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On Delay-Range-Dependent and Delay-Rate-Dependent Stability for Delayed Systems

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Abstract: It is well known that the phenomena of time delays are frequently encountered in many process and various control systems. The presence of delays can have an effect on system stability and performance, so ignoring them may lead to design flaws and incorrect analysis conclusions. Hence, the stability problem for time-delayed systems has received considerable attention in recent years. This brief focuses on the stability analysis for a class of delayed linear systems. Firstly, we construct a novel augmented Lyapunov-Krasovskii functional (LKF) which includes the lower, the upper bounds of the delay and the delay itself. Secondly, utilizing some integral inequalities and the reciprocally convex combination lemma, we obtain less conservative stability criteria formulated in form of linear matrix inequalities (LMIs). Finally, numerical examples are provided to show the effectiveness of the proposed method.

Keywords: Time Delay, Lyapunov-Krasovskii Functional (LKF), Linear Matrix Inequalities (LMIs)

1. Introduction

Time delay universally emerges in various engineering systems such as networked control systems, aircraft, process control systems and long transmission lines in pneumatic systems. It is well known that time delay often leads to the oscillation, poor performance behavior even instability. Accordingly, many researchers have paid attention to the stability problems of delayed control systems over the past few decades.

To deduce stability sufficient conditions, a variety of methods were adopted, such as integral inequalities, the model transformation, the delay decomposition approach, the free-weighting matrices approach (FWMA) and the reciprocally convex lemmas (RCLs) and so on [1-32] and references therein. Recently, the FWMA, the Jensen's or Wirtinger's integral inequalities method and the RCLs are widely used.

In Wu et al [10], the FWMA was firstly developed to investigate the stability of delayed systems. Later, He et al. [12, 13] improved and extended this approach. Park and Ko [14] further generalized the FWMA to a new Lyapunov functional. By contrast, the FWMA can keep a tradeoff between the conservatism and the computational complexity [12, 15-16]. However there is still some conservatism in such criteria and the criteria should be simplified.

The integral inequality especially Jensen's inequality [6] is regarded as another important approach. Not introducing excessive matrix variables in the stability criteria, it provides a simple stability sufficient conditions and easily to be tested. To deal with the cross terms, Sun et al. extended the Jensen's integral inequality was from single integral to double integral and obtained the Jensen's double integral inequality [2, 18]. In order to obtain less conservative stability conditions, the Wirtinger's integral inequalities [21, 22, 25, 26] were utilized to evaluate the upper bounds on the derivative of the LKF and accomplished numerous results. Kim [25] developed an infinite-series-type Jensen integral inequality. It should be pointed out that the stability criteria derived by this infinite-series-type inequality are the least conservative compared with those by any others integral inequalities so far when choosing identical LKF. However, the drawback of the infinite-series-type inequality is that it cannot produce the tightest upper bound because the parameters are not adjustable. Thus, there exist some room to reduce the conservatism of results based on integral inequalities and motivates our current work.

Recently, the convex combination method [17-20] and the RCLs [8, 23, 24, 27-29] have been introduced to discuss the stability of delayed systems. Shao [17] estimated the cross terms by using the convex combination rather than enlarging directly them as the lower and the upper bounds of the delay, respectively, and Ref. [18-20] followed this idea to cope with the stability of delayed systems. Park et al [23] presented a reciprocally convex combination lemma (RCCL), which can cope with the terms containing the delays just introducing a few slack matrices. Thus, the RCCL has become powerful tool to assess the integral terms with time-varying delays. Ref. [27-29] extended the RCCL, and developed some useful extended reciprocally convex matrix inequality to analyze the delayed control systems. However, there leaves some room to investigate further.

In this paper, a modified augmented LKF where more information about the delay includes is constructed. Some novel stability criteria are derived in terms of linear matrix inequalities (LMIs). At last, two examples are given to demonstrate the effectiveness of the proposed results.

