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Optimal Control of a Class of Parabolic Partial Fractional Differential Equations

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Abstract: In this paper, the existence of the solution of the parabolic partial fractional differential equation is studied and the solution bound estimate which are used to prove the existence of the solution of the optimal control problem in a Banach space is also studied, then apply the classical control theory to parabolic partial differential equation in a bounded domain with boundary problem. An expansion formula for fractional derivative, optimal conditions and a new solution scheme is proposed.

Keywords: Optimal Control, Fractional Order System, Expansion Formula for Fractional Derivative, Parabolic Partial Differential Equations, Functional Analysis, Interior and Neumann Boundary Controls

1. Introduction

In an optimal control problem, one adjusts control in a dynamic system to achieve a goal. The underlying system can have a variety of types of equations such as ordinary differential equations (see [1], [2]), partial differential equations [3], fractional differential equations (see [4], [5] and [6]), stochastic differential equations (see [7], [8], [9] and [10]) or Integra-partial differential equations.

Many processes in physics and engineering are described by systems of equations in which derivatives of arbitrary order appear (not necessarily integer). mention problems of describing behavior of viscoelastic diffusion – wave problems. As a matter of fact, if one wants to include memory effects, i.e., the influence of the part on the behavior of the system at present time one may use fractional derivative to describe such an effect. In principle, there are two different approaches to "fractionalization" of the dynamic of a system (see [11], [12] and [13]). In the Fust procedure, integer order derivatives are simply replaced by derivatives of real order.

In the second approach, considered to be more fundamental from the physical point of view, Functionalization is made on the level of Hamilton's principle (see [14], [15] and [16]).

This approach, however, leads to differential equations of the process involving both left and right fractional and partial fractional derivatives, thus making the effective solution procedure more difficult. For more results concerning fractional calculus and variational principles with fractional derivatives, (see [17], [18], [19] and [20]).

In this paper, considering systems of fractional partial differential equation. denoting A (α, t), B(α, t), π(t) and μ(t) are controls. denoting u(x, t) as the state. The state function u(x, t), satisfy the following partial differential fractional equation:

$$
{}_0D_x^{\beta}\,{}_0D_t^{\alpha}\, u(x,t) - A(\alpha, t)\, D(x, \beta)\, u(x,t) - B(\alpha, t)\, E(x, \beta)\frac{\partial^2 u}{\partial x \partial t} = A(\alpha, t) f(x, \beta)\frac{\partial^2 u}{\partial x^2} \tag{1}
$$

For $(x, t) \in R_T = \{(x, t): x \in \delta = [0, 1]; 0 \le t \le T\}$, A (α, t), B (α, t), E(x, β), D (x, β) and F (x, β) $\in C^1$ [0, 1] with conditions

$$
\frac{\partial u(0,t)}{\partial x} = \pi(t), \quad \frac{\partial u(1,t)}{\partial x} = \mu(t), \qquad u(x,0) = u_0(x) \tag{2}
$$

Such that $\pi(t)$, $\mu(t)$, A (α, t) and B (α, t) are called control parameters.

Note that,

$$ {}_0D_t^{\alpha}\, u(x,t) = \frac{d}{dt}\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{u(x,\tau)}{(t-\tau)^{\alpha}}\, d\tau \quad (3) $$

And,

$$ {}_0D_x^{\beta}\, u(x,t) = \frac{d}{dx}\frac{1}{\Gamma(1-\beta)}\int_0^x \frac{u(y,t)}{(t-y)^{\beta}}\, dy \quad (4) $$

The control variables and the state function affect the goal which is called the objective functional.

$$ J(A,B,\pi,\mu) = \frac{1}{2}\int_0^T [\omega_A A^2(t) + \omega_B B^2(t) + \omega_{\pi}\pi^2(t) + \omega_{\mu}\mu^2(t)]\, dt + \frac{1}{2}\int_{\delta} \omega_u [u(x,T) - u^d(x)]\, dx \quad (5) $$

using k to expand the formula for the left fractional derivative, then prove the existence of the control variables and corresponding state that achieve the maximum (or minimum) of our objective functional in Banach space. This paper is organized in the following way. In section 2 introducing an expansion formula for fractional derivatives. In section 3 studying the solution bounded estimates which are used to show the existence of optimal control.

