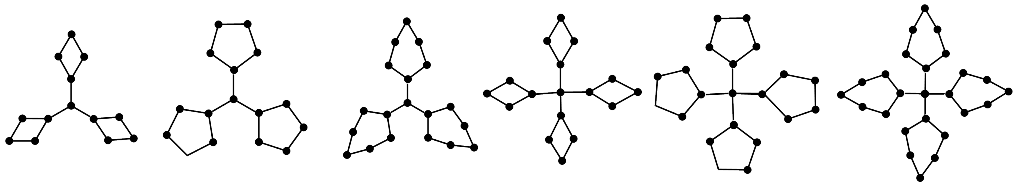

Figure 1. *Abid Waheed AW_p^q Graphs.*

3. Resemblance of Abid-Waheed Graph with Other Graphs

The Abid-Waheed of order $(AW)_3^6$ is same as the unit cell of nanostar dendrimer given in [31]. A dendrimer is generally synthesized from monomers by iterative growth and activation steps. Dendrimers were prepared in the 1980s. These unique branched polymers possess bimolecular-like properties, low polydispersity, and high versatility in terms of structure (easy to customize physicochemically), delivery route (enteral or parenteral), and application (drug delivery, diagnosis, therapy, and detection). Dendrimers have lots of applications in chemistry and biology. The graph used in this paper also helps to understand this type of property.
The generalization of abid-waheed graph is similar to the Jahangir graph [32]. The generalization of this graph can be done by increasing the number of central edges (p) and edges in the branches (q). This newly introduced graph is applicable to computer networking. The Wiener index is a distance-based index, so it tells us about the locations in the graph.

4. Fundamental Results

The main purpose of this section is to compute the values for H-polynomial and W-index of Abid Waheed graph $(AW)_p^6$ where $p \geq 1$. First of all we compute H-polynomial by calculating different distances and their cardinality in a graph then with the help of this polynomial the W-index for $(AW)_p^6$ is calculated. The number of unordered pair of nodes τ and υ at distance $\Phi(\tau,\upsilon)$ are represented by $\Phi(G,z)$. The term D(G) is topological diameter (TD) of graph G. The maximum value of z may be equal to D(G). The range of z is $1 \leq z \leq D(G)$.

4.1. Example 1

The H-polynomial and W-index of Abid-Waheed graph $(AW)_3^6$ is calculated as:-

$$H(G,k) = \sum_{z=1}^{8} \Phi(G,z)k^z \quad (5)$$

$$H(G,k) = \Phi(G,1)k + \Phi(G,2)k^2 + \Phi(G,3)k^3 + \Phi(G,4)k^4 + \Phi(G,5)k^5 + \Phi(G,6)k^6 + \Phi(G,7)k^7 + \Phi(G,8)k^8 \quad (6)$$

The topological diameter of $(AW)_3^6$ is eight. The distances with their relevant frequencies are displayed in Table 1. The range of distance $\Phi(G, z)$ is represented by $1 \leq z \leq 8$.

Table 1. *Distance* Φ*(G,z) for* $1 \leq z \leq 8$ *for each pair of vertices* $(AW)_3^6$.

$\Phi(G, z)$	Cardinality	$\Phi(G, z)$	Cardinality
Φ(G,1)	21	Φ(G,5)	30
Φ(G,2)	27	Φ(G,6)	24
Φ(G,3)	27	Φ(G,7)	12
Φ(G,4)	27	Φ(G,8)	3

$$H((AW)_3^6, k) = 21k + 27k^2 + 27k^3 + 27k^4 + 30k^5 + 24k^6 + 12k^7 + 3k^8 \tag{7}$$

The W-index for $(AW)_3^6$ can be calculated by taking the 1^{st} derivative of H($(AW)_3^6$, k) at k=1.

$$W((AW)_3^6) = 21 + 54k|_{k=1} + 81k^2|_{k=1} + 108k^3|_{k=1} + 105k^4|_{k=1} + 144k^5|_{k=1} + 84k^6|_{k=1} + 24k^7|_{k=1} \tag{8}$$

$$W((AW)_3^6) = 666 \tag{9}$$

4.2. Example 2

The H-polynomial and W-index of the Abid-Waheed graph $(AW)_4^6$ are determined as follows:-

$$H(G, k) = \sum_{z=1}^{8} \Phi(G, z) k^z \tag{10}$$

The topological diameter of $(AW)_4^6$ is eight. The graph $(AW)_4^6$ is displayed in Figure 1. The H-polynomial and W-index are computed with respect to the distance between pairs of nodes τ and υ. The distances $\Phi(G, z)$ for $1 \leq z \leq 8$ are shown in Table 2 with relative cardinalities of different distance edges.

Table 2. *Distance* Φ*(G,z) for* $1 \leq z \leq 8$ *for each pair of vertices* $(AW)_4^6$.

