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The Exact Computation of the Decompositions of the Recto Table of the Rhind Mathematical Papyrus

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Abstract: The Rhind Mathematical Papyrus contains in the fragment known as the Recto Table the division of 2 by all the odd integer numbers from 3 to 101. This table hides the secret of how it was computed by its ancient Egyptian author because, till now, there is not a set of established known assumptions which allow compute all its values without any exception. In this paper, the algorithm which computes all the Recto Table is going to be established except three cases (denominators n = 35, 91 ad 95) which are calculated using another formula and the final denominator n = 101 which is a rareness.

Keywords: History of Mathematics, Rhind Papyrus, Kahun Papyrus, Recto Table, Prime Numbers, Algorithm, Unit Fractions

1. Introduction

Rhind Papyrus is one of the most wonderful treasures of the History of Mathematics. The papyrus, very well described in sources as [9, 10, 15] or [20], was purchased by the Scottish antiquary Henry Rhind in the city of Luxor in later 1850's. A little introduction says that its author was Ahmes and it was copied around 1650 BCE from older texts dated two centuries before. Ahmes copied a series of eighty-four mathematical problems written in hieratic script including different questions about Arithmetic, Algebra and Geometry, and a very interesting table known as the Recto Table which deals with the division of 2 by all the odd numbers from 3 to 101 (the remainder of the papyrus is known as the Verso). Ancient Egyptian scribes only used the commonly called "unit fractions" which indicates the reciprocal of a natural number and also the two-thirds fraction. Thus, where we write a fraction of numerator equal to 1 and any natural number in the denominator, Ahmes only wrote a dot over the natural number. Ahmes named "What part is 2 of *n*" [22] or "Call 2 out of *n*" [1] to doubling a unit fraction which was very important in the Egyptian algorithm of multiplication: it is well known that ancient Egyptians' product was based in doubling a number so many times as getting the desired result (for examples, see [15]). Furthermore, it seems that this table could be widely known by the ancient Egyptian scribes (why not?) since one part of it is also recorded in Kahun Papyrus (see [17], 15-16) known as Kahun IV, 2. In this mathematical fragment, another scribe reproduced exactly the same table for all the odd numbers from 3 to 21 and the coincidence between two tables let us think that the divisions were not the product of an unique moment in time but they were widely accepted by scribes in their mathematical purposes.

The Recto Table can be seen in table 1 and it can be noticed that Ahmes did not copy directly the decompositions. For example, for case $n = 41$, Ahmes wrote:

$$41 \quad \frac{1}{24} \quad 1+\frac{2}{3}+\frac{1}{24} \quad \frac{1}{246} \quad \frac{1}{6} \quad \frac{1}{328} \quad \frac{1}{8}$$

corresponding to decomposition $\frac{2}{41}=\frac{1}{24}+\frac{1}{246}+\frac{1}{328}$.

The first number of the row is the denominator of the first fraction of the decomposition. Then, Ahmes computed $\frac{1}{24}$ of 41 and obtained the result $1+\frac{2}{3}+\frac{1}{24}$. Now, he proceeded to compute $\frac{1}{246}$ and $\frac{1}{328}$ of 7 and obtained, indeed, $\frac{1}{6}$ and $\frac{1}{8}$, respectively. Now, he would check that:

$$\left(1+\frac{2}{3}+\frac{1}{24}\right)+\frac{1}{6}+\frac{1}{8}=2$$

However, Ahmes should have noticed some singularity in two special cases: $n = 35$ and 91. According to reference [9] and [15], Ahmes made one unique special check in the first case and he wrote (numbers 6, 7 and 5 written in red color):

35	$\frac{1}{30}$	$1+\frac{1}{6}$	$\frac{1}{42}$	$\frac{2}{3}+\frac{1}{6}$
	7			

With these red auxiliary numbers (for more information about the red auxiliary numbers, see [15], 81-88), Ahmes would have checked that this decomposition was effectively correct: if $\frac{1}{30}$ of 210 is 7 and $\frac{1}{42}$ of 210 is 5, then $\frac{1}{30}+\frac{1}{42}$ of 210 is 12. On the other hand, $\frac{1}{35}$ of 210 is equal to 6 and its double is equal to 12, exactly the same result as 7 + 5.

In case n = 91, Ahmes wrote the word "find" before computing $\frac{1}{70}$ of 91 (see details in [1]), so he also noticed something strange here.

Hence, except these two commented cases, it would be very interesting if the values of the rest of the table had been computed following an unique algorithm of computation. If this idea was real it would have been discovered what can be considered a great achievement in ancient Egyptian Mathematics because there only are a few mathematical papyri which are extant nowadays. From here, it is understood and accepted that the text of Rhind Papyrus is not original from Ahmes but we are going to call his name here as the author for a simple exposure of the facts and the results.

Table 1. *The Recto Table.*

Decomposition	Text on the papyrus								D	d_1	d_2	d_3
$\frac{2}{3}=\frac{1}{2}+\frac{1}{6}$	$\frac{1}{2}$	$1+\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{2}$	[Reconstructed value]				2	2	---	---
$\frac{2}{5}=\frac{1}{3}+\frac{1}{15}$	$\frac{1}{3}$	$1+\frac{2}{3}$	$\frac{1}{15}$	$\frac{1}{3}$					3	3	---	---
$\frac{2}{7}=\frac{1}{4}+\frac{1}{28}$	$\frac{1}{4}$	$1+\frac{1}{2}+\frac{1}{4}$	$\frac{1}{28}$	$\frac{1}{4}$					4	4	---	---
$\frac{2}{9}=\frac{1}{6}+\frac{1}{18}$	$\frac{1}{6}$	$1+\frac{1}{2}$	$\frac{1}{18}$	$\frac{1}{2}$					---	---	---	---
$\frac{2}{11}=\frac{1}{6}+\frac{1}{66}$	$\frac{1}{6}$	$1+\frac{2}{3}+\frac{1}{6}$	$\frac{1}{66}$	$\frac{1}{6}$					6	6	---	---
$\frac{2}{13}=\frac{1}{8}+\frac{1}{52}+\frac{1}{104}$	$\frac{1}{8}$	$1+\frac{1}{2}+\frac{1}{8}$	$\frac{1}{52}$	$\frac{1}{4}$	$\frac{1}{104}$	$\frac{1}{8}$			8	4	8	---
$\frac{2}{15}=\frac{1}{10}+\frac{1}{30}$	$\frac{1}{10}$	$1+\frac{1}{2}$	$\frac{1}{30}$	$\frac{1}{2}$					---	---	---	---
$\frac{2}{17}=\frac{1}{12}+\frac{1}{51}+\frac{1}{68}$	$\frac{1}{12}$	$1+\frac{1}{3}+\frac{1}{12}$	$\frac{1}{51}$	$\frac{1}{3}$	$\frac{1}{68}$	$\frac{1}{4}$			12	3	4	---
$\frac{2}{19}=\frac{1}{12}+\frac{1}{76}+\frac{1}{114}$	$\frac{1}{12}$	$1+\frac{1}{2}+\frac{1}{12}$	$\frac{1}{76}$	$\frac{1}{4}$	$\frac{1}{114}$	$\frac{1}{6}$			12	4	6	---
$\frac{2}{21}=\frac{1}{14}+\frac{1}{42}$	$\frac{1}{14}$	$1+\frac{1}{2}$	$\frac{1}{42}$	$\frac{1}{2}$					---	---	---	---
$\frac{2}{23}=\frac{1}{12}+\frac{1}{276}$	$\frac{1}{12}$	$1+\frac{2}{3}+\frac{1}{4}$	$\frac{1}{276}$	$\frac{1}{12}$					12	12	---	---
$\frac{2}{25}=\frac{1}{15}+\frac{1}{75}$	$\frac{1}{15}$	$1+\frac{2}{3}$	$\frac{1}{75}$	$\frac{1}{3}$					---	---	---	---
$\frac{2}{27}=\frac{1}{18}+\frac{1}{54}$	$\frac{1}{18}$	$1+\frac{1}{2}$	$\frac{1}{54}$	$\frac{1}{2}$					---	---	---	---
$\frac{2}{29}=\frac{1}{24}+\frac{1}{58}+\frac{1}{174}+\frac{1}{232}$	$\frac{1}{24}$	$1+\frac{1}{6}+\frac{1}{24}$	$\frac{1}{58}$	$\frac{1}{2}$	$\frac{1}{174}$	$\frac{1}{6}$	$\frac{1}{232}$	$\frac{1}{8}$	24	2	6	8
$\frac{2}{31}=\frac{1}{20}+\frac{1}{124}+\frac{1}{155}$	$\frac{1}{20}$	$1+\frac{1}{2}+\frac{1}{20}$	$\frac{1}{124}$	$\frac{1}{4}$	$\frac{1}{155}$	$\frac{1}{5}$			20	4	5	---
$\frac{2}{33}=\frac{1}{22}+\frac{1}{66}$	$\frac{1}{22}$	$1+\frac{1}{2}$	$\frac{1}{66}$	$\frac{1}{2}$					---	---	---	---
$\frac{2}{35}=\frac{1}{30}+\frac{1}{42}$	$\frac{1}{30}$	$1+\frac{1}{6}$	$\frac{1}{42}$	$\frac{2}{3}+\frac{1}{6}$					---	---	---	---
$\frac{2}{37}=\frac{1}{24}+\frac{1}{111}+\frac{1}{296}$	$\frac{1}{24}$	$1+\frac{1}{2}+\frac{1}{24}$	$\frac{1}{111}$	$\frac{1}{3}$	$\frac{1}{296}$	$\frac{1}{8}$			24	3	8	---
$\frac{2}{39}=\frac{1}{26}+\frac{1}{78}$	$\frac{1}{26}$	$1+\frac{1}{2}$	$\frac{1}{78}$	$\frac{1}{2}$					---	---	---	---
$\frac{2}{41}=\frac{1}{24}+\frac{1}{246}+\frac{1}{328}$	$\frac{1}{24}$	$1+\frac{2}{3}+\frac{1}{24}$	$\frac{1}{246}$	$\frac{1}{6}$	$\frac{1}{328}$	$\frac{1}{8}$			24	6	8	---
$\frac{2}{43}=\frac{1}{42}+\frac{1}{86}+\frac{1}{129}+\frac{1}{301}$	$\frac{1}{42}$	$1+\frac{1}{42}$	$\frac{1}{86}$	$\frac{1}{2}$	$\frac{1}{129}$	$\frac{1}{3}$	$\frac{1}{301}$	$\frac{1}{7}$	42	2	3	7
$\frac{2}{45}=\frac{1}{30}+\frac{1}{90}$	$\frac{1}{30}$	$1+\frac{1}{2}$	$\frac{1}{90}$	$\frac{1}{2}$					---	---	---	---
$\frac{2}{47}=\frac{1}{30}+\frac{1}{141}+\frac{1}{470}$	$\frac{1}{30}$	$1+\frac{1}{2}+\frac{1}{15}$	$\frac{1}{141}$	$\frac{1}{3}$	$\frac{1}{470}$	$\frac{1}{10}$			30	3	10	---
$\frac{2}{49}=\frac{1}{28}+\frac{1}{196}$	$\frac{1}{28}$	$1+\frac{1}{2}+\frac{1}{4}$	$\frac{1}{196}$	$\frac{1}{4}$					---	---	---	---
$\frac{2}{51}=\frac{1}{34}+\frac{1}{102}$	$\frac{1}{34}$	$1+\frac{1}{2}$	$\frac{1}{102}$	$\frac{1}{2}$					---	---	---	---
$\frac{2}{53}=\frac{1}{30}+\frac{1}{318}+\frac{1}{795}$	$\frac{1}{30}$	$1+\frac{2}{3}+\frac{1}{10}$	$\frac{1}{318}$	$\frac{1}{6}$	$\frac{1}{795}$	$\frac{1}{15}$			30	6	15	---
$\frac{2}{55}=\frac{1}{30}+\frac{1}{330}$	$\frac{1}{30}$	$1+\frac{2}{3}+\frac{1}{6}$	$\frac{1}{330}$	$\frac{1}{6}$					---	---	---	---
$\frac{2}{57}=\frac{1}{38}+\frac{1}{114}$	$\frac{1}{38}$	$1+\frac{1}{2}$	$\frac{1}{114}$	$\frac{1}{2}$					---	---	---	---
$\frac{2}{59}=\frac{1}{36}+\frac{1}{236}+\frac{1}{531}$	$\frac{1}{36}$	$1+\frac{1}{2}+\frac{1}{12}+\frac{1}{18}$	$\frac{1}{236}$	$\frac{1}{4}$	$\frac{1}{531}$	$\frac{1}{9}$			36	4	9	---
$\frac{2}{61}=\frac{1}{40}+\frac{1}{244}+\frac{1}{488}+\frac{1}{610}$	$\frac{1}{40}$	$1+\frac{1}{2}+\frac{1}{40}$	$\frac{1}{244}$	$\frac{1}{4}$	$\frac{1}{488}$	$\frac{1}{8}$	$\frac{1}{610}$	$\frac{1}{10}$	40	4	8	10
$\frac{2}{63}=\frac{1}{42}+\frac{1}{126}$	$\frac{1}{42}$	$1+\frac{1}{2}$	$\frac{1}{126}$	$\frac{1}{2}$					---	---	---	---
$\frac{2}{65}=\frac{1}{39}+\frac{1}{195}$	$\frac{1}{39}$	$1+\frac{2}{3}$	$\frac{1}{195}$	$\frac{1}{3}$					---	---	---	---

