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Some Invariants of Cartesian Product of a Path and a Complete Bipartite Graph

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Abstract: A topological index of graph G is a numerical parameter related to G, which characterizes its topology and is preserved under isomorphism of graphs. Properties of the chemical compounds and topological indices are correlated. In this paper we will compute M-polynomial, first and second Zagreb polynomials and forgotten polynomial for the Cartesian Product of a path and a complete bipartite graph for all values of n and m. From the M-polynomial, we will compute many degree-based topological indices such that general Randić index, inverse Randić index, first and second Zagreb index, modified Zagreb index, Symmetric division index, Inverse sum index augmented Zagreb index and harmonic index for the Cartesian Product of a path and a complete bipartite graph. Also, we will compute the hyper- Zagreb index, the first and second multiple Zagreb index and forgotten index for the Cartesian Product of a path and a complete bipartite graph.

Keywords: M-polynomial, Topological Index, Path, Complete Graph, Cartesian Product

1. Introduction

Through this paper we consider simple connected graph, i.e. connected without loops and multiple edges. Let $G(V,E)$ be a graph with vertex and edge are sets of $V(G)$ and $E(G)$, respectively. The degree of vertex u in G is denoted by d_u, which is defined as the number of edges incident to u.

In chemical graph theory, a molecular graph is a simple graph (having no loops and multiple edges) in which atoms and chemical bonds between them are represented by vertices and edges respectively. A graph $G(V,E)$ with vertex set $V(G)$ and edge set $E(G)$ is connected if there exists a connection between any pair of vertices in G.

A topological index is a function that characterizes the topology of the graph. Most commonly known invariants of such kinds are degree-based topological indices. These are actually the numerical values that correlate the structure with various physical properties, chemical reactivity and biological activities.

Numerous graph polynomials were introduced in the literature, several of them turned out to be applicable in mathematical chemistry. There are relationships between topological indices and polynomials, and we can know a lot of indices, when we know polynomials for instance, the Hosoya polynomial [1], see also [2- 4], is the key polynomial in the area of distance-based topological indices. In particular, the Wiener index which is the first topological index introduced by chemist Harold Wiener [5, 6] can be computed as the first derivative of the Hosoya polynomial, evaluated at 1, the hyper-Wiener index [7] and the Tratch-Stankevich-Zefirov index can be obtained similarly [8].

M-polynomial [9], introduced by Deutsch and Klavzar in 2015, plays the same role in determining closed forms of many degree-based topological indices [10, 11]. The main advantage of M-polynomial is the wealth of information that it contains about degree-based graph invariants and by finding M-polynomial we can conclude many degree-based topological indices as we will find in the definitions of the article.

The Cartesian product $G\ H$ of two graphs G and H is the graph with vertex set $V(G\ H) = V(G) \times V(H)$, where two vertices $(v_1, v_2), (u_1, u_2) \in V(G\ H)$ are adjacent if and only if either $v_1u_1 \in E\ (G)$ and $v_2 = u_2$ or $v_2u_2 \in E(H)$ and $v_1 = u_1$.

A path graph P_n is a graph whose vertices can be listed in the order $v_1, v_2, \dots, v_n$ such that the edges are $\{v_i, v_{i+1}\}$, where $i = 1, 2, \dots., n-1$. A path P_n consists of two vertices of degree one and $n-2$ vertices of degree two.

complete bipartite graph $K_{(n,m)}$ is a graph whose vertices can be partitioned into subsets V_1, V_2 such that no edge has both endpoints in the same subset, and every possible edge that could connect vertices in different is part of the graph if $n=1$ then the complete bipartite graph is called star graph.

Figure 1 shows a Cartesian Product of path and complete bipartite graph P_3 $K_{(1,4)}$.

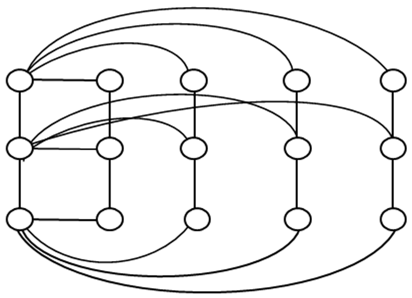

Figure 1. P_3 $K_{(1,4)}$.

2. Basic Definitions and Literature Review

Here we give some basic definitions and literature review.

Definition 2.1. The M-polynomial of G is defined in [9] as:

$$M(G,x,y) = \sum_{\delta \le i \le j \le \Delta} m_{ij}\, x^i y^j.$$

where $\delta = min\{d_v : v \in V(G)\}, \Delta = max\{d_v : v \in V(G)\}$, and $m_{ij}(G)$ is the number of edges $e = vu \in \mathrm{E(G)}$ such that $d_v = i, d_u = j$.

Gutman and Trinajstic [12] introduced first Zagreb index and second Zagreb index,

$$M_1(G) = \sum_{uv\in \mathrm{E(G)}} (d_u + d_v) \text{ and}$$
$$M_2(G) = \sum_{uv\in \mathrm{E(G)}} (d_u \times d_v)$$

respectively. For details about these indices we refer [13-17] to the readers. Both the first Zagreb index and the second Zagreb index give greater weights to the inner vertices and edges, and smaller weights to the outer vertices and edges which oppose intuitive reasoning [18]. For a simple connected graph G, the second modified Zagreb index is defined as:

$${}^{\mathrm{m}}M_2(G) = \sum_{uv\in E(G)} \frac{1}{d_u \times d_v}.$$

Randić index, [19] denoted by $R_{-1/2}(G)$ and introduced by Milan Randić in 1975, is also one of the oldest topological indices. The Randić index is defined as

$$R_{-\frac{1}{2}}(\mathrm{G}) = \sum_{uv\in \mathrm{E(G)}} \frac{1}{\sqrt{d_u d_v}}.$$

In 1998, working independently, Bollobas and Erdos [20] and Amic et al. [21] proposed the generalized Randić index and it has been studied extensively by both chemists and mathematicians [22].

