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Sufficient Conditions of Optimality for Second Order Differential Inclusions

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Abstract: In this paper we are concerned with the Bolza problem (P_C) for second order differential inclusions (SODIs). The aim is to derive sufficient conditions of optimality for a problem (P_C). The basic concept of obtaining these conditions is the locally adjoint mappings (LAMs). Besides the transversality conditions, approaches to the general problem therefore involve distinctive Euler-Lagrange and Hamiltonian kind of adjoint inclusions. Furthermore, the aim of the considered "linear" problem with SODIs is to show the reader, by example, how the obtained results can be applied in practice.

Keywords: Differential Inclusion, Cauchy, Euler-Lagrange, Adjoint, Multivalued, Second Order, Transversality

1. Introduction

Optimal control problems with partial and first order discrete and differential inclusions and their applications have been extensively developed since the 1985s (see, for example [1], [6-8], [10, 11], [13-15], [17-21]). The optimality problems accompanied with the second order discrete [16] and differential inclusions are rather complicated because of the existing higher order derivatives. As a consequence, a convenient procedure for eliminating the difficulty is to construct an adjoint SODIs and the transversality conditions. Notice that on the whole in literature are investigated only the qualitative problems with SODIs [2-5], [9, 12, 16].

The first viability result for the SODIs in the nonconvex case has been studied by Lupulescu [9] and Cernea [5] in the finite dimensional case. In [16] the existence of Lyapunov functions for SODIs the analyses are carried out by using the methodology of the viability theory. A necessary assumption on the initial states and sufficient conditions for the existence of local and global Lyapunov functions are obtained. An application is also provided.

In the paper [4] the existence of solutions for initial and boundary value problems for second order impulsive functional differential inclusions in Banach spaces are investigated. The paper [5] gives necessary and sufficient conditions ensuring the existence of solutions to the SODIs with state of constraints. Furthermore, the second order interior tangent sets are introduced and studied to obtain such conditions.

The investigated optimization problem is the logical continuation of work done in previous paper of Mahmudov [12], where is studied discrete approximation of the Bolza problem of optimal control theory given by convex and nonconvex SODIs. The main goal in [12] is to derive necessary and sufficient optimality conditions for a Cauchy problem of second order discrete and discrete approximation inclusions. Therefore, on this path the most natural approach to formulation sufficient conditions of optimality for SODIs is the use of the optimality conditions for discrete approximation problems and derivation of Euler-Lagrange and transversality conditions for Bolza problem (P_C) is implemented by passing to the formal limit as the discrete steps tend to zero.

The paper is organized in the following order:

In Section 2, the needed facts and supplementary results from the book of Mahmudov [15] are given; Hamiltonian function H and argmaximum sets of a set-valued mapping

F, the locally adjoint mapping (LAM) are introduced and the Cauchy problem for SODIs are formulated.

In Section 3 employing LAM in Hamiltonian and Euler-Lagrange forms we derive sufficient conditions of optimality for the SODIs. The posed problems and the corresponding optimality conditions are new. The sufficient conditions, including distinctive transversality ones, are proved by incorporating the Euler-Lagrange and Hamiltonian type of inclusions. As it is shown in these problems, generally, the second order adjoint inclusion involves an auxiliary adjoint variable $v^*(\cdot)$, which in the concrete problems is eliminated and the same inclusion involves only the "main" variable, that is $x^*(\cdot)$.

2. Needed Facts and Problem Statement

Necessary notions can be found in [15]. Let R^n be a n-dimensional Euclidean space, $\langle x,q\rangle$ be an inner product of elements $x,q\in R^n$, (x,q) be a pair of x,q. Let $P(R^n)$ be a family of subsets of R^n. Assume that $F:R^{2n}\to P(R^n)$ is a multivalued (set-valued) mapping from R^{2n} into $P(R^n)$. Then $F:R^n\times R^n\to P(R^n)$ is convex if its graph

$gph\, F=\{(x,p,q):q\in F(x,p)\}$ is a convex subset of R^{3n}. The multivalued mapping F is convex closed if its graph is a convex closed set in R^{3n}. F is convex-valued if $F(x,p)$ is a convex set for each $(x,p)\in dom\, F$.

Let us introduce the Hamiltonian function and argmaximum set for a multivalued mapping F

$$H_F(x,p,q^*)=\sup_q\left\{\langle q,q^*\rangle : q\in F(x,p)\right\},\ q^*\in R^n,$$

$$F(x,p;q^*)=\left\{q\in F(x,p):\langle q,q^*\rangle=H_F(x,p,q^*)\right\}$$

respectively.

