**American Journal of Theoretical and Applied Statistics**
2017; 6(5-1): 46-50
http://www.sciencepublishinggroup.com/j/ajtas
doi: 10.11648/j.ajtas.s.2017060501.17
ISSN: 2326-8999 (Print); ISSN: 2326-9006 (Online)

# On The Fractional Optimal Control Problem with Free End Point

**Mahmoud M. El-borai[1], Wagdy G. ElSayed[1], M. A. Abdou[2], M. Taha E.[2]**

[1]Faculty of Science, Alexandria University, Alexandria, Egypt
[2]Faculty of Education, Alexandria University, Alexandria, Egypt

### Email address:
m_m_elborai@yahoo.com (M. M. El-borai), Wagdygoma@alexu.edu.eg (W. G. ElSayed), abdella_777@yahoo.com (M. A. Abdou), eslam.math86@yahoo.com (M. Taha E.)

### To cite this article:
Mahmoud M. El-borai, Wagdy G. ElSayed, M. A. Abdou, M. Taha E. On The Fractional Optimal Control Problem with Free End Point. *American Journal of Theoretical and Applied Statistics.* Special Issue: Statistical Distributions and Modeling in Applied Mathematics. Vol. 6, No. 5-1, 2017, pp. 46-50. doi: 10.11648/j.ajtas.s.2017060501.17

**Received**: March 15, 2017; **Accepted**: March 16, 2017; **Published**: April 11, 2017

**Abstract:** We present a necessary optimality conditions for a class of optimal control problems. The dynamical control system involves integer and fractional order derivatives and the final time is free. Optimality conditions are obtained. Feedback control laws for linear dynamic system are obtained.

**Keywords:** Optimal Control, Fractional Differential Equation, Free Time, Lagrange Multipliers and Feedback Control

## 1. Introduction

An optimal control problem consists in the finding of control signals that make a system satisfy certain constraints while an objective functional is optimized.

In a fractional optimal control problem, at least one fractional order derivative is present in the formulation of the problem.

There is a growing interest to the modeling of physical phenomena in terms of fractional operators [Sec, e.g. 1, 2, 3, 4, 5] and another modeling of phys-phenomena in terms of integer and fractional operators [see, e.g., 6, 7, 8, 9].

This gives us the sight that, sooner or later, such problems will appear in our real-world life. We inter some basic concepts on fractional calculus. We may say that the fractional calculus is the integral and differential calculus of real order. Now we will define,

$$ {}_aD_t^{\alpha}\, x(t) \;=\; \frac{1}{\Gamma(n-\alpha)}\,\frac{d^n}{dt^n}\int_a^t \frac{x(\tau)\,dt}{(t-\tau)^{1+\alpha-n}} $$

$$ {}_tD_b^{\alpha}\, x(t) \;=\; \frac{1}{\Gamma(n-\alpha)}\,\frac{d^n}{dt^n}\int_t^b \frac{x(\tau)\,dt}{(t-\tau)^{1+\alpha-n}} $$

Where; $n-1 \le \alpha < n$, and $\Gamma$ is the gamma function.

The aim of this paper is to determine the necessary conditions for extreme of the functional of the form:

$$ J(x(t),u(t),t) = \int_a^T L(x(t),u(t),t)\,dt \;+\; \phi(x(T),T) \qquad (1) $$

Subject to dynamic constraints on the form

$$ {}_aD_t^{\alpha_p}\, x(t) \;=\; f\left(x(t),\ {}_aD_t^{\alpha_i}\, x(t),\ \ldots\ {}_aD_t^{\alpha_{p-1}}\, x(t),\, u(t),\, t\right) \qquad (2) $$