Notations: Throughout the paper, the superscript "T" denotes the transpose of a matrix; $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ stand for the set of real vector with *n*-dimensional and real matrix of size $n\times m$, respectively; $P>0 (P\ge 0)$ symbolizes a symmetric positive definite (positive semi-definite) matrix; I and 0 refer to the identity matrix and zero matrix with appropriate dimensions, respectively. Block diagonal matrix is symbolized by $diag\{\cdots\}$; $col\{x_1, x_2, \cdots, x_n\} = [x_1^T, x_2^T, \cdots, x_n^T]^T$. The symbol"*"denotes the elements induced by symmetry in a symmetric matrix. In what follows, if not explicitly stated, matrices are assumed to have compatible dimensions.

2. Problem Formulation and Preliminaries

Consider a linear delayed system:

$$\begin{aligned} &\dot{x}(t) = Ax(t) + A_d x(t-d(t)), \qquad t>0 \\ &x(\theta) = \varphi(\theta), \quad \theta\in[-h_2, 0] \end{aligned} \tag{1}$$

where $x(t)\in\mathbb{R}^n$ is the state vector; $A\in\mathbb{R}^{nxn}$ and $A_d\in\mathbb{R}^{nxn}$ are system matrices; The initial condition $\varphi(t)$ is a continuously differentiable vector-valued function; $d(t)$ is a time-varying discrete delay and satisfies:

$$0 < h_1 \le d(t) \le h_2 < +\infty \tag{2}$$

$$\dot{d}(t) \le \mu < +\infty \tag{3}$$

where h_1, h_2 and μ are constants.

To begin with, we introduce the following lemmas which play an essential role in deriving our main results.

Lemma 1 (Jensen's integral inequality [6]) For any real symmetric positive definite matrix $M\in\mathbb{R}^{n\times n}$, two scalars $\alpha\le\beta$ and a vector-valued function $\omega(t):[\alpha,\beta]\to\mathbb{R}^n$ such that the following integration are well defined, then

$$(\beta-\alpha)\int_\alpha^\beta \omega^T(t)M\omega(t)dt \ge \left(\int_\alpha^\beta \omega^T(t)dt\right)M\left(\int_\alpha^\beta \omega(t)dt\right). \tag{4}$$

Lemma 2 (Jensen's double integral inequality [18], [30]) For any real symmetric positive definite matrix $R\in\mathbb{R}^{n\times n}$, two scalars satisfying $\tau_2>\tau_1\ge 0$, and a vector-valued function $\omega(t)$ such that the following integration are well defined, then

$$\frac{\tau_2^2-\tau_1^2}{2}\int_{-\tau_2}^{-\tau_1} d\lambda \int_{t+\lambda}^{t} \omega^T(s)R\omega(s)ds \ge \left(\int_{-\tau_2}^{-\tau_1} d\lambda \int_{t+\lambda}^{t} \omega(s)ds\right)^T R\left(\int_{-\tau_2}^{-\tau_1} d\lambda \int_{t+\lambda}^{t} \omega(s)ds\right). \tag{5}$$

3. Main Results

In this section, we are to discuss stability of system (1). For simplicity and convenience, we define

$$\begin{aligned} \eta(t) = col\{&x(t), x(t-d(t)), x(t-h_1), x(t-h_2), \dot{x}(t-h_1), \dot{x}(t-h_2), \\ &\int_{t-h_1}^{t} x(s)ds, \int_{t-d(t)}^{t-h_1} x(s)ds, \int_{t-h_2}^{t-d(t)} x(s)ds\} \end{aligned}$$

and

$$e_i^T = [0_{n\times(i-1)n}, I_n, 0_{n\times(9-i)n}], i=1,2,\cdots,9\ .$$

For system (1), we have main result as follows.

Theorem 1 For any given h_1, h_2 and μ, the system (1) satisfying (2) and (3) is globally asymptotically stable if there exist symmetric positive definite matrices with appropriate dimensions

$$P=\begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} & P_{15} \\ * & P_{22} & P_{23} & P_{24} & P_{25} \\ * & * & P_{33} & P_{34} & P_{35} \\ * & * & * & P_{44} & P_{45} \\ * & * & * & * & P_{55} \end{bmatrix},$$