The general Randić index is defined as: $R_\alpha(G) = \sum_{uv\in \mathrm{E(G)}} \frac{1}{(d_u d_v)^\alpha}$ and the inverse Randić index is defined as: $RR_\alpha(G) = \sum_{uv\in \mathrm{E(G)}} (d_u\, d_v)^\alpha$. Obviously, $R_{-1/2}$ (G) is the particular case of R_α(G) when $\alpha = \frac{1}{2}$.

The augmented Zagreb index of G proposed by Furtula et al. [23] is defined as:

$$A(\mathrm{G}) = \sum_{uv\in \mathrm{E(G)}} \left\{\frac{d_u\, d_v}{d_u + d_v - 2}\right\}^3.$$

and it is useful for computing heat of formation of alkanes [23, 24].

The symmetric division index [SDD] is the one among 148 discrete Adriatic indices and is a good predictor of the total surface area for polychlorobiphenyls, see [25]. The symmetric division index of a connected graph G is defined as:

$$SDD(G) = \sum_{uv\in \mathrm{E(G)}} \left\{\frac{\min(d_u, d_v)}{\max(d_u, d_v)} + \frac{\max(d_u, d_v)}{\min(d_u, d_v)}\right\}.$$

The inverse sum index is the descriptor that was selected in [26] as a significant predictor of total surface area of octane isomers and for which the extremal graphs obtained with the help of mathematical chemistry have, particularly, a simple and elegant structure. The inverse sum index is defined as:

$$I(G) = \sum_{uv\in \mathrm{E(G)}} \frac{d_u\, d_v}{d_u + d_v}.$$

Another variant of Randić index is the harmonic index defined as:

$$H(\mathrm{G}) = \sum_{uv\in \mathrm{E(G)}} \frac{2}{d_u + d_v}.$$

As far as we know, this index first appeared in [27]. Favaron et al. [28] considered the relation between the harmonic index and the eigenvalues of graphs.

We give derivations of some well-known degree-based topological indices from M-polynomial [9] in Table 1.

Table 1. *Derivations of some degree-based topological indices from M-polynomial.*

Topological Index	**Derivation from $M(G;\, x, y)$**
First Zagreb	$(D_x + D_y)(M(G;\, x, y))_{x=y=1}$
Second Zagreb	$(D_x D_y)(M(G;\, x, y))_{x=y=1}$
Second Modified Zagreb	$(S_x S_y)(M(G;\, x, y))_{x=y=1}$
Inverse Randić	$(D_x^\alpha D_y^\alpha)$ $(\mathrm{M(G;\, x, y)})_{x=y=1}$
General Randić	$(S_x^\alpha S_y^\alpha)(M(G;\, x, y))_{x=y=1}$
Symmetric Division Index	$(D_x\, S_y + S_x\, D_y)$ $(M(G;\, x, y))_{x=y=1}$
Harmonic Index	$2\, S_x\, J(M(G;\, x, y))_{x=1}$
Inverse sum Index	$S_x\, J D_x D_y(M(G;\, x, y))_{x=1}$
Augmented Zagreb Index	$S_x^3 Q_{-2}\, J D_x^3 D_y^3(M(G;\, x, y))_{x=1}$

Where $D_x = x\frac{\partial f(x,y)}{\partial x}$, $D_y = y\frac{\partial f(x,y)}{\partial y}$, $S_x = \int_0^x \frac{f(t,y)}{t}dt$,

$$S_y = \int_0^y \frac{f(x,t)}{t}dt$$

$$J(f(x,y)) = f(x,x),\ Q_\alpha(f(x,y)) = x^\alpha f(x,y).$$

In 2013, Shirdel et al. in [29] proposed "hyper-Zagreb index" which is also a degree based index.

Definition 2.2. Let G be a simple connected graph. Then the hyper-Zagreb index of G is defined as:

$$HM(\mathrm{G}) = \sum_{uv\in \mathrm{E(G)}} [d_u + d_v]^2.$$

In 2012, Ghorbani and Azimi [30] proposed two new variants of Zagreb indices.

Definition 2.3. Let G be a simple connected graph. Then the first multiple Zagreb index of G is defined as:

$$PM_1(\mathrm{G}) = \prod_{uv\in \mathrm{E(G)}} [d_u + d_v].$$

Definition 2.4. Let G be a simple connected graph. Then the second multiple Zagreb index of G is defined as:

$$PM_2(\mathrm{G}) = \prod_{uv\in \mathrm{E(G)}} [d_u .\ d_v].$$

Definition 2.5. Let G be a simple connected graph. Then the first Zagreb polynomial of G is defined as:

$$M_1(\mathrm{G,x}) = \sum_{uv\in \mathrm{E(G)}} x^{[d_u + d_v]}.$$

Definition 2.6. Let G be a simple connected graph. Then the second Zagreb polynomial of G is defined as:

$$M_2(\mathrm{G,x}) = \sum_{uv\in \mathrm{E(G)}} x^{[d_u \cdot d_v]}.$$

Definition 2.7. Let G be a simple connected graph. Then the Forgotten polynomial of G is defined as:

$$F(\mathrm{G,x}) = \sum_{uv\in \mathrm{E(G)}} x^{[(d_u)^2 + (d_v)^2]}.$$

Definition 2.8. Let G be a simple connected graph. Then the forgotten index of G is defined as:

$$F(\mathrm{G}) = \sum_{uv\in \mathrm{E(G)}} [(d_u)^2 + (d_v)^2].$$

In [31] we previously found some indices and polynomial for the Cartesian product of two paths $P_n\ P_m$ in.