$$ x(a) = x_a, \qquad (3) $$

*Where* $0 < \alpha < 1$, $1 < \alpha_2 < 2, \ldots\ p-1 < \alpha_p \le p$, and $x \in R^n$, $u \in R^m$ are the state and control vectors respectively, and p is an n-dimension column vector. It is assumed that L is a differentiable function on the domain $[a,\infty[\times\mathbb{R}^2$ and $\phi$ is a differentiable function on the domain $[a,\infty[\times\mathbb{R}$. We assume that T is a variable number with $a < T < \infty$. We can say that our problem result from the basic functional:

$$ J\,(x(t),u(t),t) = \int_a^b L(x(t),u(t),t)\,dt $$

$$ = \int_a^{T-\varepsilon} L(x(t),u(t),t)\,dt \;+ $$

$$\int_{T+\varepsilon}^{b} L(x(t), u(t), t)\, dt + \varphi (T, x(T))$$

Where ε is a very small positive constant and a < T <b.

1. If ε tends to zero, we can say that the objective functional J (x(t), u(t),t) will be functional with fixed ends point and this problem has already been studied for different type of optimal control problems (see. e.g. [7], [10]).
2. If the objective functional J vanishes after period equal T then the objective functional J will be functional with free end point.

# 2. Fractional Necessary Conditions

The constraints of the problem are handled by introducing additional variable called Lagrange multipliers λ to define an augmented cost functional.

$$\begin{aligned} J [x(t), t, u(t)] = \int_a^T \Big[ & H (x(t), {}_aD_t^{\alpha} x(t), \ldots\ldots, \\ & {}_aD_t^{\alpha_p} x(t), \lambda(t), u(t), t) - \lambda f (x(t), \\ & {}_aD_t^{\alpha_1} x(t), \ldots\ldots, {}_aD_t^{\alpha_{p-1}} x(t))\Big]\, dt \\ & + \varphi (x(T), T), \end{aligned} \tag{4}$$

Such that

$$H(x(t), {}_aD_t^{\alpha} x(t), \ldots {}_aD_t^{\alpha_p} x(t), \lambda(t), u(t), t)$$

$$= L(x(t), u(t), t) + \lambda^T(t)$$

$f(x(t), {}_aD_t^{\alpha_1} x(t), \ldots\ldots, {}_aD_t^{\alpha_{p-1}} x(t))$ is the Hamiltonian function and $H(x(t), D_{a+}^{\alpha_1} x(t), \ldots, D_{a+}^{\alpha_p} x(t), u(t), t)$ is a function with continuous first and second partial derivatives with respect to all its arguments. It consists of all functions $x(t)$ which have continuous derivatives up to order n-1 on $[a,\infty[$ with $x^{n-1}$ absolutely continuous function. Now we consider variations $x + \delta x$, $u + \delta u$, $\lambda + \delta\lambda$ and $T + \delta T$, with $\delta x(a) = 0$ by the imposed boundary condition (3). Using the well known fact that the first variation of J must vanish when evaluated along a minimizer, we get

$$\begin{aligned} 0 = \int_a^T \Bigg[ & \frac{\partial H}{\partial x} \delta x(t) + \frac{\partial H}{\partial\, {}_aD_t^{\alpha_1} x}\, {}_aD_t^{\alpha_1} \delta x(t) + \ldots\ldots \\ & + \frac{\partial H}{\partial\, {}_aD_t^{\alpha_p} x}\, {}_aD_t^{\alpha_p} \delta x(t) + \frac{\partial H}{\partial u} \delta u(t) + \frac{\partial H}{\partial \lambda} \delta\lambda(t) \\ & - \lambda (t)\, {}_aD_t^{\alpha_p} x(t) - \delta\lambda(t)\, {}_aD_t^{\alpha_p} x(t)\Bigg]\, dt \\ & + \delta T \Big[H - \lambda\, {}_aD_T^{\alpha_p} x(t)\Big]_{t=T} + \frac{\partial \varphi}{\partial t} (T, x(T))\, \delta T \\ & + \frac{\partial \varphi}{\partial t} (T, x(T))\, \dot{x}\, \delta T \end{aligned} \tag{5}$$

With the partial derivatives of H at (x(t),u(t), λ(t)).