$Q_i\,(i=1,2,3,4,5), Z_j\,(j=1,2,3,4)$ and $R_k\,(k=1,2)$ such that for any matrices S_1 and S_2 the following LMIs hold simultaneously

$$\begin{bmatrix} Z_2 & S_1 \\ * & Z_2 \end{bmatrix} \ge 0, \begin{bmatrix} Z_4 & S_2 \\ * & Z_4 \end{bmatrix} \ge 0 \tag{6}$$

$$\begin{aligned} &\Omega + \Gamma^T \Upsilon \Gamma + e_1(Q_1 + h_1^2 Z_3 + h_{12}^2 Z_4)e_1^T + (\mu - 1)e_2 Q_3 e_2^T + e_3(Q_2 - Q_1 + Q_3)e_3^T \\ &- e_4 Q_2 e_4^T + e_5(Q_5 - Q_4)e_5^T - e_6 Q_5 e_6^T - e_7 Z_3 e_7^T - e_8 Z_4 e_8^T - 2e_8 S_2 e_9^T - e_9 Z_4 e_9^T \\ &- (e_3 - e_2) Z_2 (e_3 - e_2)^T - 2(e_3 - e_2) S_1 (e_2 - e_4)^T - (e_2 - e_4) Z_2 (e_2 - e_4)^T \\ &- (h_1 e_1 - e_7) R_1 (h_1 e_1 - e_7)^T - (h_{12} e_1 - e_8 - e_9) R_2 (h_{12} e_1 - e_8 - e_9)^T < 0. \end{aligned} \tag{7}$$

where

$$\Omega = \begin{bmatrix} \Omega_{11} & P_{11}A_d & \Omega_{13} & \Omega_{14} & P_{12} & P_{13} & \Omega_{17} & \Omega_{18} & \Omega_{19} \\ * & 0 & A_d^T P_{12} & A_d^T P_{13} & 0 & 0 & A_d^T P_{14} & A_d^T P_{15} & A_d^T P_{15} \\ * & * & \Omega_{33} & \Omega_{34} & P_{22} & P_{23} & \Omega_{37} & \Omega_{38} & \Omega_{39} \\ * & * & * & \Omega_{44} & P_{23}^T & P_{33} & -P_{45}^T & -P_{55} & -P_{55} \\ * & * & * & * & 0 & 0 & P_{24} & P_{25} & P_{25} \\ * & * & * & * & * & 0 & P_{34} & P_{35} & P_{35} \\ * & * & * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & * & * & 0 \end{bmatrix}$$

$$\Omega_{11} = P_{11}A + A^T P_{11} + P_{14} + P_{14}^T,$$

$$\Omega_{13} = P_{15} - P_{14} + A^T P_{12} + P_{24}^T,$$

$$\Omega_{14} = -P_{15} + A^T P_{13} + P_{34}^T,$$

$$\Omega_{17} = P_{44} + A^T P_{14},$$

$$\Omega_{18} = \Omega_{19} = P_{45} + A^T P_{15},$$

$$\Omega_{33} = P_{25} + P_{25}^T - P_{24} - P_{24}^T,$$

$$\Omega_{34} = -P_{25} + P_{35}^T - P_{34}^T,$$

$$\Omega_{37} = P_{45}^T - P_{44},$$

$$\Omega_{38} = \Omega_{39} = P_{55} - P_{45},$$

$$\Omega_{44} = -P_{35}^T - P_{35},$$

$$\Gamma = \begin{bmatrix} A & A_d & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Upsilon = Q_4 + h_1^2 Z_1 + h_{12}^2 Z_2 + \frac{1}{4} h_1^4 R_1 + \frac{\gamma^2}{4} R_2,$$

$$h_{12} = h_2 - h_1,\ \gamma = h_2^2 - h_1^2.$$

Proof. We first construct a Lyapunov functional as follows:

$$V(x_t) = \sum_{i=1}^{4} V_i(x_t),$$

where

$$V_1(x_t) = \xi^T(t)P\xi(t),$$

$$V_2(x_t) = \int_{t-h_1}^{t} x^T(\alpha)Q_1x(\alpha)d\alpha + \int_{t-h_2}^{t-h_1} x^T(\alpha)Q_2x(\alpha)d\alpha + \int_{t-d(t)}^{t-h_1} x^T(\alpha)Q_3x(\alpha)d\alpha$$
$$+\int_{t-h_1}^{t} \dot{x}^T(\alpha)Q_4\dot{x}(\alpha)d\alpha + \int_{t-h_2}^{t-h_1} \dot{x}^T(\alpha)Q_5\dot{x}(\alpha)d\alpha,$$