3. Main Results and Discussions

In this paper we will compute the M-polynomial, first and second Zagreb polynomials and forgotten polynomial for the Cartesian Product of a Path and a complete bipartite graph $P_n\ K_{(1,m)}$ for all values of n and m. From the M-polynomial, we will compute many degree-based topological indices such that general Randić index, inverse Randić index, first and second Zagreb index, modified Zagreb index, Symmetric division index, Inverse sum index augmented Zagreb index and harmonic index for $P_n\ K_{(1,m)}$. Also, We will compute the hyper- Zagreb index, the first and second multiple Zagreb index and the forgotten index for $P_n\ K_{(1,m)}$.

Theorem 3.1. The M-polynomial for $P_n\ K_{(1,m)}$ is:

$$M(P_n\ K_{(1,m)}, x, y) = 2m\, x^2y^3 + m(n-3)x^3y^3 + 2m\, x^2y^{m+1}$$
$$+m(n-2)x^3y^{m+2} + 2x^{m+1}y^{m+2} + (n-3)x^{m+2}y^{m+2}.$$

Proof. Clearly, we have $|V(P_n\ K_{(1,m)})| = n(m+1)$.

$$|E(P_n\ K_{(1,m)})| = 2nm + n - m - 1.$$

We can divide the edge set into following four partitions:

$$E_1(P_n\ K_{(1,m)}) = \{e = uv \in E(P_n\ K_{(1,m)}) : d_u = 2, d_v = 3\}.$$

$$E_2(P_n\ K_{(1,m)}) = \{e = uv \in E(P_n\ K_{(1,m)}) : d_u = 3, d_v = 3\}.$$

$$E_3(P_n\ K_{(1,m)}) = \{e = uv \in E(P_n\ K_{(1,m)}) :\ d_u = 2, d_v = m+1\}.$$

$$E_4(P_n\ K_{(1,m)}) = \{e = uv \in E(P_n\ K_{(1,m)}) : d_u = 3, d_v = m+2\}.$$

$$E_5(P_n\ K_{(1,m)}) = \{e = uv \in E(P_n\ K_{(1,m)}) : d_u = m+1, d_v = m+2\}.$$

$$E_6(P_n\ K_{(1,m)}) = \{e = uv \in E(P_n\ K_{(1,m)}) : d_u = m+2, d_v = m+2\}$$

$$|E_1(P_n\ K_{(1,m)})| = 2m,\ |E_2(P_n\ K_{(1,m)})| = m(n-3),$$

$$|E_3(P_n\ K_{(1,m)})| = 2m,\ |E_4(P_n\ K_{(1,m)})| = m(n-2),$$

$$\left|E_5\left(P_n\ K_{(1,m)}\right)\right| = 2,\ \left|E_6\left(P_n\ K_{(1,m)}\right)\right| = n-3.$$

Now, by definition of M-polynomial, we have:

$$M\left(P_n\ K_{(1,m)}, x, y\right) = \sum_{\delta\le i\le j\le\Delta} m_{i,j}\, x^i y^j$$

$$=\textstyle\sum_{2\le 3} m_{2,3}\, x^2y^3 + \sum_{3\le 3} m_{3,3}\, x^3y^3 + \sum_{2\le m+1} m_{2,m+1}\, x^2y^{m+1}$$

$$+\sum_{3\le m+2} m_{3,m+2}\, x^3y^{m+2} + \sum_{m+1\le m+2} m_{m+1,m+2}\, x^{m+1}y^{m+2} + \sum_{m+2\le m+2} m_{m+2,m+2}\, x^{m+2}y^{m+2}$$

$$= \sum_{u,v\in E_1\left(P_n\ K_{(1,m)}\right)} m_{2,3}\, x^2y^3 + \sum_{u,v\in E_2\left(P_n\ K_{(1,m)}\right)} m_{3,3}\, x^3y^3$$

$$+\textstyle\sum_{u,v\in E_3\left(P_n\ K_{(1,m)}\right)} m_{2,m+1}\, x^2y^{m+1} + \sum_{u,v\in E_4\left(P_n\ K_{(1,m)}\right)} m_{3,m+2}\, x^3y^{m+2}$$

$$+\sum_{u,v\in E_5\left(P_n\ K_{(1,m)}\right)} m_{m+1,m+2}\, x^{m+1}y^{m+2} + \sum_{u,v\in E_6\left(P_n\ K_{(1,m)}\right)} m_{m+2,m+2}\, x^{m+2}y^{m+2}$$

$$= \left|E_1\left(P_n\ K_{(1,m)}\right)\right| x^2y^3 + \left|E_2\left(P_n\ K_{(1,m)}\right)\right| x^3y^3 + \left|E_3\left(P_n\ K_{(1,m)}\right)\right| x^2y^{m+1} + \left|E_4\left(P_n\ K_{(1,m)}\right)\right| x^3y^{m+2}$$

$$+\left|E_5\left(P_n\ K_{(1,m)}\right)\right| x^{m+1}y^{m+2} + \left|E_6\left(P_n\ K_{(1,m)}\right)\right| x^{m+2}y^{m+2}$$

$$= 2m\, x^2y^3 + m(n-3)x^3y^3 + 2m\, x^2y^{m+1} + m(n-2)x^3y^{m+2} + 2x^{m+1}y^{m+2} + (n-3)x^{m+2}y^{m+2}.$$

Now, we compute some degree-based topological indices of the Cartesian product of a Path and a complete bipartite graph $P_n\ K_{(1,m)}$ for all values of n from this M-polynomial.

Theorem 3.2.