2. The Expansion Formulas for the Left Fractional Derivative

The expansion formula for fractional derivative, without reference to the distribution theory Let $V_n(g^{(p)})$ (see [21]), $n\in N$, denote the n-th moment of the function $g^{(p)}$, where $g^{(p)}$, $p \in N$ is the p-th derivative of g, i.e.

$$ V_n\, g^{(p)}(x,t) = \int_0^t g^{(p)}(x,\tau)\, \tau^n\, d\tau \quad (6) $$

In the following procedure, it is assumed that $u \in C^2\,[0, b]$ such that u, $u^{(1)}$ are continuous on [0, b] and $u^{(2)} \in L^1(0, b)$. By partial integration in (3)

$$ {}_0D_t^{\alpha}\, u(x,t) = \frac{1}{\Gamma(1-\alpha)} u_0(x) + \frac{1}{\Gamma(2-\alpha)} u_0^{(1)}(x) t^{1-\alpha} + \frac{1}{\Gamma(2-\alpha)}\int_0^t (t-\tau)^{1-\alpha}\, u^{(2)}(x,\tau)\, dt \quad (7) $$

where, $0 < t \le b$.

By using the binomial formula

$$ (1+z)^{\gamma} = \sum_{p=0}^{\infty}\binom{\gamma}{p} z^p = \sum_{p=0}^{\infty}\frac{(-1)^p\,\Gamma(p-\gamma)}{\Gamma(-\gamma)\, p!} z^p, \quad (8) $$

where, $|z| < 1$.

Expression (8), holds also for z = 1 if and only if $\gamma > 1$ and z = -1, $\gamma \neq 0$ if and only of $\gamma > 0$ (see [22]).

Also it is well known that:

$$ \left|\binom{\gamma}{p}\right| \le C\frac{1}{p^{1+\gamma}}, \quad \gamma \neq -1, -2, \ldots \text{ and } p \to \infty \quad (9) $$

From (8), put ($\gamma = 1 - \alpha$) the equation (7) becomes

$$ {}_0D_t^{\alpha} u(x,t) = \frac{1}{\Gamma(1-\alpha)} u_0(x) t^{-\alpha} + \frac{1}{\Gamma(2-\alpha)} u_0^{(1)}(x) t^{1-\alpha} + t^{1-\alpha}\int_0^t u^{(2)}(x,\tau)\sum_{p=1}^{\infty}\left(\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)p!}\left(\frac{\tau}{t}\right)^p\right) d\tau, \quad (10) $$

where, $t > 0$.

Integrate series in (10) term by term. Also, by using the relation

$$ \int_0^t u^{(2)}(x,\tau)\,\tau^p\, d\tau = t^p\, u^{(1)}(t) - p\int_0^t u^{(1)}\,\tau^{p-1}\, d\tau, $$

where $p \ge 1$.

Using equation (10), and by using the equation (6) obtains,

$$ {}_0D_t^{\alpha}\, u(x,t) = \frac{1}{\Gamma(1-\alpha)} u_0(x)\, t^{-\alpha} + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{p=1}^{\infty}\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)\,p!}\left(u^{(1)}(x,t) - \frac{p}{t^p} V_{p-1}(u^{(1)}(x,t))\right) + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} u^{(1)}(x,t), \quad (11) $$

where, $t > 0$.