For a convex mapping F a multifunction defined by

$$F^*\left(q^*;(x,p,q)\right):=\left\{(x^*,p^*):(x^*,p^*,-q^*)\in K^*_{gph\,F}(x,p,q)\right\}$$

is called a LAM to F at a point $(x,p,q)\in gph F$, where $K_{gphF}(x,p,q)$ is the cone of tangent directions.

The LAM to "nonconvex" mapping F is defined as follows

$$\begin{aligned}&F^*(q^*;(x,p,q)):=\left\{(x^*,p^*):H_F(x_1,p_1,q^*)\right.\\&-H_F(x,p,q^*)\le\langle x^*,x_1-x\rangle+\langle p^*,p_1-p\rangle,\\&\left.\forall(x_1,p_1)\in R^{2n}\right\},\ (x,p,q)\in gph F,\ q\in F(x,p;q^*).\end{aligned}$$

Clearly for the convex mapping $H(\cdot,\cdot,q^*)$ is concave and the latter definition of LAM coincide with the previous definition of LAM. Note that, the similar notion is given by

In Section 3 we deal with the optimization of Cauchy problem for SODIs:

$$\text{minimize } J[x(\cdot)]=\int_0^1 g(x(t),t)dt+\varphi(x(1)) \tag{1}$$

$$(\mathrm{P_C})\ x''(t)\in F(x(t),x'(t)),\ \text{a.e. } t\in[0,1], \tag{2}$$

$$x(0)=x_0,\ x'(0)=x_1 \tag{3}$$

Here $F:R^{2n}\to P(R^n)$ is multivalued mapping, g is continuous function $g(\cdot,t):R^n\to R^1$, and x_0,x_1 are fixed vectors. The optimization problem is to find the trajectory $\tilde{x}(t)$ of the problem (1) – (3) satisfying (2) almost everywhere (a.e.) on $[0,1]$ and the initial conditions (3) on [0, 1] that minimizes the Bolza functional $J[x(\cdot)]$. We label this problem as ($\mathrm{P_C}$). Here, a feasible trajectory $x(\cdot)$ is understood to be an absolutely continuous function on a time interval $[0,1]$ together with the first order derivatives for which $x''(\cdot)\in L_1^n([0,1])$. Notice that such class of functions is a Banach space, endowed with the different equivalent norms.

3. Sufficient Conditions of Optimality for SODIs

Let us formulate an adjoint differential inclusion (i) and the transversality condition (ii) for problem ($\mathrm{P_C}$) in the convex case (F and $g(\cdot,t)$, φ are convex) crucial in what follows:

(i) $\left(\dfrac{d^2x^*(t)}{dt^2}+\dfrac{dq^*(t)}{dt},\ q^*(t)\right)\in F^*\left(x^*(t);(\tilde{x}(t),\tilde{x}'(t),\tilde{x}''(t))\right)$
$-\partial_x g(\tilde{x}(t),t)\times\{0\}$, a.e. $t\in[0,1]$,

(ii) $q^*(1)+\dfrac{dx^*(1)}{dt}\in\partial\varphi(\tilde{x}(1))$, $x^*(1)=0$.

Note that formulation of the conditions (i) and (ii) appear due to a limiting process in the conditions of Theorem 4.4 [12].

According to terminology of Mordukhovich [17], Mahmudov [15] we will call the inclusion (i) as the Euler-Lagrange inclusion. We remind that our notation and terminology are generally consistent with those in Mordukhovich [17], Mahmudov [14] for first order differential inclusions.

In what follows we assume that $x^*(t)$, $t\in[0,1]$ is absolutely continuous function together with the first order derivatives and $x^{*\prime\prime}(\cdot)\in L_1^n([0,1])$. Besides $q^*(t)$, $t\in[0,1]$ is absolutely continuous and $q^{*\prime}(\cdot)\in L_1^n([0,1])$.

Moreover we get one more condition ensuring that the LAM F^* is nonempty at a given point:

(iii) $\dfrac{d^2\tilde{x}(t)}{dt^2}\in F\left(\tilde{x}(t),\tilde{x}'(t);x^*(t)\right)$, a.e. $t\in[0,1]$.

It turns out that the following assertion is true.

Theorem 3.1 Suppose that $g:R^n\times[0,1]\to R^1$ is continuous and convex with respect to x, and F is a convex mapping.