1) $M_1\left(P_n\ K_{(1,m)}\right) = 13nm - 14m + 4n + m^2n - 6.$
2) $M_2\left(P_n\ K_{(1,m)}\right) = -3m^2 + 19nm + 4nm^2 - 29m + 4n - 8.$
3) ${}^{m}M_2\left(P_n\ K_{(1,m)}\right) = (\frac{1}{9} + \frac{1}{3(m+2)})nm + \frac{m}{m+1} - \frac{2m}{3(m+2)} + \frac{2}{(m+1)(m+2)} + \frac{n-3}{(m+2)^2}.$
4) $RR_\alpha\left(P_n\, K_{(1,m)}\right) = m(2\times 6^\alpha + 9^\alpha(n-2) + 2(2m+2)^\alpha) + (3m+6)^\alpha(n-2)) + 2(m^2+3m+2)^\alpha + (n-3)(m+2)^{2\alpha}.$
5) $R_\alpha\left(P_n\ K_{(1,m)}\right) = m\left(\frac{2}{6^\alpha} + \frac{n-3}{9^\alpha} + \frac{2}{(2m+2)^\alpha} + \frac{n-2}{(3m+6)^\alpha}\right) + \frac{2}{(m^2+3m+2)^\alpha} + \frac{n-3}{(m+2)^{2\alpha}}.$
6) $SDD\left(P_n\ K_{(1,m)}\right) = m\left(\frac{13}{3} + 2(n-3) + \frac{4}{m+1} + m + 1\right) + \frac{3m}{m+2}(n-2) + \frac{m}{3}(m+2)(n-2) + \frac{2(m+1)}{m+2} + \frac{2(m+2)}{m+1} + 2(n-3).$
7) $H\left(P_n\ K_{(1,m)}\right) = m\left(\frac{4}{5} + \frac{n-3}{3} + \frac{4}{m+3} + \frac{2(n-2)}{m+5}\right) + \frac{4}{2m+3} + \frac{2(n-3)}{2m+4}.$
8) $I\left(P_n\ K_{(1,m)}\right) = m\left(\frac{12}{5} + \frac{3(n-3)}{2} + \frac{4(m+1)}{m+3} + \frac{3(n-2)(m+2)}{m+5}\right) + \frac{2(m+1)(m+2)}{2m+3} + \frac{(n-3)(m+2)}{2}.$
9) $A\left(P_n\ K_{(1,m)}\right) = m\left(32 + \frac{729}{64}(n-3) + \frac{27}{(m+3)^3}(m+2)^3(n-2)\right) + \frac{2(m+1)^3(m+2)^3}{(2m+1)^3} + \frac{m-3}{(2m+2)^3}(m+2)^6.$

Proof. Let $M\left(P_n\ K_{(1,m)}, x, y\right) = f(x,y) = 2m\, x^2y^3 + m(n-3)x^3y^3 + 2m\, x^2y^{m+1} + m(n-2)x^3y^{m+2} + 2x^{m+1}y^{m+2} + (n-3)x^{m+2}y^{m+2}.$

Then:

$$D_x f(x,y) = 4m\, x^2y^3 + 3m(n-3)x^3y^3 + 4m\, x^2y^{m+1} + 3m(n-2)x^3y^{m+2}$$
$$+2(m+1)x^{m+1}y^{m+2} + (n-3)(m+2)x^{m+2}y^{m+2}.$$

$$D_y f(x,y) = 6m\, x^2y^3 + 3m(n-3)x^3y^3 + 2m(m+1)\, x^2y^{m+1}$$
$$+ m(m+2)(n-2)x^3y^{m+2} + 2(m+2)x^{m+1}y^{m+2} + (n-3)(m+2)x^{m+2}y^{m+2}.$$

$$D_xD_y f(x,y) = 12m\, x^2y^3 + 9m(n-3)x^3y^3 + 4m(m+1)\, x^2y^{m+1}$$
$$+3m(m+2)(n-2)x^3y^{m+2} + 2(m+1)(m+2)x^{m+1}y^{m+2} + (n-3)\,(m+2)^2x^{m+2}y^{m+2}.$$

$$S_y f(x,y) = \frac{2}{3} m\, x^2 y^3 + \frac{m}{3}(n-3)x^3y^3 + \frac{2}{m+1} m\, x^2 y^{m+1}$$

$$+ \frac{m}{m+2}(n-2)x^3y^{m+2} + \frac{2}{m+2} x^{m+1}y^{m+2} + \frac{n-3}{m+2} x^{m+2}y^{m+2}.$$

$$S_x\, S_y f(x,y) = \frac{1}{3} m\, x^2 y^3 + \frac{m}{9}(n-3)x^3y^3 + \frac{1}{m+1} m\, x^2 y^{m+1}$$

$$+ \frac{m}{3(m+2)}(n-2)x^3y^{m+2} + \frac{2}{(m+1)(m+2)} x^{m+1}y^{m+2} + \frac{n-3}{(m+2)^2} x^{m+2}y^{m+2}.$$

$$S_y^\alpha f(x,y) = \frac{2}{3^\alpha} m\, x^2 y^3 + \frac{m}{3^\alpha}(n-3)x^3y^3 + \frac{2}{(m+1)^\alpha} m\, x^2 y^{m+1}$$

$$+ \frac{m}{(m+2)^\alpha}(n-2)x^3y^{m+2} + \frac{2}{(m+2)^\alpha} x^{m+1}y^{m+2} + \frac{n-3}{(m+2)^\alpha} x^{m+2}y^{m+2}.$$

$$S_x^\alpha S_y^\alpha f(x,y) = \frac{2}{6^\alpha} m\, x^2 y^3 + \frac{m}{9^\alpha}(n-3)x^3y^3 + \frac{2}{(2m+2)^\alpha} m\, x^2 y^{m+1}$$

$$+ \frac{m}{(3m+6)^\alpha}(n-2)x^3y^{m+2} + \frac{2}{(m+1)^\alpha(m+2)^\alpha} x^{m+1}y^{m+2} + \frac{n-3}{(m+2)^{2\alpha}} x^{m+2}y^{m+2}.$$