$$V_3(x_t) = \int_{-h_1}^{0} d\lambda \int_{t+\lambda}^{t} h_1\dot{x}^T(\alpha)Z_1\dot{x}(\alpha)d\alpha + \int_{-h_2}^{-h_1} d\lambda \int_{t+\lambda}^{t} h_{12}\dot{x}^T(\alpha)Z_2\dot{x}(\alpha)d\alpha$$
$$+\int_{-h_1}^{0} d\theta \int_{t+\theta}^{t} h_1x^T(\alpha)Z_3x(\alpha)d\alpha + \int_{-h_2}^{-h_1} d\theta \int_{t+\theta}^{t} h_{12}x^T(\alpha)Z_4x(\alpha)d\alpha,$$

$$V_4(x_t) = \frac{h_1^2}{2}\int_{-h_1}^{0} d\theta \int_{\theta}^{0} d\alpha \int_{t+\alpha}^{t} \dot{x}^T(s)R_1\dot{x}(s)ds + \frac{\gamma}{2}\int_{-h_2}^{-h_1} d\theta \int_{\theta}^{0} d\alpha \int_{t+\alpha}^{t} \dot{x}^T(s)R_2\dot{x}(s)ds,$$

with

$$\xi(t) = col\{x(t), x(t-h_1), x(t-h_2), \int_{t-h_1}^{t} x(s)ds, \int_{t-h_2}^{t-h_1} x(s)ds\}.$$

Calculating the time derivative of $V_1(x_t)$, $V_2(x_t)$, and $V_3(x_t)$ respectively along the solution of the system (1) yields

$$\dot{V}_1(x_t) = 2\xi^T(t)P\dot{\xi}(t) = \eta^T(t)\Omega\eta(t) \tag{8}$$

where Ω is defined in Theorem 1.

$$\begin{aligned}\dot{V}_2(x_t) \le\ & x^T(t)Q_1x(t) + x^T(t-h_1)(Q_2 - Q_1 + Q_3)x(t-h_1) + x^T(t-h_2)(-Q_2)x(t-h_2)\\ & + (\mu-1)x^T(t-d(t))Q_3x(t-d(t)) + \dot{x}^T(t)Q_4\dot{x}(t) + \dot{x}^T(t-h_1)(Q_5 - Q_4)\dot{x}(t-h_1)\\ & + \dot{x}^T(t-h_2)(-Q_5)\dot{x}(t-h_2)\end{aligned} \tag{9}$$

$$\begin{aligned}\dot{V}_3(x_t) =\ & x^T(t)(h_1^2Z_3 + h_{12}^2Z_4)x(t) + \dot{x}^T(t)(h_1^2Z_1 + h_{12}{}^2Z_2)\dot{x}(t) - \int_{t-h_1}^{t} h_1\dot{x}^T(\alpha)Z_1\dot{x}(\alpha)d\alpha\\ & - \int_{t-h_2}^{t-h_1} h_{12}\dot{x}^T(\alpha)Z_2\dot{x}(\alpha)d\alpha - \int_{t-h_1}^{t} h_1x^T(\alpha)Z_3x(\alpha)d\alpha - \int_{t-h_2}^{t-h_1} h_{12}x^T(\alpha)Z_4x(\alpha)d\alpha,\end{aligned} \tag{10}$$

Using Lemma 1 yields

$$-h_1\int_{t-h_1}^{t} \dot{x}^T(\alpha)Z_1\dot{x}(\alpha)d\alpha \le -\eta^T(t)(e_1 - e_3)Z_1(e_1 - e_3)^T\eta(t) \tag{11}$$

and

$$-h_1\int_{t-h_1}^{t} x^T(\alpha)Z_3x(\alpha)d\alpha \le -\eta^T(t)e_7Z_3e_7{}^T\eta(t) \tag{12}$$

Observe that

$$-\int_{t-h_2}^{t-h_1} h_{12}\dot{x}^T(\alpha)Z_2\dot{x}(\alpha)d\alpha = -\int_{t-h_2}^{t-d(t)} h_{12}\dot{x}^T(\alpha)Z_2\dot{x}(\alpha)d\alpha - \int_{t-d(t)}^{t-h_1} h_{12}\dot{x}^T(\alpha)Z_2\dot{x}(\alpha)d\alpha.$$