Then for the optimality of the arc $\tilde{x}(t)$ in the convex problem (1)-(3) it is sufficient that there exists a pair of absolutely continuous functions $\{x^*(t), q^*(t)\}$, $t \in [0,1]$ satisfying a.e. the Euler-Lagrange inclusion (i), (iii) and transversality condition (ii).

Proof. Obviously by Theorem 2.1 [15] $F^*(q^*,(x,p,q)) = \partial_{(x,p)} H_F(x,p,q^*)$, $q \in F(x,p;q^*)$. Then by using the Moreau-Rockafellar theorem [15, 17] and the conversation that $-\partial_x g(\cdot,t) = \partial_x(-g(\cdot,t))$ from condition (i) we obtain the adjoint differential inclusion of second order

$$\left(\frac{d^2x^*(t)}{dt^2} + \frac{dq^*(t)}{dt},\ q^*(t)\right) \in \partial_{(x,p)}\left[H_F\left(\tilde{x}(t),\tilde{x}'(t),x^*(t)\right) - \bar{g}\left(\tilde{x}(t),\tilde{x}'(t),t\right)\right],\quad \bar{g}\left(\tilde{x}(t),\tilde{x}'(t),t\right) \equiv g\left(\tilde{x}(t),t\right),\ t \in [0,1].$$

On the definition of subdifferential set of the Hamiltonian function H_F we rewrite the last relation in the form:

$$H_F\left(x(t),x'(t),x^*(t)\right) - H_F\left(\tilde{x}(t),\tilde{x}'(t),x^*(t)\right) - g(x(t),t) + g(\tilde{x}(t),t) \le \left\langle \frac{d^2x^*(t)}{dt^2} + \frac{dq^*(t)}{dt}, x(t)-\tilde{x}(t)\right\rangle + \left\langle q^*(t), x'(t)-\tilde{x}'(t)\right\rangle. \quad (4)$$

In turn by using the definition of the Hamiltonian function, (4) can be converted to the relation

$$\left\langle \frac{d^2x(t)}{dt^2}, x^*(t)\right\rangle - \left\langle \frac{d^2\tilde{x}(t)}{dt^2}, x^*(t)\right\rangle - g(x(t),t) + g(\tilde{x}(t),t) \le \left\langle \frac{d^2x^*(t)}{dt^2}, x(t)-\tilde{x}(t)\right\rangle + \frac{d}{dt}\left\langle q^*(t), x(t)-\tilde{x}(t)\right\rangle.$$

For convenience, let us rewrite the latter inequality as follows

$$g(x(t),t) - g(\tilde{x}(t),t) \ge \left\langle \frac{d^2\left(x(t)-\tilde{x}(t)\right)}{dt^2}, x^*(t)\right\rangle - \left\langle \frac{d^2x^*(t)}{dt^2}, x(t)-\tilde{x}(t)\right\rangle - \frac{d}{dt}\left\langle q^*(t), x(t)-\tilde{x}(t)\right\rangle. \quad (5)$$

Integrating (5) over the interval $[0,1]$ and taking into account that $x(\cdot)$, $\tilde{x}(\cdot)$ are feasible $(x(0) = \tilde{x}(0) = x_0)$ we can write

$$\int_0^1 [g(x(t),t) - g(\tilde{x}(t),t)]dt \ge \int_0^1 \left[\left\langle \frac{d^2\left(x(t)-\tilde{x}(t)\right)}{dt^2}, x^*(t)\right\rangle - \left\langle \frac{d^2x^*(t)}{dt^2}, x(t)-\tilde{x}(t)\right\rangle\right]dt + \left\langle q^*(0), x(0)-\tilde{x}(0)\right\rangle - \left\langle q^*(1), x(1)-\tilde{x}(1)\right\rangle = \int_0^1 \left[\left\langle \frac{d^2\left(x(t)-\tilde{x}(t)\right)}{dt^2}, x^*(t)\right\rangle - \left\langle \frac{d^2x^*(t)}{dt^2}, x(t)-\tilde{x}(t)\right\rangle\right]dt - \left\langle q^*(1), x(1)-\tilde{x}(1)\right\rangle \quad (6)$$

Let us transform the expression in the square parentheses on the right hand side of (6) as follows

$$\left\langle \frac{d^2\left(x(t)-\tilde{x}(t)\right)}{dt^2}, x^*(t)\right\rangle - \left\langle \frac{d^2x^*(t)}{dt^2}, x(t)-\tilde{x}(t)\right\rangle = \frac{d}{dt}\left\langle \frac{d\left(x(t)-\tilde{x}(t)\right)}{dt}, x^*(t)\right\rangle - \frac{d}{dt}\left\langle \frac{dx^*(t)}{dt}, x(t)-\tilde{x}(t)\right\rangle.$$