Integration by parts gives these relations (see [11], [12])

$$\int_a^T \lambda(t)\, {}_aD_t^{\alpha_p} \delta x(t)\, dt = \int_a^T \delta x(t)\, {}_tD_T^{\alpha_p} \lambda(t)\, dt,$$

*and* (6)

$$\int_a^T \frac{\partial H}{\partial\, {}_aD_t^{\alpha_p} x}\, {}_aD_t^{\alpha_p} \delta x(t)\, dt = \int_a^T {}_tD_T^{\alpha_p} \frac{\partial H}{\partial\, {}_aD_t^{\alpha_p} x} \delta x(t) dt$$

We get;

$$\begin{aligned} 0 = \int_a^T \Bigg\{ & \frac{\partial H}{\partial x} \delta x(t) + {}_tD_T^{\alpha_1} \left(\frac{\partial H}{\partial\, {}_aD_t^{\alpha_1} x}\right) \delta x \\ & + {}_tD_T^{\alpha_2} \left(\frac{\partial H}{\partial\, {}_aD_t^{\alpha_2} x}\right) \delta x + \ldots \\ & {}_tD_T^{\alpha_p} \left(\frac{\partial H}{\partial\, {}_aD_t^{\alpha_p} x}\right) \delta x(t) + \frac{\partial H}{\partial u} \delta u + \frac{\partial H}{\partial \lambda} \delta\lambda \\ & - \Big[\delta\lambda (t)\, {}_aD_t^{\alpha_p} x(t) + \delta x(t)\ \ {}_tD_T^{\alpha_p} \lambda (t)\Big] \Bigg\}\, dt \\ & + \delta T \left[H - \lambda\, {}_aD_T^{\alpha_p} x(t) + \frac{\partial \phi}{\partial t}(x(t),t) + \frac{\partial \phi}{\partial x}(x(t),t)\dot{x}\right]_{t=T} \end{aligned} \tag{7}$$

Then the necessary conditions can be shown to be:

$$\frac{\partial H}{\partial u} = 0, \tag{8}$$

$$\frac{\partial H}{\partial \lambda} = {}_aD_t^{\alpha_p} x(t) \tag{9}$$

$$\begin{aligned} \frac{\partial H}{\partial x} = {}_tD_T\, \lambda(t) - \Bigg[ & {}_tD_T^{\alpha_1} \frac{\partial H}{{}_aD_t^{\alpha_1} x} + {}_tD_T^{\alpha_2} \left(\frac{\partial H}{{}_aD_t^{\alpha_2} x}\right) \\ & + \ldots\ldots + {}_tD_T^{\alpha_p} \left(\frac{\partial H}{{}_aD_t^{\alpha_p} x}\right)\Bigg] \end{aligned} \tag{10}$$

since variation functions where chosen arbitrarily, The following condition result from T is a free time, which is called the transversely condition

$$\left[H - \lambda\, {}_aD_t^{\alpha_p} x(t) + \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x} (x(t),t)\, \dot{x}\right]_{t=T} = 0, \tag{11}$$

and this condition is vanished if T is fixed.

We can say that the system [8]–[11] is called the Hamiltonian system.

# 3. Generalizations of the Functional Conditions

In this section we will dealing with the free terminal time on the dynamical control system involving integer and fractional order derivatives.

Using the performance functional (2) subject to the new control system

$$\left(M\dot{x}(t) + {}_aD_t^{\alpha_p} x(t) = f(x(t), {}_aD_t^{\alpha_1} x(t), \ldots {}_aD_t^{\alpha_{p-1}} x(t), u(t), t\right) \quad (12)$$

Where the initial condition $x(a) = x_a$, and $0 < \alpha < 1$, $1 < \alpha_2 < 2, \ldots$ $p-1 < \alpha_p \le p$, with $M \neq 0$ is a fixed real number. Our goal is generalizing the previous works on fractional optimal control problems by considering the end time T free and the dynamical control system (12) involving integer and fractional order derivatives. For convenience, we consider the one-dimensional case. However, using similar techniques, the results can be easily extended to problems with multiple states and multiple controls.