$$D_y^\alpha f(x,y) = 2 \times 3^\alpha m\, x^2y^3 + 3^\alpha m(n-3)x^3y^3 + 2m(m+1)^\alpha\, x^2y^{m+1}.$$

$$+ m(m+2)^\alpha(n-2)x^3y^{m+2} + 2(m+2)^\alpha x^{m+1}y^{m+2} + (n-3)(m+2)^\alpha x^{m+2}y^{m+2}.$$

$$D_x^\alpha D_y^\alpha f(x,y) = 2 \times 6^\alpha m\, x^2y^3 + 9^\alpha m(n-3)x^3y^3 + 2m(2m+2)^\alpha\, x^2y^{m+1}.$$

$$+ m(3m+6)^\alpha(n-2)x^3y^{m+2} + 2(m+1)^\alpha(m+2)^\alpha x^{m+1}y^{m+2} + (n-3)(m+2)^{2\alpha} x^{m+2}y^{m+2}.$$

$$D_x\, S_y f(x,y) = \frac{4}{3} m\, x^2y^3 + m(n-3)x^3y^3 + \frac{4}{m+1} m\, x^2y^{m+1}$$

$$+ \frac{3m}{m+2}(n-2)x^3y^{m+2} + \frac{2(m+1)}{m+2} x^{m+1}y^{m+2} + (n-3)x^{m+2}y^{m+2}.$$

$$S_x\, D_y\, f(x,y) = 3m\, x^2y^3 + m(n-3)x^3y^3 + m(m+1)\, x^2y^{m+1}$$

$$+ \tfrac{m}{3}(m+2)(n-2)x^3y^{m+2} + \tfrac{2(m+2)}{m+1} x^{m+1}y^{m+2} + (n-3)x^{m+2}y^{m+2}.$$

$$Jf(x,y) = 2m\, x^5 + m(n-3)x^6 + 2m\, x^{m+3} + m(n-2)x^{m+5} + 2x^{2m+3} + (n-3)x^{2m+4}.$$

$$S_x\, Jf(x,y) = \frac{2}{5} m\, x^5 + \frac{m}{6}(n-3)x^6 + \frac{2m}{m+3}\, x^{m+3} + \frac{m(n-2)}{m+5} x^{m+5} + \frac{2}{2m+3} x^{2m+3} + \frac{n-3}{2m+4} x^{2m+4}.$$

$$JD_xD_y f(x,y) = 12m\, x^5 + 9m(n-3)x^6 + 4m(m+1)\, x^{m+3}$$

$$+3m(m+2)(n-2)x^{m+5} + 2(m+1)(m+2)x^{2m+3} + (n-3)\,(m+2)^2 x^{2m+4}.$$

$$S_x\, JD_xD_y f(x,y) = \frac{12}{5} m\, x^5 + \frac{3}{2} m(n-3)x^6 + \frac{4m(m+1)}{m+3}\, x^{m+3}$$

$$+ \frac{3m(m+2)(n-2)}{m+5} x^{m+5} + \frac{2(m+1)(m+2)}{2m+3} x^{2m+3} + \frac{(n-3)(m+2)}{2} x^{2m+4}.$$

$$D_y^3 f(x,y) = 54\, m\, x^2y^3 + 27m(n-3)x^3y^3 + 2m(m+1)^3\, x^2y^{m+1}$$

$$+ m(m+2)^3(n-2)x^3y^{m+2} + 2(m+2)^3 x^{m+1}y^{m+2} + (n-3)(m+2)^3 x^{m+2}y^{m+2}.$$

$$D_x^3 D_y^3 f(x,y) = 432\, m\, x^2y^3 + 729\, m(n-3)x^3y^3 + 16\, m(m+1)^3\, x^2y^{m+1}$$

$$+ 27\, m(m+2)^3(n-2)x^3y^{m+2} + 2(m+1)^3(m+2)^3 x^{m+1}y^{m+2} + (n-3)(m+2)^6 x^{m+2}y^{m+2}.$$

$$JD_x^3D_y^3f(x,y) = 432\,m\,x^5 + 729\,m(n-3)x^6 + 16\,m(m+1)^3\,x^{m+3}$$
$$+\,27\,m(m+2)^3(n-2)x^{m+5} + 2(m+1)^3(m+2)^3x^{2m+3} + (n-3)(m+2)^6x^{2m+4}.$$
$$Q_{-2}\,JD_x^3D_y^3f(x,y) = 432\,m\,x^3 + 729\,m(n-3)x^4 + 16\,m(m+1)^3\,x^{m+1}$$
$$+\,27\,m(m+2)^3(n-2)x^{m+3} + 2(m+1)^3(m+2)^3x^{2m+1} + (n-3)(m+2)^6x^{2m+2}.$$
$$S_x^3Q_{-2}\,JD_x^3D_y^3f(x,y) = 16\,m\,x^3 + \frac{729}{64}m(n-3)x^4 + 16\,m\,x^{m+1}$$
$$+\,\frac{27}{(m+3)^3}\,m(m+2)^3(n-2)x^{m+3}. + \frac{2}{(2m+1)^3}(m+1)^3(m+2)^3x^{2m+1} + \frac{n-3}{(2m+2)^3}(m+2)^6x^{2m+2}.$$

Now, from Table 1:

1. First Zagreb Index

$$M_1\big(P_n\ K_{(1,m)}\big) = \big(D_x + D_y\big)f(x,y)\,\big|_{x=y=1} = 13n\,m - 14m + 4n + m^2n - 6.$$

2. Second Zagreb Index

$$M_2\big(P_n\ K_{(1,m)}\big) = (D_xD_y)(f(x,y))\,\big|_{x=y=1} - 8.$$

3. Modified second Zagreb Index

$$^{\mathrm{m}}M_2\big(P_n\ K_{(1,m)}\big) = \big(S_xS_y\big)(f(x,y))\,\big|_{x=y=1} = -3m^2 + 19nm + 4nm^2 - 29m + 4n$$
$$= (\frac{1}{9} + \frac{1}{3(m+2)})nm + \frac{m}{m+1} - \frac{2m}{3(m+2)} + \frac{2}{(m+1)(m+2)} + \frac{n-3}{(m+2)^2}.$$