Then we can easily compute the integral on the right hand side of (6)

$$\int_0^1 \left[\left\langle \frac{d^2\left(x(t)-\tilde{x}(t)\right)}{dt^2}, x^*(t)\right\rangle - \left\langle \frac{d^2x^*(t)}{dt^2}, x(t)-\tilde{x}(t)\right\rangle\right]dt = \left\langle \frac{d\left(x(1)-\tilde{x}(1)\right)}{dt}, x^*(1)\right\rangle - \left\langle \frac{d\left(x(0)-\tilde{x}(0)\right)}{dt}, x^*(0)\right\rangle - \left\langle \frac{dx^*(1)}{dt}, x(1)-\tilde{x}(1)\right\rangle + \left\langle \frac{dx^*(0)}{dt}, x(0)-\tilde{x}(0)\right\rangle. \quad (7)$$

Since $x^*(1) = 0$ by condition (ii) of theorem, and $x(t)$, $t \in [0,1]$ is a feasible $(x(0) = \tilde{x}(0) = x_0$, $x'(0) = \tilde{x}'(0) = x_1)$ solution in (7), we have

$$\left\langle \frac{d\left(x(1)-\tilde{x}(1)\right)}{dt}, x^*(1)\right\rangle = \left\langle \frac{d\left(x(0)-\tilde{x}(0)\right)}{dt}, x^*(0)\right\rangle = \left\langle \frac{dx^*(0)}{dt}, x(0)-\tilde{x}(0)\right\rangle = 0$$

and so (7) can be rewritten as follows

$$\int_0^1 \left[\left\langle \frac{d^2\left(x(t)-\tilde{x}(t)\right)}{dt^2}, x^*(t)\right\rangle - \left\langle \frac{d^2x^*(t)}{dt^2}, x(t)-\tilde{x}(t)\right\rangle\right]dt = -\left\langle \frac{dx^*(1)}{dt}, x(1)-\tilde{x}(1)\right\rangle. \quad (8)$$

Thus (6) and (8) implies

$$\int_0^1 [g(x(t),t) - g(\tilde{x}(t),t)]dt \ge -\left\langle \frac{dx^*(1)}{dt}, x(1)-\tilde{x}(1)\right\rangle - \left\langle q^*(1), x(1)-\tilde{x}(1)\right\rangle. \quad (9)$$

Now remember that by the condition (ii) for all feasible arcs $x(t)$, $t \in [0,1]$ we have

$$\varphi\left(x(1)\right) - \varphi\left(\tilde{x}(1)\right) \ge \left\langle q^*(1) + \frac{dx^*(1)}{dt},\ x(1)-\tilde{x}(1)\right\rangle. \quad (10)$$

Finally adding the inequalities (9), (10) we obtain that $J[x(t)] \ge J[\tilde{x}(t)]$, $\forall x(t)$, $t \in [0,1]$ i.e. $\tilde{x}(t)$, $t \in [0,1]$ is optimal.

Below we prove that if in the problem (P_C) a mapping F depends only on x, then the adjoint inclusion involves only one conjugate variable, that is, there are no an auxiliary adjoint variable $q^*(t)$ in the conjugate SODIs. Besides, note that the condition $x^*(1)=0$ is the analog of the condition of classical optimal control theory with free right-hand endpoint constraints.

Corollary 3.1 Suppose that for the convex problem (P_C) a mapping F doesn't depend on derivative $x'(t)$ i.e. $F(x,x')\equiv F(x)$ and that the conditions of Theorem 3.1 are satisfied. Then the Euler-Lagrange SODI and transversality condition of Theorem 3.1 consist of the following

(i) $\dfrac{d^2x^*(t)}{dt^2}\in F^*\left(x^*(t);(\tilde{x}(t),\tilde{x}''(t))\right)-\partial_x g\left(\tilde{x}(t),t\right)$ a.e. $t\in[0,1]$,

(ii) $\dfrac{dx^*(1)}{dt}\in\partial\varphi\left(\tilde{x}(1)\right)$, $x^*(1)=0$,

(iii) $\dfrac{d^2\tilde{x}(t)}{dt^2}\in F\left(\tilde{x}(t);x^*(t)\right)$, a.e. $t\in[0,1]$.