We assume that the state variable x is differentiable and that the control u is piecewise continuous, the case M = 0 with fixed T has already been studied for different types of fractional order derivatives (see, e.g., [2, 5, 13, 14]. In [15] a special type of the proposed problem is also studied for fixed T.

By using (1), (3), (12) and the Hamiltonian function H, we can rewrite the initial problem as minimizing

$$J[x, u, T, \lambda] = \int_a^T \Big[H(x(t), {}_aD_t^{\alpha} x(t), \ldots, {}_aD_t^{\alpha_p} x(t), \lambda(t), t) - \lambda(t)(M\dot{x}(t) + {}_aD_t^{\alpha_p} x(t))\Big]\, dt + \varphi(T, x(T))$$

Using the previous variations, the integration by parts in (6), and the following relation

$$\int_a^T \lambda(t)\, \delta\dot{x}(t)\, dt = -\int_a^T \delta x(t)\, \dot{\lambda}(t)\, dt + \delta x(T)\, \lambda(T),$$

we get

$$\begin{aligned}
0 = \int_a^T \Big[&\frac{\partial H}{\partial x}\, \delta x(t) + {}_aD_t^{\alpha_1}\left(\frac{\partial H}{\partial\, {}_aD_t^{\alpha_1} x}\right)\delta x \\
&+ {}_tD_T^{\alpha_2}\left(\frac{\partial H}{\partial\, {}_aD_t^{\alpha_2} x}\right)\delta x(t) + \ldots \\
&+ {}_tD_T^{\alpha_p}\left(\frac{\partial H}{\partial\, {}_aD_t^{\alpha_p} x}\right)\delta x(t) + \frac{\partial H}{\partial u}\,\delta u + \frac{\partial H}{\partial \lambda}\,\delta\lambda \\
&- \delta\lambda\,(M\dot{x}(t) + {}_aD_t^{\alpha_p} x(t)) - M\dot{\lambda}(t)\,\delta x(t) \\
&- {}_tD_T^{\alpha_p}\,\lambda(t)\,\delta x(t)\Big]\, dt + \delta T\left[H - \lambda(t)(M\dot{x}(t) + {}_aD_T^{\alpha_p} x(t)\right]_{t=T} \\
&+ M\delta x(T)\,\lambda(T) + \frac{\partial\phi(T, x(T))}{\partial t}\,\delta T + \frac{\partial\phi(T, x(T))}{\partial t}\,\dot{x}(t)\delta T
\end{aligned}$$

Now, define the new variable

$$\delta x^T = [x + \delta x](T + \delta T) - x(T).$$

By using Taylor's theorem.

$$[x + \delta x](T + \delta T) - [x + \delta x](T)$$

$= \dot{x}(T)\,\delta T + O(\delta T^2)$, Where $\lim_{\zeta \to 0} \dfrac{O(\zeta)}{\zeta}$ is finite. And so $\delta x(T) = \delta x^T - \dot{x}(T)\,\delta T$. In conclusion, we arrive at the expression

$$\begin{aligned}
&\delta T\left[H(t,x,u,\lambda) - \lambda(t)\, {}_aD_t^{\alpha_p} x(t) + \frac{\partial\varphi}{\partial x}\dot{x} + \frac{\partial\varphi}{\partial t}(t, x(t))\right]_{t=T} \\
&+ \int_a^T \left[\left(\frac{\partial H}{\partial x} + {}_tD_T^{\alpha_1}\left(\frac{\partial H}{\partial\, {}_aD_t^{\alpha_1} x}\right) + {}_tD_T^{\alpha_2}\left(\frac{\partial H}{\partial\, {}_aD_t^{\alpha_2} x}\right) + {}_tD_T^{\alpha_p}\left(\frac{\partial H}{\partial\, {}_aD_t^{\alpha_p} x}\right) + M\dot{\lambda}(t) - {}_tD_T^{\alpha_p}\,\lambda(t)\right)\delta x \right. \\
&\qquad \left. + \delta\lambda\left(\frac{\partial H}{\partial \lambda} - M\dot{x}(t) - {}_aD_t^{\alpha_p}\, x(t)\right)\right] dt \\
&- \delta x_T\, [M\lambda(t)]_{t=T} = 0
\end{aligned}$$