Decomposition	Text on the papyrus								D	d_1	d_2	d_3
$\frac{2}{67}=\frac{1}{40}+\frac{1}{335}+\frac{1}{536}$	$\frac{1}{40}$	$1+\frac{1}{2}+\frac{1}{8}+\frac{1}{20}$	$\frac{1}{335}$	$\frac{1}{5}$	$\frac{1}{536}$	$\frac{1}{8}$			40	5	8	---
$\frac{2}{69}=\frac{1}{46}+\frac{1}{138}$	$\frac{1}{46}$	$1+\frac{1}{2}$	$\frac{1}{138}$	$\frac{1}{2}$					---	---	---	---
$\frac{2}{71}=\frac{1}{40}+\frac{1}{568}+\frac{1}{710}$	$\frac{1}{40}$	$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{40}$	$\frac{1}{568}$	$\frac{1}{8}$	$\frac{1}{710}$	$\frac{1}{10}$			40	8	10	---
$\frac{2}{73}=\frac{1}{60}+\frac{1}{219}+\frac{1}{292}+\frac{1}{365}$	$\frac{1}{60}$	$1+\frac{1}{6}+\frac{1}{20}$	$\frac{1}{219}$	$\frac{1}{3}$	$\frac{1}{292}$	$\frac{1}{4}$	$\frac{1}{365}$	$\frac{1}{5}$	60	3	4	5
$\frac{2}{75}=\frac{1}{50}+\frac{1}{150}$	$\frac{1}{50}$	$1+\frac{1}{2}$	$\frac{1}{150}$	$\frac{1}{2}$					---	---	---	---
$\frac{2}{77}=\frac{1}{44}+\frac{1}{308}$	$\frac{1}{44}$	$1+\frac{1}{2}+\frac{1}{4}$	$\frac{1}{308}$	$\frac{1}{4}$					---	---	---	---
$\frac{2}{79}=\frac{1}{60}+\frac{1}{237}+\frac{1}{316}+\frac{1}{790}$	$\frac{1}{60}$	$1+\frac{1}{4}+\frac{1}{15}$	$\frac{1}{70}$	$\frac{1}{3}$	$\frac{1}{316}$	$\frac{1}{4}$	$\frac{1}{790}$	$\frac{1}{10}$	60	3	4	10
$\frac{2}{81}=\frac{1}{54}+\frac{1}{162}$	$\frac{1}{54}$	$1+\frac{1}{2}$	$\frac{1}{162}$	$\frac{1}{2}$					---	---	---	---
$\frac{2}{83}=\frac{1}{60}+\frac{1}{332}+\frac{1}{415}+\frac{1}{498}$	$\frac{1}{60}$	$1+\frac{1}{3}+\frac{1}{20}$	$\frac{1}{332}$	$\frac{1}{4}$	$\frac{1}{415}$	$\frac{1}{5}$	$\frac{1}{498}$	$\frac{1}{6}$	60	4	5	6
$\frac{2}{85}=\frac{1}{51}+\frac{1}{255}$	$\frac{1}{51}$	$1+\frac{2}{3}$	$\frac{1}{255}$	$\frac{1}{3}$					---	---	---	---
$\frac{2}{87}=\frac{1}{58}+\frac{1}{174}$	$\frac{1}{58}$	$1+\frac{1}{2}$	$\frac{1}{174}$	$\frac{1}{2}$					---	---	---	---
$\frac{2}{89}=\frac{1}{60}+\frac{1}{356}+\frac{1}{534}+\frac{1}{890}$	$\frac{1}{60}$	$1+\frac{1}{3}+\frac{1}{10}+\frac{1}{20}$	$\frac{1}{356}$	$\frac{1}{4}$	$\frac{1}{534}$	$\frac{1}{6}$	$\frac{1}{890}$	$\frac{1}{10}$	60	4	6	10
$\frac{2}{91}=\frac{1}{70}+\frac{1}{130}$	$\frac{1}{70}$	$1+\frac{1}{5}+\frac{1}{10}$	$\frac{1}{130}$	$\frac{2}{3}+\frac{1}{30}$					---	---	---	---
$\frac{2}{93}=\frac{1}{62}+\frac{1}{186}$	$\frac{1}{62}$	$1+\frac{1}{2}$	$\frac{1}{186}$	$\frac{1}{2}$					---	---	---	---
$\frac{2}{95}=\frac{1}{60}+\frac{1}{380}+\frac{1}{570}$	$\frac{1}{60}$	$1+\frac{1}{2}+\frac{1}{12}$	$\frac{1}{380}$	$\frac{1}{4}$	$\frac{1}{570}$	$\frac{1}{6}$			---	---	---	---
$\frac{2}{97}=\frac{1}{56}+\frac{1}{679}+\frac{1}{776}$	$\frac{1}{56}$	$1+\frac{1}{2}+\frac{1}{8}+\frac{1}{14}+\frac{1}{28}$	$\frac{1}{679}$	$\frac{1}{7}$	$\frac{1}{776}$	$\frac{1}{8}$			56	7	8	---
$\frac{2}{99}=\frac{1}{66}+\frac{1}{198}$	$\frac{1}{66}$	$1+\frac{1}{2}$	$\frac{1}{198}$	$\frac{1}{2}$					---	---	---	---
$\frac{2}{101}=\frac{1}{101}+\frac{1}{202}+\frac{1}{303}+\frac{1}{606}$	$\frac{1}{101}$	1	$\frac{1}{202}$	$\frac{1}{2}$	$\frac{1}{303}$	$\frac{1}{3}$	$\frac{1}{606}$	$\frac{1}{6}$	---	---	---	---

2. Previous Analysis of the Recto Table

Hence, the Recto Table is very important in the History of ancient Egyptian Mathematics because it contains the results of doubling the reciprocals of all the odd numbers from 3 to 101. However, Ahmes did not double each fraction with the logical addition:

$$\frac{2}{n}=\frac{1}{n}+\frac{1}{n}$$

but decomposed each division $\frac{2}{n}$ into the sum of two, three or four different unit fractions.

In the 19th century, James Joseph Sylvester [21] found a general method to represent any fraction into the sum of a indeterminate number of unit fractions. In that paper, Sylvester recognized that he focused this matter after reading "Cantor's *Geschichte der Mathematik* which gives an account of the singular method in use among the ancient Egyptians for working with fractions". Sylvester's method applied to the fractions of the Recto Table always give the 2-terms decomposition:

$$\frac{2}{n}=\frac{1}{\frac{n+1}{2}}+\frac{1}{\frac{n(n+1)}{2}} \tag{1}$$

Thus, he did not get the desired ancient algorithm but an useful method to decompose any fraction.

Some years later, Loria [19] also gave explanation to all the decompositions of the table but it can not be generalized any universal method from his study and the same thing can be said of August Eisenlohr's paper [12].

In Chace's edition of the Rhind Papyrus [9], he classified all the decompositions into six categories:

A: when the author first takes $\frac{2}{3}$.

B: when he simply halves.

C: when at some step he gets a whole number and uses its reciprocal as a multiplier.

D: along with A (category AD) or B (category BD) when he also uses $\frac{1}{10}$ or $\frac{1}{7}$.

E: for special cases of $n = 35$, 91 and 101.

For cases A, Chace used $n = 17$ as a commented example arguing that Ahmes should have solved a completion problem (using the red auxiliary numbers) which we have three detailed examples in the Rhind Papyrus. After computing $\frac{1}{12}$ of 17 equal to $1+\frac{1}{4}+\frac{1}{6}$, Ahmes would have multiplied 17 by 3, getting 51, and it follows that $\frac{1}{51}$ of 17 is $\frac{1}{3}$ and, in the same way, he would have calculated that $\frac{1}{68}$ of 17 is $\frac{1}{4}$. Chace gave two possible ways of writing down this last step but he never stated how to choose the "correct" decomposition. In all cases, he only checked Ahmes's calculations with the known results.

An interesting study of the Recto Table was carried out by Gillings ([15], 45–80) and concluded that there were five necessary precepts which Ahmes considered in his decompositions seeking the simplest possible combination:

The first precept is that Ahmes preferred the smallest denominators of all the possible equalities and none of them had to be largest than 1.000.

Secondly, Ahmes preferred an equality of only 2 terms to one of 3 terms, and one of 3 terms to one of 4 terms, but an equality of more than a 4-terms decomposition was never used.

The third precept is that the unit fractions always had to be set down in descending order of magnitude, so the smaller numbers came first, but never the same fraction twice.

With the fourth precept, the smallness of the first denominator was the main consideration but Ahmes could accept slightly larger first denominator if it reduced the last one.

Finally, Ahmes preferred even denominators to odd ones.