By integrating by parts in (11) and rearranging the result, obtains

$$ {}_0D_t^{\alpha}\, u(x,t) = u(x,t)\frac{t^{-\alpha}}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(2-\alpha)}\left\{u^{(1)}(x,t)\left[1 + \sum_{p=1}^{\infty}\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)!\,p!}\right] t^{(1-p)} - \sum_{p=2}^{\infty}\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)(p-1)!}\left(\frac{u(x,t)}{t^p} + \frac{\tilde{V}p}{\tau^{p-1+\alpha}}\right)\right\} \quad (12) $$

Where it is established from equation (6)

$$\tilde{V}p = -(p-1)\int_0^t \tau^{p-1} g(x,\tau)\, d\tau,$$
$$where,\ p = 2,3,...$$

Note that the moments $\tilde{V}p$, p = 2, 3,… are solutions to the following system of differential equations.

$$\tilde{V}p\ (x, t) = -(p-1)\, t^{p-2}\, u(x, t),$$

$$\tilde{V}p\ (x, 0) = 0,\ p = 2, 3, \ldots$$

Now the equation (12) become

$$\begin{aligned} {}_0D_t^{\alpha}\, u(x,t) &= A(\alpha,t)\, u(x,t) \\ &\quad + B(\alpha,t)\, u^{(1)}(x,t) - C(\alpha,t) \end{aligned} \tag{13}$$

Such that

$$A(\alpha,t) = t^{-\alpha}\left[\frac{1}{\Gamma(\alpha-1)} - \sum_{p=2}^{\infty}\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)(p-1)!}\right]$$

$$B(\alpha,t) = \left[1 + \sum_{p=1}^{\infty}\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)p!}\right]\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}$$

$$C(\alpha,t) = \sum_{p=2}^{\infty}\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)(p-1)!}\,\frac{\tilde{V}p}{t^{p-1+\alpha}}$$

To keep on the functions B, A, C are continuous functions. Must be using (13) with finite number of terms. (replace ∞ by N) and the equation (13) become

$$\begin{aligned} {}_0D_t^{\alpha}\, u(x,t) &\approx A(\alpha,t)\, u(x,t) \\ &\quad + B(\alpha,t)\, u^{(1)}(x,t) - C(\alpha,t) \end{aligned} \tag{14}$$

Now

$$\begin{aligned} {}_0D_x^{\beta}\ {}_0D_t^{\alpha}\, u(x,t) &\approx A(\alpha,t)\ {}_0D_x^{\beta}\, u(x,t) \\ &\quad + B(\alpha,t)\ {}_0D_x^{\beta}\, u^{(1)}(x,t) \end{aligned}$$

Obviously;

$$\begin{aligned} {}_0D_x^{\beta}\ {}_0D_t^{\alpha}\, u(x,t) &\approx A(\alpha,t)\, D(x,\beta)\, u(x,t) \\ &\quad + A(\alpha,t)\, E(x,\beta)\frac{\partial u(x,t)}{\partial x} \\ &\quad + B(\alpha,t)\, D(x,\beta)\, u^{(1)} \\ &\quad + B(\alpha,t)\, E(x,\beta)\frac{\partial^2 u}{\partial x\partial t} - F(x,\beta) \end{aligned} \tag{15}$$

Such that D, E, A, F ∈ C^1 (0, 1)
Then,

$$\begin{aligned} &{}_0D_x^{\beta}\ {}_0D_t^{\alpha}\, u(x,t) - A(\alpha,t)\, D(x,\beta)\, u(x,t) \\ &\quad - B(\alpha,t)\, E(x,\beta)\frac{\partial^2 u}{\partial x\partial t} = \\ &\quad A(\alpha,t)\, E(x,\beta)\frac{\partial u}{\partial x} \\ &\quad + B(\alpha,t)\, D(x,\beta)\, u^{(1)} - F(x,\beta) \end{aligned} \tag{16}$$

From equation (1) the equation (16) becomes

$$\begin{aligned} &A(\alpha,t)\, E(x,\beta)\frac{\partial u}{\partial x} + B(\alpha,t)\, D(x,\beta)\, u^{(1)} \\ &\quad - F(x,\beta) = A(\alpha,t)\, f(x,\beta)\frac{\partial^2 u}{\partial x^2} \end{aligned} \tag{17}$$