Proof. Indeed in the presented case domain of the multifunction is dom $F\times\mathbb{R}^n$, which means that $q^*(t)\equiv 0$, $t\in[0,1]$ and so from conditions of Theorem 3.1 we have the needed result.

Corollary 3.2 In addition to assumptions of Theorem 3.1 let F be a closed multivalued mapping. Then the conditions (i), (iii) of Theorem 3.1 can be rewritten in term of Hamiltonian function as follows

$$\left(\frac{d^2x^*(t)}{dt^2}+\frac{dq^*(t)}{dt},\ q^*(t)\right)\in\partial_{(x,p)}H_F\left(\tilde{x}(t),\tilde{x}'(t),x^*(t)\right)$$
$$-\partial_x g\left(\tilde{x}(t),t\right)\times\{0\},$$

$$\frac{d^2\tilde{x}(t)}{dt^2}\in\partial_{q^*}H_F\left(\tilde{x}(t),\tilde{x}'(t);x^*(t)\right),\ \text{a.e. } t\in[0,1].$$

Proof. Indeed [15, Theorem 2.1] the LAM at a given point and argmaximum set are the subdifferential on and on q^* of the Hamiltonian function

$$F^*\left(q^*,(x,p,q)\right)=\partial_{(x,p)}H_F(x,p,q^*),$$
$$F(x,p;q^*)=\partial_{q^*}H_F(x,p,q^*)$$

respectively. Note that for the validity of the second formula it is taken into account Lemma 2.1 [14], which follows immediately from Theorem 1.31 [15] Then the assertions of corollary are equivalent with the conditions (i), (iii).

Note that the results of Theorem 3.1 can be generalized to the "nonconvex" case as follows.

Theorem 3.2 Let us consider the nonconvex problem (1)-(3) that is $g:R^n\times[0,1]\to R^1$ and φ are nonconvex with respect to x, and F is a nonconvex mapping. Then for the optimality of the arc $\tilde{x}(t)$, $t\in[0,1]$ in the problem (1)-(3), it is sufficient that there exist a pair of absolutely continuous functions $\{x^*(t),q^*(t)\}$, $t\in[0,1]$ satisfying the conditions:

(a) $\left(\dfrac{d^2x^*(t)}{dt^2}+\dfrac{dq^*(t)}{dt}+x^*(t),\ q^*(t)\right)\in F^*\left(x^*(t);(\tilde{x}(t),\tilde{x}'(t),\tilde{x}''(t))\right)$, a.e. $t\in[0,1]$,

(b) $g(x,t)-g(\tilde{x}(t),t)\geq\langle x^*(t),\ x-\tilde{x}(t)\rangle$, $\forall x\in R^n$,

(c) $\varphi(x)-\varphi(\tilde{x}(1))\geq\left\langle q^*(1)+\dfrac{dx^*(1)}{dt},\ x-\tilde{x}(1)\right\rangle$, $\forall x\in R^n$, $x^*(1)=0$,

(d) $\left\langle\dfrac{d^2\tilde{x}(t)}{dt^2},x^*(t)\right\rangle=H_F\left(\tilde{x}(t),\tilde{x}'(t);x^*(t)\right)$, a.e. $t\in[0,1]$.

Proof. By condition (a) and definition of LAM in the nonconvex case (see Section 2)

$$H_F\left(x(t),x'(t),x^*(t)\right)-H_F\left(\tilde{x}(t),\tilde{x}'(t),x^*(t)\right)$$
$$\leq\left\langle\frac{d^2x^*(t)}{dt^2}+\frac{dq^*(t)}{dt}+x^*(t),x(t)-\tilde{x}(t)\right\rangle+\langle q^*(t),x'(t)-\tilde{x}'(t)\rangle$$

whereas

$$\left\langle\frac{d^2x(t)}{dt^2},x^*(t)\right\rangle-\left\langle\frac{d^2\tilde{x}(t)}{dt^2},x^*(t)\right\rangle$$
$$\leq\left\langle\frac{d^2x^*(t)}{dt^2}+\frac{dq^*(t)}{dt}+x^*(t),x(t)-\tilde{x}(t)\right\rangle+\langle q^*(t),x'(t)-\tilde{x}'(t)\rangle.$$

From the latter inequality and condition (b) is justified (5). Thus, the furthest proof of theorem is similar to the one for Theorem 3.1.

We note that in the convex case, the conditions (b), (c) of Theorem 3.2 are equivalent to the conditions $x^*(t)\in\partial_x g\left(\tilde{x}(t),t\right)$ and (ii), respectively. Then in the convex case, it is easy to see that from the conditions (a), (b) of Theorem 3.2 we have the inequality (4). It means that the conditions of Theorems 3.2 in the convex case coincide with the conditions of Theorem 3.1.