According to Gillings, these five rules would have directed Ahmes' decisions in case of having some options to choose. Gillings computed all the possible combinations for each division $\frac{2}{n}$ and tried to agree Ahmes' election with them. Therefore, among the great number of possible combinations in 2, 3 or 4-terms decompositions which Ahmes could calculate, he followed the five precepts as the rules of a game. All this reasoning and the number of these possible combinations were analyzed and discussed later by Bruckheimer and Salomon [8], who checked Gillings' results with a computer program. They stated that there were approximately 28.000 possible combinations of unit fractions whose denominators are less than 1.000. Furthermore, they pointed out that Ahmes did not always use irreducible expressions although he usually did. Thus, in some decompositions it is possible to add two unit fractions to get a third unit fraction. For example, Ahmes decomposed:

$$\frac{2}{95}=\frac{1}{60}+\frac{1}{380}+\frac{1}{570}$$

but he could notice that:

$$\frac{1}{380}+\frac{1}{570}=\frac{1}{228}\Rightarrow\frac{2}{95}=\frac{1}{60}+\frac{1}{228}$$

(this example will be seen later).

Bruckheimer and Salomon also said that Ahmes could replace a single odd denominator by two even ones, and argued that the inclusion of these reducible expressions introduces a lot of "noise" in such a comparative study.

Three years later, Van der Waerden [22] divided all the decompositions in five groups. The first one was based in decomposition:

$$\frac{2}{3q}=\frac{1}{q}\left(\frac{1}{2}+\frac{1}{6}\right)=\frac{1}{2q}+\frac{1}{6q} \tag{2}$$

for each q = 1, ..., 33. This result is a direct consequence of a more general rule which states that every odd integer n can always be written in a 2-terms decomposition ([2] and [18]) corresponding to formula (1):

$$\frac{2}{n}=\frac{1}{\frac{n+1}{2}}+\frac{1}{\frac{n(n+1)}{2}}$$

Therefore, if $n = kp$ where p is a prime number and k is a natural number, then doubling the reciprocals of multiples of p = 3, 5, 7 and 11 Ahmes had to use this rule:

$$\frac{2}{kp}=\frac{1}{k}\left(\frac{1}{\frac{p+1}{2}}+\frac{1}{\frac{p(p+1)}{2}}\right) \tag{3}$$

The second group described by Van der Waerden contains all fractions decomposed using the Egyptian algorithm of division; the third derived from the second because all the results are computed from multiplying the decompositions of the second group by an appropriate number; the fourth was decomposed using the known as red auxiliary numbers and, finally; the fifth group only contained the non regular cases with denominators n = 35, 91 and 101.

Another important check of the Recto Table was made by Milo Gardner [13], also based in previous studies, who stated that Ahmes should have computed mentally a formula like:

$$\frac{2}{pq}=\frac{2}{A}\cdot\frac{A}{pq}$$

for an appropriate A. For example, for case $p=3$, $q=7$, then $A=p+1=4$ ad Ahmes obtained:

$$\frac{2}{21}=\frac{2}{4}\cdot\frac{4}{21}=\frac{1}{2}\left(\frac{3}{21}+\frac{1}{21}\right)=\frac{1}{2}\left(\frac{1}{7}+\frac{1}{21}\right)=\frac{1}{14}+\frac{1}{42}$$

However, in this case, as n = 21 is a multiple of 3, it seems more plausible that Ahmes would use a rule like (2) taking profit of previous results.

Gardner also published in a blog [14] the explanation of his new theory which consisted in the multiplication of the original fraction by an appropriate "unity": $\frac{2}{n}=\frac{2}{n}\cdot\frac{A}{A}=\frac{A_0+q_1+q_2+...+q_t}{nA}=\frac{1}{D}+\frac{1}{d_1n}+\frac{1}{d_2n}+...+\frac{1}{d_tn}$ in which, first of all, $2A$ was additively portioned into t+1 integers A_0, q_1, q_2, ..., q_t. Furthermore, all these integer numbers divided nA. Thus, for example:

$$\frac{2}{7}=\frac{2}{7}\cdot\frac{4}{4}=\frac{7+1}{28}=\frac{1}{4}+\frac{1}{28},\ \frac{2}{23}=\frac{2}{23}\cdot\frac{12}{12}=\frac{23+1}{276}=\frac{1}{12}+\frac{1}{276}$$

$$\text{or }\frac{2}{31}=\frac{2}{31}\cdot\frac{20}{20}=\frac{31+5+4}{620}=\frac{1}{20}+\frac{1}{124}+\frac{1}{155}$$

The existence of such number A is developed from Ahmes' resolution of problem RMP 36 of the Rhind Papyrus (see reference [14] and [15]). The problem consists in find a number x (written in modern algebraic notation) which satisfies that:

$$3x+\frac{x}{3}+\frac{x}{5}=1$$

Ahmes thought this equation as:

$$\frac{45x+5x+3x}{15}=1\Rightarrow x=\frac{15}{53}$$

and using the red auxiliary numbers he computed:

$$x=\frac{15}{53}=\frac{15}{53}\cdot\frac{4}{4}=\frac{60}{212}=\frac{53+4+2+1}{212}=\frac{1}{4}+\frac{1}{53}+\frac{1}{106}+\frac{1}{212}$$

Gardner [14] refuses former studies about the election of *A* as Kevin Gong's [16] in which we can see the 3-terms decomposition:

$$\frac{5}{53}=\frac{5}{53}\cdot\frac{12}{12}=\frac{46+12+2}{636}=\frac{1}{6}+\frac{1}{23}+\frac{1}{138}$$

because Gong's argument is not related with Rhind Papyrus in any way. But Gardner's reconstruction of the Recto Table (given in [14]) is based in an election of *A* which depends on Ahmes' mental calculus. Hence, with this argument it is impossible to know why Ahmes wrote:

$$\frac{2}{53}=\frac{2}{53}\cdot\frac{30}{30}=\frac{53+5+2}{1.590}=\frac{1}{30}+\frac{1}{318}+\frac{1}{795}$$

and not:

$$\frac{2}{53}=\frac{2}{53}\cdot\frac{42}{42}=\frac{53+21+7+3}{2.226}=\frac{1}{42}+\frac{1}{106}+\frac{1}{318}+\frac{1}{742}$$

Gardner did not explain why Ahmes chose 2, 3 or 4-term decompositions and the special cases of the table either so Gardner's method provides a very practical property of all the unit fractions of the Recto Table but did not assure an algorithm of election of each value.

After all that, it seems that the historians of Mathematics changed their point of view to reconstruct of the Recto Table. In general, Ahmes computed three kinds of decompositions and in all them, the denominators were related to the initial one in some of the unit fractions. Thus, Ahmes considered 2-terms decompositions:

$$\frac{2}{n}=\frac{1}{D}+\frac{1}{d_1 n} \tag{4}$$

and also 3-terms and 4-terms decompositions, which can be respectively considered as:

$$\frac{2}{n}=\frac{1}{D}+\frac{1}{d_1 n}+\frac{1}{d_2 n} \tag{5}$$

$$\frac{2}{n}=\frac{1}{D}+\frac{1}{d_1 n}+\frac{1}{d_2 n}+\frac{1}{d_3 n} \tag{6}$$

where D, d_1, d_2 and d_3 are appropriate numbers (see Table 1, with D, d_1, d_2 and d_3 only given for prime denominators n). With these new assumptions, the general research has focused the study in the determination of the exact value of D which produces Ahmes' election. In this way. Abdulaziz [1] started his "modern methods for reconstructing the 2:*n*-table" dividing the entries of the table into two groups. The group G_1 consisted of the 29 entries that are expressed as the sum of two unit fractions along with denominator n = 95, because it is a reducible decomposition as it has been already noticed. The group G_2 consistd of the remaining 20 entries, which are expressed as a 3-term or 4-term decompositions. Using only techniques explicitly mentioned in the Rhind Papyrus, Abdulaziz's method did not distinguish between prime and composite numbers and the method of decomposing the elements of G_2 tried to be a natural extension of the method used to decompose the elements of G_1. First of all, Abdulaziz showed how the majority of the elements of G_1 were decomposed using rule (4), where D is the largest number for which

$$Q=\frac{n-D}{D}=\frac{1}{2},\ \frac{2}{3},\ \frac{4}{5},\ \frac{5}{6}\ \text{or}\ \frac{11}{12} \tag{7}$$

If decomposition (4) is considered (with $d_1 = D$), then M, R and Q are defined (in our own notation):

$$M=\frac{n}{D},\ R=2-M=2-\frac{n}{D}=\frac{2D-n}{D},$$

$$M=1+Q \Rightarrow Q=M-1=\frac{n}{D}-1=\frac{n-D}{D} \tag{8}$$

If $n = 3q$ is a multiple of 3, then $Q = R = \frac{1}{2}$ and $D = 2q$, therefore decomposition (2) is obtained. In the same way, the case in which $Q=\frac{2}{3}$ and $R=\frac{1}{3}$ corresponds to the denominators $n = 5q$ multiples of 5 and then $D = 3q$, although cases n = 35, 55 and 95 were not calculated in this manner. But Abdulaziz explains that this procedure is a special case of a more general method which consists in starting from $n-1$ downward, to consider D as the first number such that M is equivalent to

$$M=1+\frac{k-1}{k}\text{, for a certain } k \tag{9}$$

Using (9), Abdulaziz calculated case n = 15 as an example, and determined that only $D = 10$ ($k = 2$) and $D = 9$ ($k = 3$) yield two possible decompositions:

$$\frac{2}{15}=\frac{1}{10}+\frac{1}{2?15}=\frac{1}{10}+\frac{1}{30}\ \text{and}\ \frac{2}{15}=\frac{1}{9}+\frac{1}{3\cdot 15}=\frac{1}{9}+\frac{1}{45}$$

As it can be noticed, the second option has odd denominators (following the fifth precept) so, according to Abdulaziz, Ahmes chose the first one which could be calculated using (2). Abdulaziz also gave explanation to special cases n = 35, 55, 91 and 95.

For group G_2, Abdulaziz said that Ahmes kept the condition that Q is reducible but allowed R to be the sum of two or three unit fractions. If still no decomposition was found, then Ahmes allowed Q to be the sum of three or four unit fractions. Then, for each denominator n, Abdulaziz listed the possible 3-terms decompositions and then, if necessary, the 4-terms decompositions with denominators less than 1.000. He stated that if the explained rules were applied, Ahmes' decompositions ere obtained in almost every case but he always needed Gillings' precepts to choose. For example, in case n = 13 the fifth precept is necessary to choose Ahmes' decomposition of the two possible ones. For n = 17, Abdulaziz obtained three possible 3-terms decompositions but one was immediately rejected because $d_2 = 12$ and Ahmes did not like $d_j > 10$ for j = 1, 2 or 3 (we will talk about this upper bound below). Since the other two decompositions:

$$\frac{2}{17} = \frac{1}{12} + \frac{1}{51} + \frac{1}{68} \text{ and } \frac{2}{17} = \frac{1}{10} + \frac{1}{85} + \frac{1}{170}$$

have an odd fraction each, Ahmes preferred the one with the larger first denominator or the smaller last one. Thus, we are still under the supposed Ahmes' preferences and Abdulaziz could reduce the number of possible combinations to only 1.225 (although if rule (2) is acceptable for denominators multiple of 3 then there only are 143 acceptable decompositions for the 32 nonmultiples of 3). Abdulaziz concluded that Ahmes did not need to consider every value of D since he would realize that a number with too few divisors, especially a prime number, does not make a good choice for D. In fact, D is always multiple of either 10 or 12 for every element of G_2 except n = 13, 43, 97 and 101. In this sense, Dorsett [11] concluded in another paper that D had to be multiple of 4 or 6 starting with 12 for $n \geq 7$. However, Dorsett arrived to this conclusion looking for an appropriate number u with which the last fraction of:

$$\frac{2}{n} = \frac{n+u}{D \cdot n} = \frac{1}{n} + \frac{u}{D \cdot n}$$

could be written as $u = q_1 + q_2$ or $u = q_1 + q_2 + q_3$, with q_1, q_2 and q_3 factors of D. Then, for example, Ahmes should had chosen D = 12 and u = 1 in case n = 23:

$$\frac{2}{23} = \frac{23+1}{12 \cdot 23} = \frac{1}{12} + \frac{1}{12 \cdot 23} = \frac{1}{12} + \frac{1}{276}$$