4. Inverse Randic Index

$$RR_\alpha\big(P_n\ K_{(1,m)}\big) = \big(D_x^\alpha D_y^\alpha\big)f(x,y)\,\big|_{x=y=1}$$
$$= m\big(2\times 6^\alpha + 9^\alpha(n-2) + 2(2m+2)^\alpha + (3m+6)^\alpha(n-2)\big) + 2(m^2+3m+2)^\alpha + (n-3)(m+2)^{2\alpha}.$$

5. Generalized Randic Index

$$R_\alpha\big(P_n\ K_{(1,m)}\big) = \big(S_x^\alpha S_y^\alpha\big)(f(x,y))\,\big|_{x=y=1}$$
$$= m\left(\frac{2}{6^\alpha} + \frac{n-3}{9^\alpha} + \frac{2}{(2m+2)^\alpha} + \frac{n-2}{(3m+6)^\alpha}\right) + \frac{2}{(m^2+3m+2)^\alpha} + \frac{n-3}{(m+2)^{2\alpha}}.$$

6. Symmetric Division Index

$$SDD\big(P_n\ K_{(1,m)}\big) = \big(D_xS_y + S_xD_y\big)(f(x,y))\,\big|_{x=y=1}$$
$$= m\left(\frac{13}{3} + 2(n-3) + \frac{4}{m+1} + m + 1\right) + \frac{3m}{m+2}(n-2) + \frac{m}{3}(m+2)(n-2) + \frac{2(m+1)}{m+2} + \frac{2(m+2)}{m+1} + 2(n-3).$$

7. Harmonic Index

$$H\big(P_n\ K_{(1,m)}\big) = 2S_x\,J(f(x,y))\,\big|_{x=1} = m\left(\frac{4}{5} + \frac{n-3}{3} + \frac{4}{m+3} + \frac{2(n-2)}{m+5}\right) + \frac{4}{2m+3} + \frac{2(n-3)}{2m+4}.$$

8. Inverse Sum Index

$$I\big(P_n\ K_{(1,m)}\big) = S_x\,JD_xD_y\,(f(x,y))\,\big|_{x=1}$$
$$= m\left(\frac{12}{5} + \frac{3(n-3)}{2} + \frac{4(m+1)}{m+3} + \frac{3(n-2)(m+2)}{m+5}\right) + \frac{2(m+1)(m+2)}{2m+3} + \frac{(n-3)(m+2)}{2}.$$

9. Augmented Zagreb Index

$$A\left(P_n\ K_{(1,m)}\right) = \left(S_x^3 Q_{-2} J D_x^3 D_y^3\right)(f(x,y))\big|_{x=1}$$

$$= m\left(32 + \frac{729}{64}(n-3) + \frac{27}{(m+3)^3}(m+2)^3(n-2)\right) + \frac{2(m+1)^3(m+2)^3}{(2m+1)^3} + \frac{m-3}{(2m+2)^3}(m+2)^6.$$

Theorem 3.3 The hyper-Zagreb index of $P_n\ K_{(1,m)}$ is:

$$HM\left(P_n\ K_{(1,m)}\right) = 50m + 36m(n-2) + 2m(m+3)^2 + m(n-2)(m+5)^2 + 2(2m+3)^2 + (n-3)(2m+4)^2.$$

Proof. By the definition of the hyper-Zagreb index

$$HM\left(P_n\ K_{(1,m)}\right) = \sum_{uv\in E(P_n\ K_{(1,m)})} [d_u + d_v]^2$$

$$= \sum_{uv\in E_1(P_n\ K_{(1,m)})} [d_u + d_v]^2 + \sum_{uv\in E_2(P_n\ K_{(1,m)})} [d_u + d_v]^2 + \sum_{uv\in E_3(P_n\ K_{(1,m)})} [d_u + d_v]^2 + \sum_{uv\in E_4(P_n\ K_{(1,m)})} [d_u + d_v]^2$$

$$+ \sum_{uv\in E_5(P_n\ K_{(1,m)})} [d_u + d_v]^2 + \sum_{uv\in E_6(P_n\ K_{(1,m)})} [d_u + d_v]^2$$

$$= \left|E_1\left(P_n\ K_{(1,m)}\right)\right|[2+3]^2 + \left|E_2\left(P_n\ K_{(1,m)}\right)\right|[3+3]^2 + \left|E_3\left(P_n\ K_{(1,m)}\right)\right|[m+3]^2 + \left|E_4\left(P_n\ K_{(1,m)}\right)\right|[m+5]^2 + \left|E_5\left(P_n\ K_{(1,m)}\right)\right|[2m+3]^2 + \left|E_6\left(P_n\ K_{(1,m)}\right)\right|[2m+4]^2$$

$$= 50m + 36m(n-2) + 2m(m+3)^2 + m(n-2)(m+5)^2 + 2(2m+3)^2 + (n-3)(2m+4)^2.$$

Theorem 3.4. The first multiple Zagreb index of $P_n\ K_{(1,m)}$ is:

$$PM_1\left(P_n\ K_{(1,m)}\right) = 5^{2m} \times 6^{m(n-2)} \times (m+3)^{2m} \times (m+5)^{m(n-2)} \times (2m+3)^2 \times (2m+4)^{n-3}.$$