$\Phi(G, z)$	Cardinality	$\Phi(G, z)$	Cardinality
Φ(G,1)	28	Φ(G,5)	60
Φ(G,2)	38	Φ(G,6)	48
Φ(G,3)	44	Φ(G,7)	24
Φ(G,4)	52	Φ(G,8)	6

$$H(G, k) = \Phi(G, 1)k + \Phi(G, 2)k^2 + \Phi(G, 3)k^3 + \Phi(G, 4)k^4 + \Phi(G, 5)k^5 + \Phi(G, 6)k^6 + \Phi(G, 7)k^7 + \Phi(G, 8)k^8 \tag{11}$$

$$H(G, k) = 28k + 38k^2 + 44k^3 + 52k^4 + 60k^5 + 48k^6 + 24k^7 + 6k^8 \tag{12}$$

The W-index of graph $(AW)_4^6$ can be calculated by taking the first derivatives of $H((AW)_4^6, k)$ at k=1.

$$W((AW)_4^6) = 28 + 76k|_{k=1} + 132k^2|_{k=1} + 208k^3|_{k=1} + 300k^4|_{k=1} + 288k^5|_{k=1} + 168k^6|_{k=1} + 48k^7|_{k=1} \tag{13}$$

$$W((AW)_4^6) = 1248 \tag{14}$$

Theorem 4.1 Let $G \cong (AW)_p^6$ is the simple connected graph where $p \geq 1$, then the H-polynomial is:-

$$H(G,k) = 7pk + (\frac{p^2+15p}{2})k^2 + (2p^2+3p)k^3 + (4p^2-3p)k^4 + (5p^2-5p)k^5 + (4p^2-4p)k^6 \quad (15)$$

$$+ (2p^2-2p)k^7 + (\frac{p^2-p}{2})k^8 \quad (16)$$

Proof Let $G \cong (AW)_p^6$ is the AW graph with $p \geq 1$. Cardinalities of nodes and links of $(AW)_p^6$ are 6p+1 and 7p respectively. The structure of $(AW)_p^6$ for $p \geq 1$ and $q = 6$ can be observed by the help of Figure 1. To prove this theorem, see information given in Table 3 about all distances of the generalized abid-waheed $(AW)_p^6$ graph.

Table 3. *Distance* Φ*(G,z) for* $1 \leq z \leq 8$ *for each pair of vertices* $(AW)_p^6$.

$\Phi(G,z)$	Cardinality	$\Phi(G,z)$	Cardinality
Φ(G,1)	7p	Φ(G,5)	$5p^2$-5p
Φ(G,2)	$\frac{p^2+15p}{2}$	Φ(G,6)	$4p^2-4p$
Φ(G,3)	$2p^2+3p$	Φ(G,7)	$2p^2-2p$
Φ(G,4)	$4p^2-3p$	Φ(G,8)	$\frac{p^2-p}{2}$

Formula for the H-polynomial is;

$$H(G,k) = \sum_{z=1}^{D(G)} \Phi(G,z)k^z \quad (17)$$

$$H(G,k) = \Phi(G,1)k + \Phi(G,2)k^2 + \Phi(G,3)k^3 + \Phi(G,4)k^4 + \Phi(G,5)k^5 + \Phi(G,6)k^6 + \Phi(G,k)k^7 + \Phi(G,8)k^8 \quad (18)$$

By utilizing Table 3, values are:-

$$H(G,k) = 7pk + (\frac{p^2+15p}{2})k^2 + (2p^2+3p)k^3 + (4p^2-3p)k^4 + (5p^2-5p)k^5 + (4p^2-4p)k^6 \quad (19)$$

$$+ (2p^2-2p)k^7 + (\frac{p^2-p}{2})k^8 \quad (20)$$

Theorem 4.2 Let $G \cong (AW)_p^6$ is represents the Abid-Waheed graph for every integer $p \geq 1$, then W-index of $(AW)_p^6$ is:-

$$W((AW)_p^6) = 90p^2 - 48p \quad (21)$$

Proof The first derivative of H-polynomial at k=1 gives the W-index of graph, so the W-index of $(AW)_p^6$ can be calculated as:-

$$W(G) = \frac{\partial H(G,k)}{\partial k}|_{k=1} \quad (22)$$

$$= 7p + (\frac{p^2+15p}{2})2k|_{k=1} + (2p^2+3p)(3k^2)|_{k=1} + (4p^2-3p)(4k^3)|_{k=1} \quad (23)$$

$$+ (5p^2-5p)(5k^4)|_{k=1} + (4p^2-4p)(6k^5)|_{k=1} + (2p^2-2p)(7k^6)|_{k=1} + \frac{(p^2-p)}{2}(8k^7)|_{k=1} \quad (24)$$

After some easy calculations:-

$$W((AW)_p^6) = 90p^2 - 48p \quad (25)$$

5. Conclusion

Graph theory has a wide range of applications in almost every field of science, like chemistry, biology, computer networking, electrical engineering, etc. The graph used in this paper is used in computer networking. In the paper, we obtained the H-polynomial and W-index of $(AW)_p^6$ for all integer $p \geq 1$.

$$H((AW)_p^6, k) = 7pk + \frac{1}{2}(p^2 + 15p)k^2 + (2p^2 + 3p)k^3 + (4p^2 - 3p)k^4 \quad (26)$$

$$+ (5p^2 - 5p)k^5 + (4p^2 - 4p)k^6 + (2p^2 - 2p)k^7 + \frac{1}{2}(p^2 - p)k^8 \quad (27)$$

$$W((AW)_p^6) = 90p^2 - 48p \quad (28)$$

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