But, why he did not choose D = 16 or 18, and the respective u = 16 or 18, in the same case?

$$\frac{2}{23} = \frac{23+9}{16 \cdot 23} = \frac{1}{16} + \frac{9}{16 \cdot 23} = \frac{1}{16} + \frac{8+1}{16 \cdot 23} =$$

$$\frac{1}{16} + \frac{8}{16 \cdot 23} + \frac{1}{16 \cdot 23} = \frac{1}{16} + \frac{1}{2 \cdot 23} + \frac{1}{16 \cdot 23}$$

$$\frac{2}{23} = \frac{23+13}{18 \cdot 23} = \frac{1}{18} + \frac{13}{18 \cdot 23} = \frac{1}{18} + \frac{9+3+1}{18 \cdot 23} =$$

$$\frac{1}{18} + \frac{9}{18 \cdot 23} + \frac{3}{18 \cdot 23} + \frac{1}{18 \cdot 23} = \frac{1}{18} + \frac{1}{2 \cdot 23} + \frac{1}{6 \cdot 23} + \frac{1}{18 \cdot 23}$$

Dorsett got all the decompositions of the recto Table but he did not explain how to discard the rest of the possible decompositions in each case.

Recently, a new analysis of the Recto Table has been carried out by Bréhamet [5] based upon his former papers ([3] and [4]). Like Dorsett, Bréhamet focused his attention in a good election of D. The proposed method starts with three "operations":

First operation: discovery of a unique 2-terms solutions, if n is a prime number.

Second operation: for a sub-project from 9 to 99, realize that a mini-table (the "Mother-table"), with just four numbers, enables to derive all the composite denominators n by using a multiplicative operation (already suggested by [15]). This mini-table is:

$$\frac{2}{3} = \frac{1}{2} + \frac{1}{2 \cdot 3} = \frac{1}{2} + \frac{1}{6},$$

$$\frac{2}{5} = \frac{1}{3} + \frac{1}{3 \cdot 5} = \frac{1}{3} + \frac{1}{15},$$

$$\frac{2}{7} = \frac{1}{4} + \frac{1}{4 \cdot 7} = \frac{1}{4} + \frac{1}{28},$$

$$\frac{2}{11} = \frac{1}{6} + \frac{1}{6 \cdot 11} = \frac{1}{6} + \frac{1}{66}$$

Notice that this decompositions can be computed using formula (1). Thus, if n is composite, then formula (3) should be generally used. There also is the special case n = 23 which we will see below.

Third operation: the rest denominators n from 13 to 97 decompose into 3 (or 4 terms if necessary).

Bréhamet's hypothesis has its initial point in finding an appropriate D in formula (5) and (6) and it seems to be "rather simple" if a table of odd numbers $2p+1$, for $p \geq 1$ is established, as a sum of two numbers $q_1 + q_2$, with $2p \geq q_1 > q_2$. The first possible candidate for D starts at an initial value $D^0 = (n+1)/2$ and we can search for general solutions of the form:

$$D^n = D^0 + p \tag{10}$$

whence

$$2D^n - n = 2p + 1 = q_1 + q_2 \tag{11}$$

Thus:

$$2D^n - n = q_1 + q_2 \Rightarrow \frac{2}{n} = \frac{1}{D^n} + \frac{q_1}{D^n \cdot n} + \frac{q_2}{D^n \cdot n} = \frac{1}{D^n} + \frac{1}{d_1 \cdot n} + \frac{1}{d_2 \cdot n} \tag{12}$$

where $d_1 = \dfrac{D^n}{q_1}$ and $d_2 = \dfrac{D^n}{q_2}$, and one of them must be odd and the other must be even (already stated by Bruins [6]). Hence, D must also be even. Bréhamet suggested that from this table of doublets q_1 and q_2, a new table of trials was built where this time doublets were selected if q_1, q_2 divide

$$\left[\frac{n+q_1+q_2}{2}\right] \tag{13}$$

and this provided a possible D^n and the corresponding d_1 and d_2. Now, for each given *p*, the table of trials defined by (11) where q_1 and q_2 divides D^n is bounded by a $p_{\max}$ since it can be proved that no solution can be found beyond $p=(n-3)/2$. For all that, the decompositions into 3-terms fractions lead to a total of trials with 71 possibilities and it seems that Ahmes' election would have been related to choose trials with a low difference $\Delta_q = q_1 - q_2$ and with the imposition of a Top-flag[1] $T_f = 10$ with which $d_1, d_2 \le T_f$ (then, there only are 16 possible trials). It is true that this method explains all the possible 3-terms decompositions but leaves the election of the chosen trials to some unknown hypotheses. For example, in case *n* = 13, there are two possible combinations:

1. If $\Delta_q = 1$, then $2p+1=3$, $p=1$, $q_1=2$, $q_2=1$, $D^n=8$, and $\frac{2}{13}=\frac{1}{8}+\frac{1}{4\cdot 13}+\frac{1}{8\cdot 13}$.
2. If $\Delta_q = 3$, then $2p+1=7$, $p=3$, $q_1=5$, $q_2=2$, $D^n=10$, and $\frac{2}{13}=\frac{1}{10}+\frac{1}{2\cdot 13}+\frac{1}{5\cdot 13}$.

In this case, Ahmes chose the first with lower Δ_q although d_1 and d_2 are respectively greater (4 > 2 and 8 > 5). However, it no seems to be a established criteria for the election of case *n* = 31, in which we have that if $\Delta_q = 1$, then there are two possible combinations with $T_f = 10$:

1. $2p+1=5$, $p=2$, $q_1=3$, $q_2=2$, $D^n=18$, and $\frac{2}{31}=\frac{1}{18}+\frac{1}{6\cdot 31}+\frac{1}{9\cdot 31}$.
2. $2p+1=9$, $p=4$, $q_1=5$, $q_2=4$, $D^n=20$, and $\frac{2}{31}=\frac{1}{20}+\frac{1}{4\cdot 31}+\frac{1}{5\cdot 31}$.

Ahmes chose the second case. If now the condition was that with the same lower Δ_q, d_1 and d_2 had to be respectively smaller (4 < 6 and 5 < 9), what had happened with case *n* = 71? The only two possible combinations with $\Delta_q = 1$ (and $T_f = 10$) are:

1. $2p+1=9$, $p=4$, $q_1=5$, $q_2=4$, $D^n=40$, and $\frac{2}{71}=\frac{1}{40}+\frac{1}{8\cdot 71}+\frac{1}{10\cdot 71}$
2. $2p+1=13$, $p=6$, $q_1=7$, $q_2=6$, $D^n=42$, and $\frac{2}{71}=\frac{1}{42}+\frac{1}{6\cdot 71}+\frac{1}{7\cdot 71}$.

The election here was the first option although d_1 and d_2 are respectively greater (8 > 6 and 10 > 7) and the Top-flag is reached. Bréhamet said that in case *n* = 71, "there is no convincing arithmetical argumentation, then the choice could have been the simplicity and direct observation".

Another interesting observation is case *n* = 23 again. Bréhamet got the unique solution $\Delta_q = 7$, $2p+1=9$, $p=4$, $q_1=8$, $q_2=1$, $D^n=16$, and:

$$\frac{2}{23}=\frac{1}{16}+\frac{1}{2\cdot 23}+\frac{1}{16\cdot 23}$$

It seems plausible that Ahmes rejected it because $d_2 = 16 > 10 = T_f$. But then, why accepted the case *n* = 53? In the table of 71 possibilities, he could choose among four possible options, namely:

1. $\frac{2}{53}=\frac{1}{28}+\frac{1}{14\cdot 53}+\frac{1}{28\cdot 53}$, with $\Delta_q = 1$.
2. $\frac{2}{53}=\frac{1}{30}+\frac{1}{6\cdot 53}+\frac{1}{15\cdot 53}$, with $\Delta_q = 3$.
3. $\frac{2}{53}=\frac{1}{30}+\frac{1}{5\cdot 53}+\frac{1}{30\cdot 53}$, with $\Delta_q = 5$.
4. $\frac{2}{53}=\frac{1}{36}+\frac{1}{2\cdot 53}+\frac{1}{36\cdot 53}$, with $\Delta_q = 17$.

Since the second has the lowest possible $d_2 = 15 > 10 = T_f$, perhaps Ahmes decided to choose this although the Top-flag was not respected. But now, we can ask us again why case *n* = 23 was not elected then. And what happened with case *n* = 43? Ahmes obtained the unique case (among the 71 possibilities):

$$\frac{2}{43}=\frac{1}{30}+\frac{1}{2\cdot 43}+\frac{1}{15\cdot 43}$$

but Bréhamet suggested that Ahmes rejected because $\Delta_q = 13$ is too large, and then he began to look for a 4-term decomposition.

In general, it seems plausible that if Ahmes did not obtain a 3-terms decomposition or the difference between d_2 and T_f was too large (or he did not like the 3-term possible decomposition), then he looked for a 4-term decomposition. Again, Bréhamet obtained a table of 71 possible 4-term combinations in which, for example, we find decomposition:

$$\frac{2}{23}=\frac{1}{20}+\frac{1}{2\cdot 23}+\frac{1}{4\cdot 23}+\frac{1}{10\cdot 23}$$

Why Ahmes did not choose it if the Top-flag is respected? Bréhamet did not have any argument to answer this question.

In this case, Bréhamet looked for three numbers q_1, q_2

1 This name for the upper bound is given by Bréhamet (2017).

and q_3 with which:

$$2D^n - n = 2p + 1 = q_1 + q_2 + q_3 \tag{14}$$

in a similar argument than (11). However, there are still cases not really stated in the paper. For example, case n = 29. Among the 71 possible 3-terms decomposition, there are three possible combinations:

$$\frac{2}{29} = \frac{1}{16} + \frac{1}{8\cdot 29} + \frac{1}{16\cdot 29},\ \frac{2}{29} = \frac{1}{18} + \frac{1}{3\cdot 29} + \frac{1}{18\cdot 29},\text{ and}$$
$$\frac{2}{29} = \frac{1}{20} + \frac{1}{2\cdot 29} + \frac{1}{20\cdot 29}$$

with respectively $\Delta_q = 1$, $\Delta_q = 5$ and $\Delta_q = 9$. In the three options, the Top-flag is not respected and Ahmes had to look for a 4-term decomposition and he found three more options:

$$\frac{2}{29} = \frac{1}{24} + \frac{1}{2\cdot 29} + \frac{1}{6\cdot 29} + \frac{1}{8\cdot 29},$$
$$\frac{2}{29} = \frac{1}{20} + \frac{1}{4\cdot 29} + \frac{1}{5\cdot 29} + \frac{1}{10\cdot 29}\text{ and}$$
$$\frac{2}{29} = \frac{1}{30} + \frac{1}{2\cdot 29} + \frac{1}{3\cdot 29} + \frac{1}{5\cdot 29}$$

and he chose the first one. This case corresponds to $q_1 = 12$, $q_2 = 4$ and $q_3 = 3$, he second to $q_1 = 5$, $q_2 = 4$ and $q_3 = 2$, meanwhile the third to $q_1 = 15$, $q_2 = 10$ and $q_3 = 6$. Bréhamet argued that the reason why Ahmes chose this first option is because it implies the lowest difference between q_2 and q_3 and this can be considered a new precept or previous condition.