Such that;

$$D(x,\beta) = x^{-\beta}\left[\frac{1}{\Gamma(1-\beta)} - \sum_{p=2}^{N}\frac{\Gamma(P-1+\beta)}{\Gamma(\beta-1)(P-1)!}\right]$$

$$E(x,\beta) = \left[1 + \sum_{p=1}^{N}\frac{\Gamma(P-1+\alpha)}{\Gamma(\alpha-1)P!}\right]\frac{x^{1-\alpha}}{\Gamma(2-\alpha)}$$

$$F(x,\beta) = \sum_{p=2}^{N}\frac{\Gamma(P-1+\beta)}{\Gamma(\beta-1)(P-1)!}\,\frac{\widehat{M}_p}{x^{P-1+\beta}}$$

Where it is established from equation (6),

$$\widehat{M}_P = -(P-1)\int_0^x y^{P-1}\, g(y,t)\, dy, \qquad p = 2,3,...$$

Note that; $\widehat{M}_P$ is a solution to the following system of differential equations

$$\frac{\partial}{\partial x}\widehat{M}_P\ (x, t) = -(P-1)\, x^{P-2}\, u(x,t),$$

$$(o, t) = 0,\ P = 2, 3,\ldots\ \widehat{M}_P$$

Now the equation (17) can be write on the form

$$\begin{aligned} \frac{\partial u}{\partial t} &= \ell(\alpha,t)\, N(x,\beta)\frac{\partial u}{\partial x} \\ &\quad + \ell(\alpha,t)\, Z(x,\beta)\frac{\partial^2 u}{\partial x^2} \\ &\quad + K(x,\beta)\, H(\alpha,t) \end{aligned}$$

Such that;

$$\ell(\alpha, t) = \frac{A(\alpha, t)}{B(\alpha, t)}$$

$$N(x, \beta) = \frac{E(x, \beta)}{D(x, \beta)}$$

$$Z(x, \beta) = \frac{f(x, \beta)}{D(x, \beta)}$$

$$K(x, \beta) = \frac{F(x, \beta)}{D(x, \beta)}, \quad H(\alpha, t) = \frac{1}{\beta(\alpha, t)}$$

Suppose that

$$N(x, \beta) = \frac{\partial}{\partial x} Z(x, \beta)$$

Then

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\ell(\alpha, t) Z(x, \beta) \frac{\partial u}{\partial x}\right) + K(x, \beta)\, H(\alpha, t)$$

noting that the dynamic system in (1) replace to the following class of linear parabolic partial differential equations.

$$\frac{\partial u}{\partial t}(x, t) = \frac{\partial}{\partial x}\left(\ell(\alpha, t) Z(x, \beta) \frac{\partial u}{\partial x}\right) + K(x, \beta) H(\alpha, t) R_T \tag{18}$$

For $(x, t) \in R_T$, $\{(x, t): x \in \delta = [0, 1]; 0 < t < T\}$ with condition

$$\frac{\partial u}{\partial x}(0, t) = \pi(t), \quad \frac{\partial u}{\partial x}(1, t) = \mu(t)$$

$$u(x, 0) = u0(x)$$

Where $K(x, \beta)$ are positive geometry parameters and the controls $H(\alpha, t)$, $\ell(\alpha, t)$, $\pi(t)$ and $\mu(t)$ represent the diffusivity, interior and (Neumann) boundary controls. (see [23], [24], [25] and [26])

3. Existence of Time Optimal Control for a Class of Partial Fractional Differential Equations

Consider the control system in (18) and the state function u(x, t) affect the goal, which is called the objective functional

$$J(\pi, \mu, A, B) = \frac{1}{2}\int_0^T \omega_\gamma \gamma^2(t) + \omega_\mu \mu^2(t) + \omega_A A^2(\alpha, t) + \omega_B B^2(\alpha, t) + \frac{1}{2}\int_\delta \omega_u \left|u(x, T) - u^d(x)\right|^2 dx$$

Now seeking to prove the existence of the control variable and corresponding state that achieve the maximum (or minimum) of our objective functional in Banach space.