Proof. By the definition of the first multiple Zagreb index:

$$PM_1\left(P_n\ K_{(1,m)}\right) = \prod_{uv\in E(P_n\ K_{(1,m)})} [d_u + d_v] = \prod_{uv\in E_1(P_n\ K_{(1,m)})} [d_u + d_v] \times \prod_{uv\in E_2(P_n\ K_{(1,m)})} [d_u + d_v]$$

$$+ \prod_{uv\in E_3(P_n\ K_{(1,m)})} [d_u + d_v] \times \prod_{uv\in E_4(P_n\ K_{(1,m)})} [d_u + d_v] + \prod_{uv\in E_5(P_n\ K_{(1,m)})} [d_u + d_v] \times \prod_{uv\in E_6(P_n\ K_{(1,m)})} [d_u + d_v]$$

$$= [2+3]^{\left|E_1(P_n\ K_{(1,m)})\right|} \times [3+3]^{\left|E_2(P_n\ K_{(1,m)})\right|} + [m+3]^{\left|E_3(P_n\ K_{(1,m)})\right|} \times [m+5]^{\left|E_4(P_n\ K_{(1,m)})\right|}$$

$$+ [2m+3]^{\left|E_5(P_n\ K_{(1,m)})\right|} \times [2m+4]^{\left|E_6(P_n\ P_m)\right|}$$

$$= 5^{2m} \times 6^{m(n-2)} \times (m+3)^{2m} \times (m+5)^{m(n-2)} \times (2m+3)^2 \times (2m+4)^{n-3}.$$

Theorem 3.5. The second multiple Zagreb index of $P_n\ K_{(1,m)}$ is:

$$\mathrm{P}M_2\left(P_n\ K_{(1,m)}\right) = 6^{2m} \times 9^{m(n-2)} \times (2m+2)^{2m} \times (3m+6)^{m(n-2)}$$

$$\times (m^2+3m+2)^2 \times (m^2+4m+4)^{n-3}.$$

Proof. By the definition of the second multiple Zagreb index:

$$PM_2\left(P_n\ K_{(1,m)}\right) = \prod_{uv\in E(P_n\ K_{(1,m)})} [d_u \times d_v] = \prod_{uv\in E_1(P_n\ K_{(1,m)})} [d_u \times d_v] \times \prod_{uv\in E_2(P_n\ K_{(1,m)})} [d_u \times d_v]$$

$$\times \prod_{uv\in E_3(P_n\ K_{(1,m)})} [d_u \times d_v] \times \prod_{uv\in E_4(P_n\ K_{(1,m)})} [d_u \times d_v] \times \prod_{uv\in E_5(P_n\ K_{(1,m)})} [d_u \times d_v] \times \prod_{uv\in E_6(P_n\ K_{(1,m)})} [d_u \times d_v]$$

$$=[2\times 3]^{|E_1(P_n\ K_{(1,m)})|}\times[3\times 3]^{|E_2(P_n\ K_{(1,m)})|}\times[2(m+1)]^{|E_3(P_n\ K_{(1,m)})|}\times[3(m+2)]^{|E_4(P_n\ K_{(1,m)})|}$$

$$\times[(m+1)(m+2)]^{|E_5(P_n\ K_{(1,m)})|}\times[(m+2)(m+2)]^{|E_6(P_n\ K_{(1,m)})|}$$

$$=6^{2m}\times 9^{m(n-2)}\times(2m+2)^{2m}\times(3m+6)^{m(n-2)}\times(m^2+3m+2)^2\times(m^2+4m+4)^{n-3}.$$

Theorem 3.6. The first Zagreb polynomial of $P_n\ K_{(1,m)}$ is:

$$M_1(P_n\ K_{(1,m)},x)=2m\,x^5+m(n-3)x^6+2m\,x^{m+3}+m(n-2)x^{m+5}+2x^{2m+3}+(n-3)x^{2m+4}.$$

Proof. By the definition of the first Zagreb polynomial:

$$M_1(P_n\ K_{(1,m)},x)=\sum_{uv\in E(P_n\ K_{(1,m)})}x^{[d_u+d_v]}=\sum_{uv\in E_1(P_n\ K_{(1,m)})}x^{[d_u+d_v]}+\sum_{uv\in E_2(P_n\ K_{(1,m)})}x^{[d_u+d_v]}$$

$$+\sum_{uv\in E_3(P_n\ K_{(1,m)})}x^{[d_u+d_v]}+\sum_{uv\in E_4(P_n\ K_{(1,m)})}x^{[d_u+d_v]}+\sum_{uv\in E_5(P_n\ K_{(1,m)})}x^{[d_u+d_v]}+\sum_{uv\in E_6(P_n\ K_{(1,m)})}x^{[d_u+d_v]}$$

$$=|E_1(P_n\ K_{(1,m)})|x^5+|E_2(P_n\ K_{(1,m)})|x^6+|E_3(P_n\ K_{(1,m)})|x^{m+3}+|E_4(P_n\ K_{(1,m)})|x^{m+5}$$

$$+|E_5(P_n\ K_{(1,m)})|x^{2m+3}+|E_6(P_n\ K_{(1,m)})|x^{2m+4}$$

$$=2m\,x^5+m(n-3)x^6+2m\,x^{m+3}+m(n-2)x^{m+5}+2x^{2m+3}+(n-3)x^{2m+4}.$$

Theorem 3.7. The Then second Zagreb polynomial of $P_n\ K_{(1,m)}$ is:

$$M_2(P_n\ K_{(1,m)},x)=2m\,x^6+m(n-3)x^9+2m\,x^{2m+2}+m(n-2)x^{3m+6}+2x^{m^2+3m+2}+(n-3)x^{m^2+4m+4}.$$