Finally, apart from cases n = 23 and n = 29, Bréhamet also points out case n = 53 for which Ahmes chose the commented 3-terms decomposition. Gillings noticed that there was a possible 4-term decomposition, namely:

$$\frac{2}{53} = \frac{1}{42} + \frac{1}{2\cdot 53} + \frac{1}{6\cdot 53} + \frac{1}{14\cdot 53}$$

Why Ahmes did not choose it if the last $d_3 = 14$ in it is lower than the chosen $d_2 = 15$? Bréhamet did not give a satisfactory answer either.

Thus, we can see that all these former studies on the Recto Table depend on some choices made by Ahmes/the real author which are not clearly explained. Since the five Gillings' precepts to Bréhamet's great trial, the 3-term and 4-term decompositions continue hiding some clues to can explain the reason why Ahmes preferred one decomposition against another without taking care of his own established rules.

3. The Reason for the Top-Flag

It is obvious that all the values d_1, d_2 and d_3 of the Recto Table are below this Top-flag $T_f = 10$ except cases n = 23 and 53. Thus, it is plausible to think that this upper bound was adopted to facilitate scribe's work. Bréhamet [5] argued in a very correct way that since there are infinite possible decompositions of one given fraction in unit fractions, it should have been the necessity of limiting the highest possible denominator $d_2 n$ in the 3-term decompositions or $d_3 n$ in the 4-term ones. This could be the real reason why Ahmes decided to look for a 4-term decomposition for case n = 29, for example. As it has been seen, the three possible options obtained with Bréhamet's method do not respect the Top-flag and neither do the five more 3-terms decompositions obtained by Gillings [15] which are not computed with the same algorithm. Finally, Ahmes chose the commented 4-terms decomposition.

Then, what happened with case n = 23 and 53? As we have seen, case n = 53 is the only 3-terms decomposition in which $d_2 = 15 > 10 = T_f$. Abdulaziz [1] argued that Ahmes should have searched for a 4-term decomposition and he would obtain:

$$\frac{2}{53} = \frac{1}{36} + \frac{1}{4\cdot 53} + \frac{1}{6\cdot 53} + \frac{1}{9\cdot 53}\text{ or}$$
$$\frac{2}{53} = \frac{1}{48} + \frac{1}{2\cdot 53} + \frac{1}{3\cdot 53} + \frac{1}{16\cdot 53}$$

In the first option, Ahmes should have deal with the problem that Q (see (8)) would have been the only case that it would have decomposed in three unit fractions, namely:

$$Q = \frac{53-36}{36} = \frac{17}{36} = \frac{1}{4} + \frac{1}{6} + \frac{1}{18}\ \text{(for example)}$$

and this would be the only case in all the table (all the computed Q were decomposed in only two unit fractions), so Ahmes rejected it. For the second option, since $d_3 = 16$, Ahmes preferred the chosen 3-terms decomposition. This argument was more reasoned than Gillings' calculations [15] which showed that the 3-term decomposition found in Rhind Papyrus was the only available one in front of the twenty-three possible 4-terms decompositions for this case. Furthermore, all these 4-term decompositions contain very high odd numbers except:

$$\frac{2}{53} = \frac{1}{42} + \frac{1}{2\cdot 53} + \frac{1}{6\cdot 53} + \frac{1}{14\cdot 53}$$

Gillings thought that if this last equality did come to Ahmes' attention, he must have thought very deeply before deciding.

For n = 23, Ahmes decomposed this fraction in a 2-terms decomposition meanwhile cases n = 13, 17 and 19 were decomposed in a 3-terms one. Why this denominator is so special? Gillings said that the main reason is that is the only 2-terms decomposition and all the possible 3-terms decompositions have last terms much greater than 276. On the other hand, Abdulaziz placed it in group G_1 and after showing four possible decompositions (two 4-terms decomposition, one of 3-terms and the chosen 2-terms one), said that clearly the last one was "the most attractive". However, there was no mention to the reason why Ahmes could not use the same method to decompose case $n = 13$, 17, 19 or 53, for example.

Now, it comes to my mind one more important question related with special case $n = 71$. We can not assure that Ahmes did not look for a different method for these cases and although there is not any solution which respects this upper bound. In fact, he could choose:

$$\frac{2}{53}=\frac{1}{36}+\frac{1}{4\cdot 53}+\frac{1}{6\cdot 53}+\frac{1}{9\cdot 53}$$

and the reason argued by Abdulaziz for not choosing it seems to be very weak because the implication of the violation of the Top-flag by Ahmes.

4. 1st Step: The Composite Denominators in the Recto Table

As it has been said, all the fractions with composite denominator seems to have been computed from the "Mother-table":

$$\frac{2}{3}=\frac{1}{2}+\frac{1}{2\cdot 3}=\frac{1}{2}+\frac{1}{6},\ \frac{2}{5}=\frac{1}{3}+\frac{1}{3\cdot 5}=\frac{1}{3}+\frac{1}{15},$$
$$\frac{2}{7}=\frac{1}{4}+\frac{1}{4\cdot 7}=\frac{1}{4}+\frac{1}{28} \text{ and } \frac{2}{11}=\frac{1}{6}+\frac{1}{6\cdot 11}=\frac{1}{6}+\frac{1}{66}.$$

I agree that in order to follow the global rule explained in (4), (5) and (6), Ahmes (or, remember, the real authors of the table), scribes had to start his task looking for a 2-terms decomposition of the type:

$$\frac{2}{n}=\frac{1}{D}+\frac{1}{d_1 n} \quad (4)$$

and this decomposition is unique because multiplying (4) by $d_1 n$, there is an unique D which satisfies the equation:

$$2D=n+1\Rightarrow D=\frac{n+1}{2} \quad (15)$$

which of course satisfies (1). Then, $d_1=\frac{n+1}{2}$ too.

Therefore, from these first four 2-terms relations, Ahmes could compute all the decompositions of the fractions whose denominators are multiples of 3, 5, 7 and 11:

$$\frac{2}{3m}=\frac{1}{2m}+\frac{1}{6m},\ m = 1, 2, 3, ..., 33 \quad (16)$$

$$\frac{2}{7m}=\frac{1}{4m}+\frac{1}{28m},\ m = 1, 7 \text{ and } 11 \quad (17)$$

$m = 3$ and 9 follows (16).

$$\frac{2}{11m}=\frac{1}{6m}+\frac{1}{66m},\ m = 1 \text{ and } 5 \quad (18)$$

$m = 3$ and 9 follow (16).
$m = 7$ follows (17).

$$\frac{2}{5m}=\frac{1}{3m}+\frac{1}{15m},\ m = 1, 5, 13 \text{ and } 17 \quad (19)$$

$m = 3$, 9 and 15 follows (16).
$m = 11$ follows (18).

Although there are some former studies which demonstrates that these are indeed the correct 2-term decompositions for all these composite denominators, it seems more probable to think that they were computed multiplying the results of the "Mother-table", as in formula (2). Furthermore, it seems that the values were calculated following the order given here, except $n = 35$ ($m = 5$ in (17) or $m = 7$ in (19)), 91 ($m = 13$ in (17)) and 95 ($m = 19$ in (19)).

As we have seen, case $n = 35$ was probably a problem for Ahmes because he decided to check the given decomposition. Gillings [15] said that the scribe chose the simplest decomposition from the 1.458 possibilities. However, he could compute one of the other three 2-term options:

$$\frac{2}{35}=\frac{1}{18}+\frac{1}{18?35}\ (d_1=18>10=T_f),$$
$$\frac{2}{35}=\frac{1}{20}+\frac{1}{4\cdot 35} \text{ or } \frac{2}{35}=\frac{1}{21}+\frac{1}{3\cdot 35}$$

Apart from the first one, Abdulaziz [1] discarded the third because it has two odd denominator (following Gillings' precepts) and then, Ahmes chose the elected one because $D = 30$ is greater than 20. However, both $n = 35$ and 91 can be seen as the solution of the same proposed initial problem although we can not know why Ahmes did not follow the "Mother-table". If the initial approach had been a rule like:

$$\frac{2}{pq}=\frac{1}{Dp}+\frac{1}{Dq} \quad (20)$$

then the scribe would obtained:

$$D=\frac{p+q}{2} \quad (21)$$

Hence:

1. If $p=5$ and $q=7$, then $D=6$ and $\frac{2}{35}=\frac{1}{6\cdot 5}+\frac{1}{6\cdot 7}=\frac{1}{30}+\frac{1}{42}$

2. If $p = 7$ and $q = 13$, then $D = 10$ and $\frac{2}{91} = \frac{1}{10 \cdot 7} + \frac{1}{10?13} = \frac{1}{70} + \frac{1}{130}$.

Could they be a contamination of another similar table with an alternative to formula (17)? In this case, this variation should be:

$$\frac{2}{7m} = \frac{1}{7\left(\frac{7+m}{2}\right)} + \frac{1}{m\left(\frac{7+m}{2}\right)} \tag{22}$$

which must obviously demand an odd value for *m*. Thus, the possible table would contain the values of Table 2. However, it would not have any convincing argument to defend the use of formula (17) to calculate decomposition for $n = 77$ in spite of the decomposition of this new table. Thus, the only thing that it can be assured is that cases $n = 35$ and 91 are computed using the same method and they are not different cases as suggested by some of the former studies on the Recto Table.

Table 2. *possible table for decompositions of 2/(7m).*

m	$\frac{m+7}{2}$	$7\left(\frac{7+m}{2}\right)$	$m\left(\frac{7+m}{2}\right)$	Decomposition	
1	4	28	4	$\frac{2}{7} = \frac{1}{4} + \frac{1}{28}$	It is the same as calculated with (17).
3	5	35	15	$\frac{2}{21} = \frac{1}{15} + \frac{1}{35}$	$n = 21 = 3 \cdot 7$ was calculated using (16).
5	6	42	30	$\frac{2}{35} = \frac{1}{30} + \frac{1}{42}$	Chosen by the scribe.
7	7	49	49	$\frac{2}{49} = \frac{1}{49} + \frac{1}{49}$	It has no sense in the Recto Table.
9	8	56	72	$\frac{2}{63} = \frac{1}{56} + \frac{1}{72}$	$n = 63 = 9 \cdot 7$ was calculated using (16).
11	9	63	99	$\frac{2}{77} = \frac{1}{63} + \frac{1}{99}$	Ahmes used (17).
13	10	70	130	$\frac{2}{91} = \frac{1}{70} + \frac{1}{130}$	Chosen by the scribe.