Using Lemma 1 again, we have

$$-\int_{t-h_2}^{t-d(t)} h_{12}\dot{x}^T(\alpha)Z_2\dot{x}(\alpha)d\alpha \le -\frac{h_2-h_1}{h_2-d(t)}\left(\int_{t-h_2}^{t-d(t)}\dot{x}(\alpha)d\alpha\right)^T Z_2\left(\int_{t-h_2}^{t-d(t)}\dot{x}(\alpha)d\alpha\right)$$

and

$$-\int_{t-d(t)}^{t-h_1} h_{12}\dot{x}^T(\alpha)Z_2\dot{x}(\alpha)d\alpha \le -\frac{h_2-h_1}{d(t)-h_1}\left(\int_{t-d(t)}^{t-h_1}\dot{x}(\alpha)d\alpha\right)^T Z_2\left(\int_{t-d(t)}^{t-h_1}\dot{x}(\alpha)d\alpha\right).$$

Thus

$$\begin{aligned}&-\int_{t-h_2}^{t-h_1} h_{12}\dot{x}^T(\alpha)Z_2\dot{x}(\alpha)d\alpha\\ &\le -\frac{h_2-h_1}{h_2-d(t)}\int_{t-h_2}^{t-d(t)}\dot{x}^T(\alpha)d\alpha\, Z_2\int_{t-h_2}^{t-d(t)}\dot{x}(\alpha)d\alpha\\ &\quad -\frac{h_2-h_1}{d(t)-h_1}\int_{t-d(t)}^{t-h_1}\dot{x}^T(\alpha)d\alpha\, Z_2\int_{t-d(t)}^{t-h_1}\dot{x}(\alpha)d\alpha.\end{aligned}$$

From the reciprocally convex combination lemmas [23] and [27], there exists a matrix S_1 with appropriate dimension such that

$$\begin{bmatrix} Z_2 & S_1 \\ * & Z_2 \end{bmatrix} \ge 0$$

then

$$\begin{aligned}&-\frac{h_2-h_1}{h_2-d(t)}\int_{t-h_2}^{t-d(t)}\dot{x}^T(\alpha)d\alpha\, Z_2\int_{t-h_2}^{t-d(t)}\dot{x}(\alpha)d\alpha-\frac{h_2-h_1}{d(t)-h_1}\int_{t-d(t)}^{t-h_1}\dot{x}^T(\alpha)d\alpha\, Z_2\int_{t-d(t)}^{t-h_1}\dot{x}(\alpha)d\alpha\\ &\le -\begin{bmatrix}\int_{t-d(t)}^{t-h_1}\dot{x}(\alpha)d\alpha\\ \int_{t-h_2}^{t-d(t)}\dot{x}(\alpha)d\alpha\end{bmatrix}^T\begin{bmatrix} Z_2 & S_1 \\ * & Z_2 \end{bmatrix}\begin{bmatrix}\int_{t-d(t)}^{t-h_1}\dot{x}(\alpha)d\alpha\\ \int_{t-h_2}^{t-d(t)}\dot{x}(\alpha)d\alpha\end{bmatrix}.\end{aligned}$$

Namely

$$\begin{aligned}&-\frac{h_2-h_1}{h_2-d(t)}\int_{t-h_2}^{t-d(t)}\dot{x}^T(\alpha)d\alpha\, Z_2\int_{t-h_2}^{t-d(t)}\dot{x}(\alpha)d\alpha-\frac{h_2-h_1}{d(t)-h_1}\int_{t-d(t)}^{t-h_1}\dot{x}^T(\alpha)d\alpha\, Z_2\int_{t-d(t)}^{t-h_1}\dot{x}(\alpha)d\alpha\\ &\le -\eta^T(t)[(e_3-e_2)Z_2(e_3-e_2)^T+2(e_3-e_2)S_1(e_2-e_4)^T+(e_2-e_4)Z_2(e_2-e_4)^T]\eta(t).\end{aligned} \tag{13}$$