Let the optimal controls $\lambda = (\pi, \mu, A, B)^T$ as follows:

Let {γn}, {μn}, {An}, and {Bn} be four minimizing sequences, such that

$$\lim_{n\to\infty} J(\pi_n, \mu_n, A_n, B_n) = \inf_{\gamma,\mu,A,B} J(\pi, \mu, A, B)$$

Assume that un = u (γn, μn, An, Bn) be corresponding a solution of system (18)

This system existence and satisfies the bound estimate

$$\int_{R_T}\left(|u_n|^2 + \left|\frac{\partial u_n}{\partial x}\right|^2 + \left|\frac{\partial u_n}{\partial t}\right|^2\right) dx\, dt \le M_{2'}$$

Where M2 is a constant independent of n (see [27] and [28]). By the week convergence theory [29], can extract weakly convergent sequences.

$$\frac{\partial u_n}{\partial t} \to \frac{\partial u}{\partial t}, \quad u_n \to u,$$

$$(\pi n, \mu n, An, Bn) \to (\pi, \mu, A, B)$$

Weakly in L2 (0, T; (H2 (δ))\), L2 (0, T; H2 (δ)) and L∞ (0, T) respectively. Then the sequence {un} admits a subsequence which converges strongly in (L2 (δ)).

Therefore it possible to show the existence of the optimal controls,

$$J(\lambda^*) \le \inf_n \int_\delta u_n(x, T)\, dx + \inf_n \int_0^T \left(\omega_\gamma \gamma_n^2 + \omega_\mu \mu_n^2 + \omega_A A_n^2 + \omega_B B_n^2\right) dt \le \lim_{n\to\infty} \inf_n J(\lambda_n)$$

4. Conclusion

In this paper, explaining the formula for fractional derivative, introducing a new solution scheme for the partial fractional optimal control.

This paper is organized in the following way. In section 2 introduced an expansion formula for fractional derivatives. In section 3 studied the existence of the solution of the parabolic partial differential equation and studied the solution bounded estimate of the optimal control problem in a Banach space.

5. New Research

We will work on deducing necessary conditions for a state / control / terminal time triplet to be optimal in stochastic fractional optimal control problems, such that the dynamic control system involves stochastic and fractional order derivative and the terminal time is free or constant. We will do that by solving typical problems using these conditions.

References

[1] A. Debbouche and M. M. El-Borai, "Weak almost periodic and optimal mild solutions of fractional evolution equations". Electronic Journal of Differential Equations, vol. 2009, pp. 1-8, 2009.

[2] M. M. El-Borai, "Some probability densities and fundamental solutions of fractional evolution equations". Chaos, Solitons & Fractals, vol. 14, pp. 433-440, 2002.

[3] M. M. El-Borai, K. E. S. El-Nadi, and E. G. El-Akabawy, "On some fractional evolution equations". Computers and mathematics with applications, vol. 59, pp. 1352-1355, 2010.

[4] O. P. Agrawal, A general formulation and solution scheme for fractional optimal control problems. Nonlinear Dynam. 38 (2004), no. 1-4, 323-337.

[5] O. P. Agrawal, A formulation and numerical scheme for fractional optimal control problems. J. Vib. Control 14 (2008), no. 9-10, 1291-1299.

[6] O. P. Agrawal, O. Defterli and D. Baleanu, Dumitru Fractional optimal control problems with several state and control variables. J. Vib. Control 16 (2010), no. 13, 1967-1976.

[7] G. S. F. Frederico and D. F. M. Torres, Fractional optimal control in the sense of Coputo and the fractional Noether's theorem. Int. Math. Forum 3 (2008), no. 9-12, 479-493.

[8] G. S. F. Frederico and D. F. M. Torres. Fractional conservation laws in optimal control theory, Nonlinear Dynam. 53 (2008), no. 3, 215-222.

[9] C. Tricaud and Y. Chen. An approximate method for numerically solving fractional order optimal control problems of general form. Comput. Math. Appl. 59 (2010), no. 5, 1644-1655.