Finally, another singular case is $m = 19$ in (19). However, this decomposition was computed from the one with denominator equal to 19:

$$\frac{2}{19} = \frac{1}{12} + \frac{1}{76} + \frac{1}{114} \Rightarrow \frac{2}{95} = \frac{2}{5 \cdot 19} = \frac{1}{5}\left(\frac{1}{12} + \frac{1}{76} + \frac{1}{114}\right) = \frac{1}{60} + \frac{1}{380} + \frac{1}{570}$$

Bruins [6] and Gillings [15] pointed out that Ahmes should have known that:

$$\frac{1}{380} + \frac{1}{570} = \frac{1}{228}$$

and then:

$$\frac{2}{95} = \frac{1}{60} + \frac{1}{380} + \frac{1}{570} = \frac{1}{60} + \frac{1}{228}$$

If this fact was true, then this third singular case must be added to cases $n = 35$ and 91 because:

$$\frac{2}{95} = \frac{1}{60} + \frac{1}{380} + \frac{1}{570} = \frac{1}{60} + \frac{1}{228} = \frac{1}{12}\left(\frac{1}{5} + \frac{1}{19}\right)$$

follows formula (20) for $p = 5$, $q = 19$ and $D = 12$.

So, for the moment, we can conclude that all the fractions with a composite number in the denominator was computed as a 2-term decomposition using formulas (16), (17), (18) and (19), except cases $n = 35$, 91 and 95, whose decomposition was calculated using formula (20).

5. 2nd Step: 3-Terms Decompositions

After the 1st step, Ahmes should have looked for a relation of the type (5):

$$\frac{2}{n} = \frac{1}{D} + \frac{1}{d_1 n} + \frac{1}{d_2 n}, \text{ with } d_1 d_2 = kD \tag{5}$$

for a certain *k*, and $d_1 < d_2$, or what is the same:

$$2kD = n + d_1 + d_2, \text{ with } d_1 d_2 = kD \tag{23}$$

Since now, the former studies have focused the attention in establishing some initial hypothesis and getting a good election of the first denominator D. In both cases, there always are cases which not seem to adapt to the a priori rules or precepts and then, there always are special and singular cases which only depend on Ahmes' predilections. Thus, all these commented studies gives convincing explanations if some exceptions are permitted. However, none of them gets their initial aim. In the best possible method [5], Ahmes had to choose among 71 3-terms possible decompositions and 71 more 4-term decompositions which could have been reduced with some preconceived ideas. But the main aim of this kind of study is not giving to Ahmes the right of choice but he got his results directly in some concrete way. After all these previous reasonings, I think that the lack of success of these works is having fixed the attention in finding an appropriate D and not on the other possible denominators which, till now, are always direct consequences of the chosen D. But if we have a look to the last four columns of Table 1 where values of D, d_1, d_2 and d_3 are given for each n prime, it can be noticed that there is not any pair $\{d_1, d_2\}$ in the 3-terms decompositions and any triplet $\{d_1, d_2\ d_3\}$ in the 4-term decomposition which appear twice. Hence, accepting the fact that after computing all the composite denominators n in a 2-term decomposition, Ahmes began to look for a 3-term decomposition for all the prime denominators using some reasoning equivalent to (4), (5) and (6), the only established initial condition is the Top-flag $d_1, d_2 \le T_f = 10$.

With this assumption, how Ahmes could proceed? First of all, we can suppose that d_1 and d_2 are coprime, with $d_1 d_2 = kD$. If not, it would exist d = g.c.d.(d_1, d_2) ≠ 1, so $d_1 = ds_1$, $d_2 = ds_2$ and $kD = d_1 d_2 = ds_1 \cdot ds_2 = d^2 s_1 s_2$. In this case, we would consider that s_1 and s_2 are the new d_1 and d_2, respectively, and $k = d^2$. However, one can consider the case $d_1 = 1$ as a possible election as we will see below.

Secondly, let us see what happens if we consider a decomposition of the kind:

$$\frac{2}{n} = \frac{1}{M}\left(\frac{1}{D} + \frac{1}{d_1 n} + \frac{1}{d_2 n}\right) \Rightarrow M \cdot 2kD = n + d_1 + d_2 \quad (24)$$

$$\text{If } M = 1,\ n + d_1 + d_2 = 2kD \Rightarrow n = 2(d_1 d_2) - (d_1 + d_2) = n_1 \quad (25)$$

$$\text{If } M = 2,\ n + d_1 + d_2 = 4kD \Rightarrow n = 4(d_1 d_2) - (d_1 + d_2) = n_1 + 2d_1 d_2 = n_2 \quad (26)$$

$$\text{If } M = 3,\ n + d_1 + d_2 = 6kD \Rightarrow n = 6(d_1 d_2) - (d_1 + d_2) = n_2 + 2d_1 d_2 = n_3 \quad (27)$$

In general, it can be considered that for all the possible values for *M*, we have that:

$$n_M = n_{M-1} + 2d_1 d_2 \quad (28)$$

so the cases with denominators n = 13 ($M = 4$), 19 ($M = 2$), 41 ($M = 2$), 53 ($M = 3$) and 71 ($M = 2$) can be reduced to easy trials and Ahmes could find them only adding $2d_1 d_2$ to some previous results. Table 3 provide all the possible results considering that d_1 and d_2 are coprime, and formula (28). Furthermore, formula (24) directly implies that d_1 and d_2 must be of different parity. Then, the table is very easy and fast to compute and we have marked out the denominators n chosen by Ahmes (T), the composite denominators ©, the not-chosen denominators (nc) and the denominators which later will be decomposed in four unit fractions (4t). Of course, the denominators $n > 101$ have not been written.

Table 3. *Possible n from d_1 and d_2.*

C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
d_1	d_2	$2d_1d_2$	d_1+d_2	$n=n_1$(25)	$n=n_2$(28)	$n=n_3$(28)	$n=n_4$(28)
1	2	4	3	1	5	9 (C)	13 (T)
1	3	6	4	2	8 (C)	14 (C)	20 (C)
1	4	8	5	3	11	19 (nc)	27 (C)
2	3	12	5	7	19 (T)	31 (nc)	43 (4t)
2	5	20	7	13 (nc)	33 (C)	53 (T)	73 (4t)
2	7	28	9	19 (nc)	47 (nc)	75 (C)	----
2	9	36	11	25 (C)	61 (4t)	97 (nc)	----
3	4	24	7	17 (T)	41 (T)	65 (C)	89 (4t)
3	8	48	11	37 (T)	85 (C)	----	----
3	10	60	13	47 (T)	----	----	----
4	5	40	9	31 (T)	71 (T)	----	----
4	7	56	11	45 (C)	----	----	----
4	9	72	13	59 (T)	----	----	----
5	6	60	11	49 (C)	----	----	----
5	8	80	13	67 (T)	----	----	----
6	7	84	13	71 (nc)	----	----	----
7	8	112	15	97 (T)	----	----	----
7	10	140	17	----	----	----	----
8	9	144	17	----	----	----	----
9	10	180	19	----	----	----	----

From table 3, we can extract some possible conclusions:

1. Except the case $n = n_4 = 13$, which Ahmes could have chosen because it is the result of a very quick calculation, it can be seen that column C_7 has three possible denominators

(19, 31 and 97) which were not chosen and column C_8 contains denominators 43, 73 and 89 which were decomposed in four unit fractions (marked as (4t)). Thus, it can be concluded that Ahmes did not compute C_7 and C_8 as a priority. Probably, he didn't have computed C_7 if $n = 53$ had appeared in C_5 or C_6. So Ahmes computed columns C_5 or C_6 and made his trials and after that, he computed C_7, found $n = 53$ and discarded cases n = 19, 31 and 97 of C_7 because he had chosen another decompositions found in previous columns.

2. Case $n = 19$ can explain a possible order in computing the table because Ahmes chose $n_2 = 19$ and not $n_1 = 19$. It can be because $n_2 = 19$ appeared before (over) $n_1 = 19$, or really because $\Delta_d = d_2 - d_1$ is less in $n_2 = 19$ (as suggested by Bréhamet [5]). For the same reason, Ahmes chose $n_2 = 71$ and not $n_1 = 71$.

3. Case $n_2 = 61$ was not chosen. The reason can be that $d_2 = 9$ and $M = 2$, so the denominator in the decomposition would be equal to 18?61 and $18 > T_f$. As $n = 61$ will be easily decomposed in four unit fractions, Ahmes would have discarded it here. Again, case $n = 53$ also implies $d_2 = 5$ and $M = 3$, so $15 > T_f$ but, as we will see below, $n = 53$ will not be decomposed in four unit fractions with this method. Thus, Ahmes had to choose this decomposition.

Hence, after these easy and fast calculations, Ahmes computed in this order:

$$M = 4,\ d_1 = 1,\ d_2 = 2\ \Rightarrow n = 13 \text{ and } \frac{2}{13} = \frac{1}{4}\left(\frac{1}{1\cdot 2} + \frac{1}{1\cdot 13} + \frac{1}{2\cdot 13}\right) = \frac{1}{8} + \frac{1}{52} + \frac{1}{104}$$

$$M = 2,\ d_1 = 2,\ d_2 = 3\ \Rightarrow n = 19 \text{ and } \frac{2}{19} = \frac{1}{2}\left(\frac{1}{2\cdot 3} + \frac{1}{2\cdot 19} + \frac{1}{3\cdot 19}\right) = \frac{1}{12} + \frac{1}{76} + \frac{1}{114}$$

$$M = 1,\ d_1 = 3,\ d_2 = 4\ \Rightarrow n = 17 \text{ and } \frac{2}{17} = \frac{1}{1}\left(\frac{1}{3\cdot 4} + \frac{1}{3\cdot 17} + \frac{1}{4\cdot 7}\right) = \frac{1}{12} + \frac{1}{51} + \frac{1}{68}$$

$$M = 2,\ d_1 = 3,\ d_2 = 4\ \Rightarrow n = 41 \text{ and } \frac{2}{41} = \frac{1}{2}\left(\frac{1}{3\cdot 4} + \frac{1}{3\cdot 41} + \frac{1}{4\cdot 41}\right) = \frac{1}{24} + \frac{1}{246} + \frac{1}{328}$$

$$M = 1,\ d_1 = 3,\ d_2 = 8\ \Rightarrow n = 37 \text{ and } \frac{2}{37} = \frac{1}{1}\left(\frac{1}{3\cdot 8} + \frac{1}{3\cdot 37} + \frac{1}{8\cdot 37}\right) = \frac{1}{24} + \frac{1}{111} + \frac{1}{296}$$

$$M = 1,\ d_1 = 3,\ d_2 = 10\ \Rightarrow n = 47 \text{ and } \frac{2}{47} = \frac{1}{1}\left(\frac{1}{3\cdot 10} + \frac{1}{3\cdot 47} + \frac{1}{10\cdot 47}\right) = \frac{1}{30} + \frac{1}{141} + \frac{1}{470}$$

$$M = 1,\ d_1 = 4,\ d_2 = 5\ \Rightarrow n = 31 \text{ and } \frac{2}{31} = \frac{1}{1}\left(\frac{1}{4\cdot 5} + \frac{1}{4\cdot 31} + \frac{1}{5\cdot 31}\right) = \frac{1}{20} + \frac{1}{124} + \frac{1}{155}$$

$$M = 2,\ d_1 = 4,\ d_2 = 5\ \Rightarrow n = 71 \text{ and } \frac{2}{71} = \frac{1}{2}\left(\frac{1}{4\cdot 5} + \frac{1}{4\cdot 71} + \frac{1}{5\cdot 71}\right) = \frac{1}{40} + \frac{1}{568} + \frac{1}{710}$$

$$M = 1,\ d_1 = 4,\ d_2 = 9\ \Rightarrow n = 59 \text{ and } \frac{2}{59} = \frac{1}{1}\left(\frac{1}{4\cdot 9} + \frac{1}{4\cdot 59} + \frac{1}{9\cdot 59}\right) = \frac{1}{36} + \frac{1}{236} + \frac{1}{531}$$

$$M = 1,\ d_1 = 5,\ d_2 = 8\ \Rightarrow n = 67 \text{ and } \frac{2}{67} = \frac{1}{1}\left(\frac{1}{5\cdot 8} + \frac{1}{5\cdot 67} + \frac{1}{8\cdot 67}\right) = \frac{1}{40} + \frac{1}{335} + \frac{1}{536}$$

$$M = 1,\ d_1 = 7,\ d_2 = 8\ \Rightarrow n = 97 \text{ and } \frac{2}{97} = \frac{1}{1}\left(\frac{1}{7\cdot 8} + \frac{1}{7\cdot 97} + \frac{1}{8\cdot 97}\right) = \frac{1}{56} + \frac{1}{679} + \frac{1}{776}$$

$$M = 3,\ d_1 = 2,\ d_2 = 5\ \Rightarrow n = 53 \text{ and } \frac{2}{53} = \frac{1}{3}\left(\frac{1}{2\cdot 5} + \frac{1}{2\cdot 53} + \frac{1}{5\cdot 53}\right) = \frac{1}{30} + \frac{1}{318} + \frac{1}{795}$$