Proof. By the definition of the second Zagreb polynomial:

$$M_2(P_n\ K_{(1,m)},x)=\sum_{uv\in E(P_n\ K_{(1,m)})}x^{[d_u\times d_v]}=\sum_{uv\in E_1(P_n\ K_{(1,m)})}x^{[d_u\times d_v]}+\sum_{uv\in E_2(P_n\ K_{(1,m)})}x^{[d_u\times d_v]}$$

$$+\sum_{uv\in E_3(P_n\ K_{(1,m)})}x^{[d_u\times d_v]}+\sum_{uv\in E_4(P_n\ K_{(1,m)})}x^{[d_u\times d_v]}+\sum_{uv\in E_5(P_n\ K_{(1,m)})}x^{[d_u\times d_v]}+\sum_{uv\in E_6(P_n\ K_{(1,m)})}x^{[d_u\times d_v]}$$

$$=|E_1(P_n\ K_{(1,m)})|x^6+|E_2(P_n\ K_{(1,m)})|x^9+|E_3(P_n\ K_{(1,m)})|x^{2m+2}+|E_4(P_n\ K_{(1,m)})|x^{3m+6}$$

$$+|E_5(P_n\ K_{(1,m)})|x^{m^2+3m+2}+|E_6(P_n\ K_{(1,m)})|x^{m^2+4m+4}$$

$$=2m\,x^6+m(n-3)x^9+2m\,x^{2m+2}+m(n-2)x^{3m+6}+2x^{m^2+3m+2}+(n-3)x^{m^2+4m+4}.$$

Theorem 3.8. The forgotten polynomial of $P_n\ K_{(1,m)}$ is:

$$F(P_n\ K_{(1,m)})=2m\,x^{13}+m(n-3)x^{18}+2m\,x^{4+(m+1)^2}$$

$$+m(n-2)x^{9+(m+2)^2}+2x^{(m+1)^2+(m+2)^2}+(n-3)x^{2(m+2)^2}.$$

Proof. By the definition of the forgotten polynomial:

$$F(P_n\ K_{(1,m)},x)=\sum_{uv\in E(P_n\ K_{(1,m)})}x^{[(d_u)^2+(d_v)^2]}=\sum_{uv\in E_1(P_n\ K_{(1,m)})}x^{[(d_u)^2+(d_v)^2]}+\sum_{uv\in E_2(P_n\ K_{(1,m)})}x^{[(d_u)^2+(d_v)^2]}$$

$$+\sum_{uv\in E_3(P_n\ K_{(1,m)})}x^{[(d_u)^2+(d_v)^2]}+\sum_{uv\in E_4(P_n\ K_{(1,m)})}x^{[(d_u)^2+(d_v)^2]}+\sum_{uv\in E_5(P_n\ K_{(1,m)})}x^{[(d_u)^2+(d_v)^2]}+\sum_{uv\in E_6(P_n\ K_{(1,m)})}x^{[(d_u)^2+(d_v)^2]}$$

$$=|E_1(P_n\ K_{(1,m)})|x^{13}+|E_2(P_n\ K_{(1,m)})|x^{18}+|E_3(P_n\ K_{(1,m)})|x^{4+(m+1)^2}+|E_4(P_n\ K_{(1,m)})|x^{9+(m+2)^2}$$

$$+|E_5(P_n\ K_{(1,m)})|x^{(m+1)^2+(m+2)^2}+|E_6(P_n\ K_{(1,m)})|x^{2(m+2)^2}$$

$$= 2m\,x^{13} + m(n-3)x^{18} + 2m\,x^{4+(m+1)^2} + m(n-2)x^{9+(m+2)^2} + 2x^{(m+1)^2+(m+2)^2} + (n-3)x^{2(m+2)^2}.$$

Theorem 3.9. The forgotten index or F-index of $P_n\ K_{(1,m)}$ is:

$$F\left(P_n\ K_{(1,m)}\right) = 26m + 18m(n-2) + 2m(4+(m+1)^2)$$

$$+m(n-2)(9+(m+2)^2) + 2((m+1)^2+(m+2)^2) + 2(n-3)(m+2)^2.$$

Proof. By the definition of the forgotten index:

$$F(G) = \sum_{uv\in E(P_n\ K_{(1,m)})} [(d_u)^2 + (d_v)^2] = \sum_{uv\in E_1(P_n\ K_{(1,m)})} [(d_u)^2 + (d_v)^2] + \sum_{uv\in E_2(P_n\ K_{(1,m)})} [(d_u)^2 + (d_v)^2]$$

$$+\sum_{uv\in E_3(P_n\ K_{(1,m)})}[(d_u)^2 + (d_v)^2] + \sum_{uv\in E_4(P_n\ K_{(1,m)})}[(d_u)^2 + (d_v)^2]$$

$$+\sum_{uv\in E_5(P_n\ K_{(1,m)})}[(d_u)^2 + (d_v)^2] + \sum_{uv\in E_6(P_n\ K_{(1,m)})}[(d_u)^2 + (d_v)^2]$$

$$= 13\left|E_1\left(P_n\ K_{(1,m)}\right)\right| + 18\left|E_2\left(P_n\ K_{(1,m)}\right)\right|$$

$$+\left|E_3\left(P_n\ K_{(1,m)}\right)\right|(4+(m+1)^2) + \left|E_4\left(P_n\ K_{(1,m)}\right)\right|(9+(m+2)^2)$$

$$+\left|E_4\left(P_n\ K_{(1,m)}\right)\right|(9+(m+2)^2) + \left|E_5\left(P_n\ K_{(1,m)}\right)\right|((m+1)^2+(m+2)^2)$$

$$+\left|E_6\left(P_n\ K_{(1,m)}\right)\right|((m+2)^2+(m+2)^2)$$

$$= 26m + 18m(n-2) + 2m(4+(m+1)^2) + m(n-2)(9+(m+2)^2) + 2((m+1)^2+(m+2)^2) + 2(n-3)(m+2)^2.$$

4. Conclusion

In this paper, many topological indices and polynomials were calculated. The most important which was the M-polynomial, through which many indices were calculated through processes of integration and derivation of M-polynomial, and we can calculated indices and polynomials for some special graph.

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