[10] C. Tricaud and Y. Chen. Time Optimal Control of Systems with Fractional Dynamics, Int. J. Differ. Equ. Appl., Volume 2010 (2010). Article ID 461048, 16 pages.

[11] M. M. El-Borai, W. G. Elsayed and R. M. Al-Masroub, Exact Solutions for Some Nonlinear Fractional Parabolic Equations, Inter. J. Adv. Eng. Res. (IJAER), vol. 10, No. III, Sep. 2015, 106-122.

[12] M. M. El-Borai, W. G. Elsayed and F. N. Ghaffoori, On the Cauchy Problem for Some Parabolic Fractional Partial Differential Equations with Time Delays, J. Math. & System Scie. 6(2016), 194-199.

[13] M. M. El-Borai, W. G. Elsayed and R. M. Al-Masroub, Exact Solutions for Some Nonlinear Partial Differential Equations via Extended (G'/G) – Expansion Method, Inter. J. Math. Trends and Tech. (IJMTT) – Vol. 36, No. 1-Aug 2016, 60-71.

[14] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo. Theory and Applications of Fractional Differential Equations. Elsevier, North-Holland Mathematics Studies, 2006, 204.

[15] M. M. El-Borai, W. G. Elsayed and M. Taha, On The Fractional Optimal Control Problem wit free End Point, American Journal of Theoretical and Applied Statistics, ISSN: 2326-8999; (2017).

[16] Zoran D, Nebosa Petrovacki, Optimality condition and a solution scheme for fractional optimal control problems, Struct. Multidisc Optim. 10.1007/s00158-008-0307-7, (2009)

[17] Atanackovic TM, Stankovic B (2004) An expansion formula for fractional derivatives and its applications. Fractional Calculus and Applied Analysis 7(3): 365–378.

[18] Atanackovic TM, Stankovic B (2007a) On a class of differential equations with left and right fractional derivatives. ZAMM, Z Angew Math Mech 87: 537–546.

[19] Atanackovic TM, Stankovic B (2007b) On a differential equation with left and right fractional derivatives. Fractional Calculus Applied Analysis 10: 138–150.

[20] Mahmoud M. El-borai, M. A. Abdou, E. M. Youssef, On some Approximate analytical solution for mathematical model of carcinogenesis using Adomian decomposition method.

[21] Dummit, D. S., Statistics and probability, Prentice Hall, John Wiley & Sons, Hoboken, NJ.

[22] Knapp, W.,' Advanced Real Analysis, Birkhauser', Boston, 2005.

[23] Yuan L, Agrawal OP (2002) A numerical scheme for dynamic systems containing fractional derivatives. J Vib Acoust 124: 321–324.

[24] M. El-Borai, K. El-Nadi, O. Mostafa, and H. Ahmed, "Numerical methods for some nonlinear stochastic differential equations," JOURNAL-KOREA SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, vol. 9, p. 79, 2005.

[25] M. M. El-Borai, "The fundamental solutions for fractional evolution equations of parabolic type," International Journal of Stochastic Analysis, vol. 2004, pp. 197-211, 2004.

[26] M. M. El-Borai, "On some fractional differential equations in the Hilbert space," Discrete and Continuous Dynamical Systems. Series A, pp. 233-240, 2005.

[27] M. M. El-Borai, A. El-Banna, and W. H. Ahmed, "Optimal Control of a Class of Parabolic Partial Differential Equations," International Journal of Advanced Computing, vol. 36, 2013.

[28] M. M. El-Borai, A.-Z. H. El-Banna, and W. H. Ahmed, "On Some Fractional-Integro Partial Differential Equations," International Journal of Basic & Applied Sciences, vol. 13, 2013.

[29] Gelfand, I. M.; Fomin, S. V. Silverman, Richard A., Calculus of variations. Mineola, New York: p. 3. ISBN 978-0486414485. (2000).

[30] Knapp, A. W., Advanced Real Analysis, Birkhauser, Boston, 2005.