Furthermore, these results show that Ahmes didn't start his search looking for D^0 and $D^n = D^0 + p$ (formula (10)) but the value of *D* only depends on the found d_1 and d_2.

Of course, we must point out that all this modern mathematical formulation was not available for Ahmes and he should have computed all these values using a similar or equivalent method easy to implement.

Now, Ahmes only had to deal with cases *n* = 23, 29, 43, 61, 73, 79, 83 and 89, which he had not found without C_8 in Table 3.

6. 3rd Step: 4-Terms Decompositions

Thus, Ahmes looked for a relation of the type:

$$\frac{2}{n} = \frac{1}{D} + \frac{1}{d_1 n} + \frac{1}{d_2 n} + \frac{1}{d_3 n}, \text{ with } d_1 d_2 d_3 = kD \tag{5}$$

for a certain *k*, and $d_1 < d_2 < d_3$, or what is the same,:

$$2kD = n + d_2 d_3 + d_3 d_1 + d_1 d_2 \tag{29}$$

The results obtained by Ahmes are not random and we can state that a similar procedure as the 3-terms decomposition was made. Hence, Ahmes had to deal with an equality as (29) and it is obvious (observing the obtained results) that he began his procedure supposing that among d_1, d_2 and d_3, there are two consecutive numbers. For simplicity, let us suppose that d_1 and d_2 are consecutive and then d_3 does not necessary have to be less than them. Then, (29) is transformed in:

$$2P_{12} d_3 = n + S_{12}\, d_3 + P_{12} \tag{30}$$

where $P_{12} = d_1 d_2 = d_1(d_1+1)$ and $S_{12} = d_1 + d_2 = 2d_1 + 1$. Equivalently:

$$(2d_3 - 1)P_{12} = n + S_{12} d_3 \tag{31}$$

In this way, an expression similar to (25) is obtained, since we have:

$$n = (2d_3 - 1)P_{12} - S_{12} d_3 \tag{32}$$

Furthermore, a general expression could be considered adding the parameter *M*:

$$\frac{2}{n} = \frac{1}{M}\left(\frac{1}{D} + \frac{1}{d_1 n} + \frac{1}{d_2 n} + \frac{1}{d_3 n}\right) \Rightarrow M \cdot 2P_{12} d_3 = n + S_{12} d_3 + P_{12} \tag{33}$$

$$\text{Again, if } M = 1,\ n = n_1 = 2P_{12} d_3 - S_{12} d_3 - P_{12} = (2d_3 - 1)P_{12} - S_{12} d_3 \tag{34}$$

$$\text{If } M = 2,\ n_2 = 4P_{12}d_3 - S_{12}d_3 - P_{12} = 2P_{12}d_3 + 2P_{12}d_3 - S_{12}d_3 - P_{12} = n_1 + 2P_{12}d_3 \tag{35}$$

$$\text{If } M = 3,\ n_3 = 6P_{12}d_3 - S_{12}d_3 - P_{12} = 2P_{12}d_3 + 4P_{12}d_3 - S_{12}d_3 - P_{12} = n_2 + 2P_{12}d_3 \tag{36}$$

In general, we can consider that for all the possible values for *M*, we have that:

$$n_M = n_{M-1} + 2P_{12} d_3 \tag{37}$$

Hence, with these assumptions, Ahmes did not have any difficulty in finding the missing denominators. One first attempt would have been the special case $d_3 = 1$ because it implies that $M \neq 1$ and $2d_3 - 1 = 1$, so according to (34) we have $n_1 = P_{12} - S_{12}$. If we look at Table 4 which the corresponding results obtained with two consecutive d_1 and d_2, and $d_3 = 1$, it can be concluded that $n_2 = 29$ was obtained and the calculus didn't continued beyond $d_1 = 3$ or 4, because $n_2 = 79$ was not chosen here. Hence, case *n* = 29 is like case *n* = 13 in the 3-terms decompositions and it seems to make sense if it is considered as a result of a quick calculation (they are the only cases in which there is a divisor $d_j = 1$). For all that, Ahmes computed:

$$M = 2,\ d_1 = 3,\ d_2 = 4,\ d_3 = 1 \Rightarrow n = 29:$$

$$\frac{2}{29} = \frac{1}{2}\left(\frac{1}{3\cdot 4} + \frac{1}{1\cdot 29} + \frac{1}{3\cdot 29} + \frac{1}{4\cdot 29}\right) = \frac{1}{24} + \frac{1}{58} + \frac{1}{174} + \frac{1}{232}$$

Table 4. *Possible n from $d_3 = 1$ and consecutive d_1 and d_2.*

C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}
d_3	d_1	d_2	P_{12}	S_{12}	$2d_3$-1	$(2d_3\text{-}1)P_{12}$	$S_{12}d_3$	$2P_{12}d_3$	$n=n_1$ (34)	$n=n_2$ (35)
1	2	3	6	5	1	6	5	12	1	13 (nc)
	3	4	12	7		12	7	24	5	29 (T)
	4	5	20	9		20	9	40	11	51 = 3· 17
	5	6	30	11		30	11	60	19 (nc)	79 (nc)
	6	7	42	13		42	13	84	29 (nc)	----
	7	8	56	15		56	15	112	41 (nc)	----
	8	9	72	17		72	17	144	55 = 5· 11	----
	9	10	90	19		90	19	180	71 (nc)	----

Now, Ahmes considered the case $d_1 = 2$, $d_2 = 3$ and all the possible results with $1 < d_3 \le 10$ (Table 5) and obtained denominators n = 43 and 89. So he had 4-terms decompositions:

$$M = 1\,, d_1 = 2\,, d_2 = 3\,, d_3 = 7 \Rightarrow n = 43:$$

$$\frac{2}{43} = \frac{1}{1}\left(\frac{1}{2\cdot 3\cdot 7} + \frac{1}{2\cdot 43} + \frac{1}{3\cdot 43} + \frac{1}{7\cdot 43}\right) = \frac{1}{42} + \frac{1}{86} + \frac{1}{129} + \frac{1}{301}$$

$$M = 2\,, d_1 = 2\,, d_2 = 3\,, d_3 = 5 \Rightarrow n = 89:$$

$$\frac{2}{89} = \frac{1}{2}\left(\frac{1}{2\cdot 3\cdot 5} + \frac{1}{2\cdot 89} + \frac{1}{3\cdot 89} + \frac{1}{5\cdot 89}\right) = \frac{1}{60} + \frac{1}{356} + \frac{1}{534} + \frac{1}{890}$$

Furthermore:

1. Ahmes obtained $n_2 = 146 = 2?73$. Then, he would compute:

$$\frac{2}{146} = \frac{2}{2\cdot 73} = \frac{1}{2}\left(\frac{1}{2\cdot 3\cdot 8} + \frac{1}{2\cdot 146} + \frac{1}{3\cdot 146} + \frac{1}{8\cdot 146}\right)$$

Therefore: $\dfrac{2}{73} = \dfrac{1}{48} + \dfrac{1}{4\cdot 73} + \dfrac{1}{6\cdot 73} + \dfrac{1}{16\cdot 73}$.

This decomposition does not preserve the Top-flag $T_f = 10$ so it must be rejected. In general, if $n_2 = 2n$ for a certain n, we have:

$$\frac{2}{2\cdot n} = \frac{1}{2}\left(\frac{1}{d_1\cdot d_2\cdot d_3} + \frac{1}{d_1\cdot 2n} + \frac{1}{d_2\cdot 2n} + \frac{1}{d_3\cdot 2n}\right) \Rightarrow$$

$$\frac{2}{n} = \frac{1}{d_1\cdot d_2\cdot d_3} + \frac{1}{2d_1\cdot n} + \frac{1}{2d_2\cdot n} + \frac{1}{2d_3\cdot n} \tag{38}$$

Thus, all the considered factors d_1 , d_2 and d_3 are doubled in the decomposition. Then, if $2d_3 > T_f$, the trial must be rejected.

2. Case $n_1 = 29$ is not chosen because it has already been computed in the quick calculation of Table 4.

Table 5. *Possible n from consecutive $d_1 = 2$, $d_2 = 3$ and $d_3 > 1$.*

C1	C2	C3	C4	C5	C6	C7	C8	C9	C10	C11
d_1	d_2	P_{12}	S_{12}	d_3	$2d_3$-1	$(2d_3\text{-}1)P_{12}$	$S_{12}d_3$	$2P_{12}d_3$	$n=n_1$ (34)	$n=n_2$ (35)
2	3	6	5	2	3	18	10	24	$8 = 2^3$	$32 = 2^5$
				3	5	30	15	36	$15 = 3\cdot 5$	$51 = 3\cdot 17$
				4	7	42	20	48	$22 = 2\cdot 11$	$70 = 2\cdot 5\cdot 7$
				5	9	54	25	60	29 (nc)	89 (T)
				6	11	66	30	72	$36 = 2^2\cdot 3^2$	$108 = 2^2\cdot 3^3$
				7	13	78	35	84	43 (T)	127 prime
				8	15	90	40	96	$50 = 2\cdot 5^2$	$146 = 2\cdot 73$
				9	17	102	45	108	$57 = 3\cdot 19$	$165 = 3\cdot 5\cdot 11$

Finally, Ahmes considered consecutive $d_1 = 3$, $d_2 = 4$, and $d_1 = 4$, $d_2 = 5$ (Table 6) and obtained denominators n = 61, 73, 79 and 83. It must be noticed that in cases n = 79 and 83, Ahmes obtained $n_1 = 2?79$ and $2?83$, respectively. This "2" is factor $k = 2$ in (6) and is the greatest common divisor of d_3 and one of the pair $\{d_1, d_2\}$.