Similarly, there exists a matrix S_2 with appropriate dimension such that

$$\begin{bmatrix} Z_4 & S_2 \\ * & Z_4 \end{bmatrix} \ge 0,$$

then

$$-\int_{t-h_2}^{t-h_1} h_{12}x^T(\alpha)Z_4x(\alpha)d\alpha \le -\eta^T(t)[e_8Z_4e_8^T+2e_8S_2e_9^T+e_9Z_4e_9^T]\eta(t) \tag{14}$$

Additionally,

$$\dot{V}_4(x_t)=\dot{x}^T(t)(\frac{h_1^4}{4}R_1+\frac{\gamma^2}{4}R_2)\dot{x}(t)-\frac{h_1^2}{2}\int_{-h_1}^{0}d\theta\int_{t+\theta}^{t}\dot{x}^T(s)R_1\dot{x}(s)ds-\frac{\gamma}{2}\int_{-h_2}^{-h_1}d\theta\int_{t+\theta}^{t}\dot{x}^T(s)R_2\dot{x}(s)ds \tag{15}$$

On basis of Lemma 2, one obtains

$$-\frac{h_1^2}{2}\int_{-h_1}^{0}d\theta\int_{t+\theta}^{t}\dot{x}^T(s)R_1\dot{x}(s)ds \le -\eta^T(t)[(h_1e_1-e_7)R_1(h_1e_1-e_7)^T]\eta(t) \tag{16}$$

and

$$-\frac{\gamma}{2}\int_{-h_2}^{-h_1}d\theta\int_{t+\theta}^{t}\dot{x}^T(s)R_2\dot{x}(s)ds \le -\eta^T(t)[(h_{12}e_1-e_8-e_9)R_2(h_{12}e_1-e_8-e_9)^T]\eta(t). \tag{17}$$

Combining with (8)-(17), one concludes

$$\dot{V}(x_t) \le \eta^T(t)\Sigma\eta(t), \tag{18}$$

where

$$\begin{aligned}\Sigma = {} & \Omega+\Gamma^T\Upsilon\Gamma+e_1(Q_1+h_1^2Z_3+h_{12}^2Z_4)e_1^T+(\mu-1)e_2Q_3e_2^T+e_3(Q_2-Q_1+Q_3)e_3^T\\ & -e_4Q_2e_4^T+e_5(Q_5-Q_4)e_5^T-e_6Q_5e_6^T-e_7Z_3e_7^T-e_8Z_4e_8^T-2e_8S_2e_9^T-e_9Z_4e_9^T\\ & -(e_3-e_2)Z_2(e_3-e_2)^T-2(e_3-e_2)S_1(e_2-e_4)^T-(e_2-e_4)Z_2(e_2-e_4)^T\\ & -(h_1e_1-e_7)R_1(h_1e_1-e_7)^T-(h_{12}e_1-e_8-e_9)R_2(h_{12}e_1-e_8-e_9)^T.\end{aligned}$$

According to the Lyapunov stability theory, if (6) and (7) hold, then $\Sigma<0$, which ensure that system (1) is globally asymptotically stable. This completes the proof of Theorem 1.

Remark 1 In Ref. [3-4, 18], the cross term

$$-\int_{-h_2}^{-h_1}d\theta\int_{t+\theta}^{t}\dot{x}^T(s)R_2\dot{x}(s)ds$$

arising in the derivative of $V(x_t)$ was enlarged directly by the Jensen's double integral inequality. However, those similar terms existing in the derivative of $V(x_t)$ such as

$$-\int_{t-h_2}^{t-h_1}h_{12}x^T(s)Z_4x(s)ds,\quad -\int_{t-h_2}^{t-h_1}h_{12}\dot{x}^T(s)Z_2\dot{x}(s)ds.$$

For such integral terms, the authors of Ref. [3-4] and [18] firstly divided them into sum of two parts respectively. Each part was then enlarged by the Jensen's integral inequality rather than being enlarged each term by directly the Jensen's integral inequality. Specifically, that is

$$\begin{aligned}-h_{12}\int_{t-h_2}^{t-h_1}x^T(\alpha)Z_4x(\alpha)d\alpha &= -h_{12}\int_{t-h_2}^{t-d(t)}x^T(\alpha)Z_4x(\alpha)d\alpha-h_{12}\int_{t-d(t)}^{t-h_1}x^T(\alpha)Z_4x(\alpha)d\alpha\\ &\le -\frac{h_{12}}{h_2-d(t)}\int_{t-h_2}^{t-d(t)}x^T(s)dsZ_4\int_{t-h_2}^{t-d(t)}x(s)ds-\frac{h_{12}}{d(t)-h_1}\int_{t-d(t)}^{t-h_1}x^T(s)dsZ_4\int_{t-d(t)}^{t-h_1}x(s)ds\end{aligned}$$