Now, we define a Hamiltonian system as:

$$\begin{cases}
{}_tD_T^{\alpha}\,\lambda(t) - M\dot{\lambda}(t) = \dfrac{\partial H}{\partial x} + {}_tD_T^{\alpha_1}\left(\dfrac{\partial H}{\partial\, {}_aD_t^{\alpha_1} x}\right) \\
\quad + {}_tD_T^{\alpha_2}\left(\dfrac{\partial H}{\partial\, {}_aD_t^{\alpha_2} x}\right) + \\
\quad \ldots\ {}_tD_T^{\alpha_p}\left(\dfrac{\partial H}{\partial\, {}_aD_t^{\alpha_p} x}\right), \\
M\dot{x}(t) + {}_tD_T^{\alpha}\, x(t) = \dfrac{\partial H}{\partial \lambda}(t, x(t), u(t), \lambda(t)),
\end{cases} \quad (13)$$

for all $t \in [a, T]$,
the stationary condition

$$\frac{\partial H}{\partial u} = 0 \quad (14)$$

For all $t \in [a, T]$,
And the transversely conditions (if T and x(T) are free)

$$[H - \lambda(t)\, {}_aD_t^{\alpha_p} x(t) + \frac{\partial\phi}{\partial x}\,\dot{x}(t) + \frac{\partial\phi}{\partial x}(t, x(t))]_{t=T} = 0, \quad (15)$$

$$[M\,\lambda(t)]_{t=T} = 0. \quad (16)$$

A few Remarks:

1. If T and *x(T)* are constant there is no transversely conditions.
2. If T is constant only there is no transversely condition in (15) and the transversely condition become

$$[M\,\lambda(t)]_{t=T} = 0.$$

3. If x(T) is constant only there is no transversely

condition in (16) and the transversely condition become

4. $\left[ \mathrm{H} - \lambda \ {}_a D_t^{\alpha_p} x(t) + \frac{\partial \varphi}{\partial t} \dot{x}(t) + \frac{\partial \varphi}{\partial x}(t, x(t)) \right]_{t=T} = 0$

we note that when we tacked a new minimization on x ($\delta x^T$) the transversely condition in (15) become itself transversely condition in (11), and this means that after this minimization the integer order derivative in the dynamical control system dose not effect on the transversely condition.

# 4. Linear Feedback

Feedback control laws can be found for a linear dynamic systems, where a quadratic performance index J is to minimized with free end final time T (see [16], [17]). The statement of the problem is to determine u(t) that takes the system from an initial state $x_a$ at time a to an unspecified point at time T. The path during a and T must minimize the quadratic functional.

$$\tilde{J} = \frac{1}{2} \int_a^T (x^T Qx + u^T Ru)\, dt\,, \tag{17}$$

Where Q is an (n ×n) symmetric positive semi-definite matrix and R is an (m × m) symmetric and positive definite matrix, subject to

$${}_a D_t^{\alpha_1} x(t) = Ax + Bu, \tag{18}$$

Where A is an (n × n) matrix of real numbers, x is an (n × 1) state vector, B is an (n × m) matrix and u is an (m × 1) control vector.