Table 6. *Possible n from consecutive $d_1 = 3$, $d_2 = 4$, and $d_1 = 4$, $d_2 = 5$, and $d_3 > 1$.*

C1	C2	C3	C4	C5	C6	C7	C8	C9	C10	C11
d_1	d_2	P_{12}	S_{12}	d_3	$2d_3$-1	$(2d_3\text{-}1)P_{12}$	$S_{12}d_3$	$2P_{12}d_3$	$n=n_1$ (34)	$n=n_2$ (35)
3	4	12	7	5	9	108	35	120	73 (T)	193 prime
				6	11	132	42	144	$90 = 2\cdot 3^2\cdot 5$	$234 = 2\cdot 3^2\cdot 13$
				7	13	156	49	168	107 prime	$275 = 5^2\cdot 11$
				8	15	180	56	192	$124 = 2^2\cdot 31$	$316 = 2^2\cdot 79$
				9	17	204	63	216	$141 = 3\cdot 47$	$357 = 3\cdot 7\cdot 17$
				10	19	228	70	240	$158 = 2\cdot 79$ (T)	$398 = 2\cdot 199$
4	5	20	9	2	3	60	18	80	$42 = 2\cdot 3\cdot 7$	$122 = 2\cdot 61$ (T)
				6	11	220	54	240	$166 = 2\cdot 83$ (T)	$406 = 2\cdot 7\cdot 29$
				7	13	260	63	280	197 prime	$477 = 3^2\cdot 53$
				8	15	300	72	320	$228 = 2^2\cdot 3\cdot 19$	$548 = 2^2\cdot 137$
				9	17	340	81	360	$259 = 7\cdot 37$	619 prime
				10	19	380	90	400	$290 = 2\cdot 5\cdot 29$	$690 = 2\cdot 3\cdot 5\cdot 23$

Therefore, having this little singularity in mind, Ahmes had the next 4-terms decompositions:

$$M = 2\,, d_1 = 4\,, d_2 = 5\,, d_3 = 2 \Rightarrow n = 2?61:$$

$$\frac{2}{61}=\frac{1}{2}\left(\frac{2}{2\cdot 4\cdot 5}+\frac{1}{2\cdot 61}+\frac{1}{4\cdot 61}+\frac{1}{5\cdot 61}\right)=\frac{1}{40}+\frac{1}{244}+\frac{1}{488}+\frac{1}{610}$$

$$M=1\,,d_1=3\,,d_2=4\,,d_3=5\Rightarrow n=73:$$

$$\frac{2}{73}=\frac{1}{1}\left(\frac{1}{3\cdot 4\cdot 5}+\frac{1}{3\cdot 73}+\frac{1}{4\cdot 73}+\frac{1}{5\cdot 73}\right)=\frac{1}{60}+\frac{1}{219}+\frac{1}{292}+\frac{1}{365}$$

$$M=1\,,d_1=3\,,d_2=4\,,d_3=10\Rightarrow n=2?79:$$

$$\frac{2}{79}=\frac{1}{1}\left(\frac{2}{3\cdot 4\cdot 10}+\frac{1}{3\cdot 79}+\frac{1}{4\cdot 79}+\frac{1}{10\cdot 79}\right)=\frac{1}{60}+\frac{1}{237}+\frac{1}{316}+\frac{1}{790}$$

$$M=1\,,d_1=4\,,d_2=5\,,d_3=6\Rightarrow n=2?83:$$

$$\frac{2}{83}=\frac{1}{1}\left(\frac{2}{4\cdot 5\cdot 6}+\frac{1}{4\cdot 83}+\frac{1}{5\cdot 83}+\frac{1}{6\cdot 83}\right)=\frac{1}{60}+\frac{1}{332}+\frac{1}{415}+\frac{1}{498}$$

7. 4th Step: The Case *n* = 23

In all this argumentation, denominator n = 23 has not appeared. We have seen that previous analysis had found other 3-terms decomposition and 4-terms decomposition for this case, as for example:

$$\frac{2}{23}=\frac{1}{16}+\frac{1}{2\cdot 23}+\frac{1}{16\cdot 23}=\frac{1}{16}+\frac{1}{46}+\frac{1}{368},$$

$$\frac{2}{23}=\frac{1}{18}+\frac{1}{2\cdot 23}+\frac{1}{6\cdot 23}+\frac{1}{18\cdot 23}=\frac{1}{18}+\frac{1}{46}+\frac{1}{138}+\frac{1}{414}$$

$$\frac{2}{23}=\frac{1}{20}+\frac{1}{2\cdot 23}+\frac{1}{4\cdot 23}+\frac{1}{10\cdot 23}=\frac{1}{20}+\frac{1}{46}+\frac{1}{92}+\frac{1}{230}$$

the third of which is recorded by Bréhamet and preserves the Top-flag. Gillings [15] said that eighteen 3-term decompositions were possible although all of them have last terms much greater than 276 = 12· 23. However, the main reason to argue that n = 23 must be decomposed in a 2-terms unique decomposition is that 23 is a primer number. If so, why denominators n = 13, 17 and 19 are not decomposed in the same kind of 2-term decomposition, namely (6)? In the procedure presented here, n = 23 has not been a possible result for Ahmes' computation because the conditions for the elections are not met. Hence, it is very plausible to think that this lack was filled with this unique 2-terms possible decomposition:

$$\frac{2}{23}=\frac{1}{12}+\frac{1}{12\cdot 23}=\frac{1}{12}+\frac{1}{276}$$

and cases n = 13, 17 and 19 were not affected by this desperate final calculus. As n = 23 is the unique case not obtained by Ahmes, he was content to use the only easy resource he had left. So he did not have to choose among some possibilities and did not have to discard the *a priori* correct 4-terms decomposition:

$$\frac{2}{23}=\frac{1}{20}+\frac{1}{2\cdot 23}+\frac{1}{4\cdot 23}+\frac{1}{10\cdot 23}=\frac{1}{20}+\frac{1}{46}+\frac{1}{92}+\frac{1}{230}$$

He simply did not find it among his calculations.

8. Case *n* = 101

Case n = 101 must clearly be treated separately because it is the only case which reproduces the same denominator in the decomposition:

$$\frac{2}{101}=\frac{1}{101}+\frac{1}{2\cdot 101}+\frac{1}{3\cdot 101}+\frac{1}{6\cdot 101}=\frac{1}{101}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{6}\right)$$

This is the only possible decomposition for this case according to Gillings [15] but it should be specified that it is the only 4-term decomposition. In Table 3, Ahmes should have easily found n = 101 in column C_6 just below n = 71. Then:

$$M=2,\ d_1=4,\ d_2=7\ \Rightarrow n=101\ \text{ and}$$

$$\frac{2}{101}=\frac{1}{2}\left(\frac{1}{4\cdot 7}+\frac{1}{4\cdot 101}+\frac{1}{7\cdot 101}\right)=\frac{1}{56}+\frac{1}{8\cdot 101}+\frac{1}{14\cdot 101}$$

which did not respect the Top-flag. Perhaps, Ahmes was very lucky and did not find it. Therefore, since he was perfectly aware of

$$1=\frac{1}{2}+\frac{1}{3}+\frac{1}{6}$$

he computed directly this decomposition.

9. Conclusions

As it has been seen, the Recto Table is one of the most controversial document in the History of Mathematics because Ahmes did not tell us how it was computed. If we look at all the values of the divisions of 2 by each prime number n from 3 to 101, we notice that all these values (except perhaps cases n = 35, 91, 95 and 101) are organized as a global set and it seems that there must be an universal algorithm which could explain them. In this sense, Ahmes chose a concrete 2, 3 or 4-term decomposition for each division and this election does not seem to be random but former studies about them let some values to Ahmes' preferences. Probably, the solving of problem RMP 36 guided these studies to a convenient decomposition of a most appropriate numerator into the sum of some factors. Then, there always are denominators which depend on some established precepts *a priori* but on some others when it is necessary. If this was true, we should recognize that the Recto Table is not calculated in one unique way but depends on, for example, Gillings' precepts. However, if there is one thing which can be said certainly is that Ahmes (or the real ancient Egyptian scribes who designed the table) managed the unit fractions and the red auxiliary numbers very easily. Hence, if his purpose was to decompose these division in expressions like:

$$\frac{2}{n}=\frac{1}{D}+\frac{1}{d_1 n}\text{, with } d_1=D \tag{4}$$

$$\frac{2}{n}=\frac{1}{D}+\frac{1}{d_1 n}+\frac{1}{d_2 n}\text{, with } d_1 d_2=kD \tag{5}$$

$$\frac{2}{n}=\frac{1}{D}+\frac{1}{d_1 n}+\frac{1}{d_2 n}+\frac{1}{d_3 n}\text{,with } d_1 d_2 d_3=kD \tag{6}$$

where D, d_1 , d_2 and d_3 are appropriate numbers (except cases n = 35, 91, 95 and 101), and considering the special case n = 23, he should have managed some kind of unique algorithm to obtain all the tabulated values. As we have seen, (4) is determined by:

$$2D=n+1 \Rightarrow D=\frac{n+1}{2} \tag{15}$$

On other hand, (5) and (6) can be transformed, respectively, in:

$$2kD=n+d_1+d_2 \tag{23}$$

and

$$2kD=n+d_2 d_3+d_3 d_1+d_1 d_2 \tag{29}$$

and we have seen that Ahmed should have known these equivalences using the red auxiliary numbers. Therefore, each denominator n can be determined by:

$$n=2\left(d_1 d_2\right)-\left(d_1+d_2\right) \tag{25}$$

and

$$n=(2d_3-1)P_{12}-S_{12}d_3 \tag{32}$$

respectively, where $P_{12}=d_1 d_2=d_1(d_1+1)$ and $S_{12}=d_1+d_2=2d_1+1$. Ahmes could perfectly have considered coprime divisors and, with the appearance of a convenient M (see formulas (24) and (33)) and a Top-flag $T_f=10$, have reconstructed all the tabulated values computing them directly from these factors. For example, considering formula (25), Ahmes would have easily calculate:

$$d_1=3\text{, } d_2=4 \;\Rightarrow n=2\left(d_1 d_2\right)-\left(d_1+d_2\right)=24-7=17$$

Hence:

$$24=17+7=17+4+3 \Rightarrow \frac{24}{17?12}=\frac{17+4+3}{17?12}=\frac{1}{12}+\frac{1}{3?17}+\frac{1}{4?17}$$

and this is the unique 3-terms decomposition which can be constructed with $d_1=3$ and $d_2=4$ (and $M=1$). Thus, as we have been demonstrated, Ahmes was able to calculate the rest of the tabulated decompositions and he could not choose among possibilities but he had to keep the obtained results in the order in which they appeared. Finally, case n = 23 never appeared so he used (4), as he could do with all the other denominators, and filled the lack of this value in the considered algorithm.

Out of the algorithm, cases n = 35, 91 and 95 were calculated using algorithm:

$$\frac{2}{pq}=\frac{1}{Dp}+\frac{1}{Dq} \tag{20}$$

with:

$$D=\frac{p+q}{2} \tag{21}$$

Considering these three singular cases, it could be plausible that the Recto Table had been contaminated for another table or another method of computation for any unknown reason. In fact, Ahmes checked values n = 35 and 91 in his copy like if he was not aware of the tabulated results.

Hence, now it is possible to assure that all the decomposition of the Recto Table (except cases n = 35, 91, 95 and 101) of the Rhind Papyrus and the fragment of the Kahun Papyrus are the results of the same and unique project.

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