Suppose now we have so-called "linear" convex problem for the SODIs:

minimize $J[x(\cdot)]=\int_0^1 g(x(t),t)dt+\varphi(x(1))$,

$$x''(t)\in F(x(t),x'(t)),\ \text{a.e. } t\in[0,1],$$
$$x(0)=x_0,\ x'(0)=x_1, \tag{11}$$
$$F(x,p)=A_1x+A_2p+BU$$

where $g(\cdot,t)$ and φ are continuously differentiable convex functions, A_i, $i=1,2$ and B are $n\times n$ and $n\times r$ matrices, respectively, U is a convex closed subset of R^r. The problem is to find a controlling parameter $\tilde{u}(t)\in U$ such that the arc

$\tilde{x}(t)$ corresponding to it minimizes $J[x(\cdot)]$

Theorem 3.3 The arc $\tilde{x}(t)$ corresponding to the controlling parameter $\tilde{u}(t)$ minimizes $J(x(\cdot))$ in the problem (11) if there exists an absolutely continuous function $x^*(t)$ satisfying the second order adjoint differential equation, the transversality condition and Pontryagin's maximum principle [19]:

$$\frac{d^2x^*(t)}{dt^2} = -A_2^* \frac{d\,x^*(t)}{dt} + A_1^* x^*(t) - g'(\tilde{x}(t),t) \text{ a.e. } t \in [0,1],$$

$$\frac{dx^*(1)}{dt} = \varphi'(\tilde{x}(1)), \quad x^*(1) = 0,$$

$$\langle B\tilde{u}(t),\, x^*(t)\rangle = \sup_{u\in U}\langle Bu,\, x^*(t)\rangle.$$

Proof. In this problem we are proceeding on the basic of Theorem 3.1. Thus by using Theorem 3.1 we can establish the following result

$$\begin{aligned} &\frac{d^2x^*(t)}{dt^2} + \frac{dq^*(t)}{dt} = A_1^* x^*(t) - g'(\tilde{x}(t),t), \\ &q^*(t) = A_2^* x^*(t). \end{aligned} \tag{12}$$

Differentiating second equation of (12) and substituting into first equation we immediately have the second order adjoint differential equation. The rest of the conditions of theorem is immediate from the conditions (ii), (iii) of Theorem 3.1. Note that here is taken into account that $x^*(1) = 0$ implies $q^*(1) = A_2^* x^*(1) = 0$. The proof is completed.

4. Conclusion

In this paper is presented a new method for solving a Bolza problem with second-order differential inclusions which are often used to describe various processes in science and engineering. This approach plays a much more important role in derivation of second-order adjoint discrete-approximate inclusions. Thus, a sufficient conditions of optimality for such problems are deduced. There has been a significant development in the study of optimization for differential equations and inclusions in recent years [2, 5, 7, 19]. Finally, it is concluded that the proposed method is reliable for solving the various optimization problems with second -order differential inclusions. Theoretical analysis and practical results show that our method is simple and easy to implement and is efficient for computing optimal solution of the second order differential inclusions.

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Biography

Prof. Dr. Elimhan Mahmudov have more than fifty SCI papers devoted to Advances in Pure Mathematics, Convex Analysis, Approximation Theory, Mathematical Methods of Optimal Control Theory. He has devoted majority of his time to textbooks and monographs, entitled “Mathematical Analysis and Its Applications” (Papatya, 2002), “Approximation and Optimization of Discrete and Differential Inclusions” (Elsevier, 2011), “Single Variable Differential and Integral Calculus” (Atlantis Press, Springer, 2013), “Optimization of Ordinary Differential Inclusions and Duality”(Academic Publishing, Germany, 2013). He was invited for lectures at the International Conferences in United Kingdom, France, Switzerland, Germany, Poland, Russia, Austria, Italy, Turkey, etc.

Assoc. Prof. Dr. Gulgun Kayakutlu was in the Management Committee of MIRIAD FW6 Project 2005-2008, on regional innovation. She developed strong international team-work skills while working for the OECD and the International Energy Agency for over 7 years in Paris (1982-1990). Hence she works in the Energy field and wrote a book named “Intelligence in Energy” published by Elsevier. Kayakutlu has already supervised fourteen graduate thesis and published more than twenty research articles in SCI and SSCI reviews on Energy Optimization and Intelligence fields.