not

$$-h_{12}\int_{t-h_2}^{t-h_1}x^T(s)Z_4x(s)ds \le -\int_{t-h_2}^{t-h_1}x^T(s)Z_4\int_{t-h_2}^{t-h_1}x(s).$$

Such approach can reduce conservatism in a way, as claimed in the literature, for example Ref. [17]. Therefore, it is unavoidable that such inconsistent tackle will lead to some conservatism. Differently from aforementioned previous literature, we cope with the term $-\int_{-h_2}^{-h_1}d\theta\int_{t+\theta}^{t}\dot{x}^T(s)R_2\dot{x}(s)ds$ as follows:

$$-\int_{-h_2}^{-h_1}d\theta\int_{t+\theta}^{t}\dot{x}^T(s)R_2\dot{x}(s)ds=-\int_{-h_2}^{-d(t)}d\theta\int_{t+\theta}^{t}\dot{x}^T(s)R_2\dot{x}(s)ds-\int_{-d(t)}^{-h_1}d\theta\int_{t+\theta}^{t}\dot{x}^T(s)R_2\dot{x}(s)ds.$$

Subsequently, we enlarge each part lying on the right-hand side of the above equality by Lemma 2. In short, we surmount the shortcoming iteming from such inconsistent tackle and obtain some less conservative conditions compared with those in the literature.

Remark 2 The conservatism reduction of the proposed criteria derived in this paper is thanks to twofold. The first aspect has been stated in Remark 1; the second one is our modified augmented LKF, where the lower and the upper bounds of the delay and the delay derivative are fully exploited by chosen the terms like

$$\int_{t-h_2}^{t-h_1} x^T(\alpha)Q_2x(\alpha)d\alpha\,,\ \int_{t-d(t)}^{t-h_1} x^T(\alpha)Q_3x(\alpha)d\alpha\,,\ \int_{-h_2}^{-h_1} d\theta\int_{t+\theta}^{t} h_{12}x^T(\alpha)Z_4x(\alpha)d\alpha\,.$$

Numerical examples are to validate such assertion further in section 4.

When the delay derivative may be unknown or does not exist, one can obtain a delay-dependent and delay-rate-independent stability criterion by setting $Q_3=0$ in Theorem 1. That is:

Corollary 1 For given h_1, h_2, the system (1) subject to (2) is globally asymptotically stable if there exist symmetric positive definite matrices with appropriate dimensions $P=[P_{ij}]_{5\times5}$, $Q_i(i=1,2,4,5)$, $Z_j(i=1,2,3,4)$ and $R_k(k=1,2)$ such that for any matrices S_1 and S_2 the following LMIs hold simultaneously

$$\begin{bmatrix} Z_2 & S_1 \\ * & Z_2 \end{bmatrix} \ge 0,\ \begin{bmatrix} Z_4 & S_2 \\ * & Z_4 \end{bmatrix} \ge 0,$$

$$\begin{aligned}&\Omega+\Gamma^T\Upsilon\Gamma+e_1(Q_1+h_1^2Z_3+h_{12}^2Z_4)e_1^T+e_3(Q_2-Q_1)e_3^T-e_4Q_2e_4^T\\&+e_5(Q_5-Q_4)e_5^T-e_6Q_5e_6^T-e_7Z_3e_7^T-e_8Z_4e_8^T-2e_8S_2e_9^T-e_9Z_4e_9^T\\&-(e_3-e_2)Z_2(e_3-e_2)^T-2(e_3-e_2)S_1(e_2-e_4)^T-(e_2-e_4)Z_2(e_2-e_4)^T\\&-(h_1e_1-e_7)R_1(h_1e_1-e_7)^T-(h_{12}e_1-e_8-e_9)R_2(h_{12}e_1-e_8-e_9)^T<0.\end{aligned}$$

and notations Ω, Υ, Γ are defined in Theorem 1.

4. Numerical Examples

In this section, two examples are presented to illustrate the proposed stability criteria in this paper.

Example 1. Consider the system (1) with

$$A=\begin{bmatrix} -2.0 & 0.0 \\ 0.0 & -0.9 \end{bmatrix},\ A_1=\begin{bmatrix} -1.0 & 0.0 \\ -1.0 & -1.0 \end{bmatrix}.$$

For various μ and unknown μ, the allowable upper bound on delay h_2, which ensures the globally asymptotic stability of the system (1) for given lower bound h_1 are listed in Table 1 and Table 2, respectively. From Table 1, it is clear that the proposed method can give much larger h_2 than those in [3, 17, 18] in most of cases.

From Table 2, the proposed criterion is less conservative than those reported in [17-20] except the case when $h_1=1.0$.

Table 1. *Allowable upper bounds h_2 with different h_1 and μ for Example 1.*

h_1	Methods	μ=0.30	μ=0.50	μ=0.90
2.0	[17]	2.6972	2.5048	2.5048
	[18]	3.0129	2.5663	2.5663
	[3]	3.0129	2.6099	2.6099
	Theorem 1	3.0327	2.6137	2.6137
3.0	[17]	3.2591	3.2591	3.2591
	[18]	3.3408	3.3408	3.3408
	[3]	3.3891	3.3891	3.3891
	Theorem 1	3.3912	3.3912	3.3912
4.0	[17]	4.0744	4.0744	4.0744
	[18]	4.1690	4.1690	4.1690
	[3]	4.1978	4.1978	4.1978
	Theorem 1	4.1749	4.1749	4.1749

Table 2. *Allowable upper bounds h_2 with various h_1 and unknown μ for Example 1.*

Methods	h_1	1.0	2.0	3.0	4.0
[17]	h_2	1.8737	2.5049	3.2591	4.0744
[18]	h_2	1.9008	2.5663	3.3408	4.169
[19] (N=1)	h_2	1.9422	2.5383	3.2749	4.0787
[20] (N=1)	h_2	1.9422	2.5383	3.2749	4.0787
Corollary 1	h_2	1.9127	2.5729	3.3517	4.1758

Example 2. Consider the system (1) with

$$A=\begin{bmatrix} 0.0 & 1.0 \\ -1.0 & -2.0 \end{bmatrix},\ A_1=\begin{bmatrix} 0.0 & 0.0 \\ -1.0 & 1.0 \end{bmatrix}$$

For various μ and given lower bound h_1, the allowable upper bound on delay h_2, which guarantees the globally asymptotic stability of the system (1) is listed in Table 3. For $h_1=0$ and various μ, the allowable upper bound on delay h_2, which ensures the globally asymptotic stability of the system (1) can be obtained in Table 4. From Tables 3 and 4, the proposed method in this paper gives larger h_2 in most of cases than those in the literature.

Table 3. *Allowable upper bounds h_2 with various h_1 and $\mu=0.3$ for Example 2.*

Methods	h_1	0.30	0.50	0.80	1.0
[17]	h_2	2.2224	2.2278	2.2388	2.2474
[18]	h_2	2.2634	2.2858	2.3078	2.3167
Theorem 1	h_2	2.2729	2.2932	2.3137	2.3228

Table 4. *Allowable upper bounds h_2 with various μ and $h_1=0$ for Example 2.*

Methods	μ	0.10	0.20	0.50	0.80
[21]	h_2	6.5906	3.6728	1.4118	1.2759
[32]	h_2	7.1480	4.4660	2.3521	1.7682
[31]	h_2	7.1765	4.5438	2.4963	1.9225
Theorem 1	h_2	7.1489	4.4832	2.4970	1.8284

5. Conclusions

In this paper, we develop the delay-range-dependent stability criterion for delayed systems. The Jensen's integral inequality, together with the reciprocally convex lemma, was employed and the derivative of the LKF was estimated more tightly. As a result, a novel stability criterion is derived and as by-product a delay-rate-independent criterion is also obtained. Two numerical examples are provided to substantiate the validity of the proposed method.

Although the proposed stability criteria do not remarkably have reduction in conservativeness, they are significant since there are fewer decisive variables including them and less computational complexity to test the proposed criteria.

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