Now the necessary conditions for optimality becomes

$$\left( \frac{\partial H}{\partial x} \right)^T = {}_t D_T^{\alpha_1} \lambda(t)\,, \tag{19}$$

$$\left( \frac{\partial H}{\partial u} \right)^T = 0\,, \tag{20}$$

$$\left( \frac{\partial H}{\partial \lambda} \right)^T = {}_a D_t^{\alpha_1} x(t) \tag{21}$$

The solution of the problem (17) and (18) can be obtained using equations (19) - (21).

Where

$$H = \frac{1}{2} x^T Qx + \frac{1}{2} u^T Ru + \lambda^T (Ax + Bu).$$

Solving (19), we get

$$Qx + A^T \lambda = {}_t D_T^{\alpha_1} \lambda(t). \tag{22}$$

Then

$$\lambda(t) = \frac{1}{\Gamma(\alpha_1)} \int_t^T (\theta\text{-}t)^{\alpha_1 - 1} (A^T \lambda(t) + Qx(\theta))\, d\theta \tag{23}$$

We apply the method of successive approximations to solve the integral equation (22). To do this we set

$$\lambda_{n+1}(t) = \frac{1}{\Gamma(\alpha_1)} \int_t^T \left(\theta - t\right)^{\alpha_1 - 1} A^T \lambda_n(\theta)\, d\theta + \frac{1}{\Gamma(\alpha_1)} \int_t^T \theta - t)^{\alpha_1 - 1} Qx\,(\theta)\, d\theta \Big)$$

The zero approximation $\lambda_0(t)$ is taken to be zero. Thus it is easy to see that

$$\left\| \lambda_{n+1}(t) \text{-} \lambda_n(t) \right\| \le \frac{\left\| A^T \right\|^n \left\| Qx\,(\theta) \right\|}{(\Gamma(\alpha_1))^{n-1}\ \Gamma(2\alpha_1 +)} (T\text{-}t)^{(n+1)\alpha_1}$$

Where $\|.\|$ is a suitable norm of the matrix. Thus the required solution of equation (22) is given by

$$\lambda(t) = \sum_{n=0}^{\infty} (\lambda_{n+1}(t) - \lambda_n(t))\,.$$

From the equation (20) the feedback control low for this control problem can be written as

# 5. Conclusion

This paper is devoted to discussion some aspects of the fractional order optimal control problems in which the dynamic control system involves integer partial and fractional order derivative with free/ constant terminal time. Necessary conditions for a state / control / terminal time triplet to be optimal are obtained. Linear feedback control laws are obtained.

# Acknowledgments

We would like to thank the referees for their careful reading of the paper and their valiable comments.

# New Research

We will work on deducing necessary conditions for a state / control / terminal time triplet to be optimal in stochastic fractional optimal control problems, such that the dynamic control system involves stochastic and fractional order derivative and the terminal time is free or constant. We will do that by solving typical problems using these conditions.

# References

[1] A. Debbouche and M. M. El-Borai, "Weak almost periodic and optimal mild solutions of fractional evolution equations". Electronic Journal of Differential Equations, vol. 2009, pp. 1-8, (2009).

[2] Mahmoud M. El-Borai, "Some probability densities and fundamental solutions of fractional evolution equations". Chaos, Solitons & Fractals, vol. 14, pp. 433-440, (2002).

[3] Mahmoud. M. El-Borai, Khairia El-Said El-Nadi, and E. G. El-Akabawy, "On some fractional evolution equations". Computers and mathematics with applications, vol. 59, pp. 1352-1355, (2010).

[4] O. P. Agrawal, A general formulation and solution scheme for fractional optimal control problems. Nonlinear Dynam. 38 (2004), no. 1-4, 323-337.

[5] Mahmoud M. El-Borai, W. G. Elsayed and F. N. Ghaffoori, On the Cauchy Problem for Some Parabolic Fractional Partial Differential Equations with Time Delays, J. Math. & System Scie. 6 (2016), 194-199.

[6] Mahmoud M. El-Borai, W. G. Elsayed and R. M. Al-Masroub, Exact Solutions for Some Nonlinear Partial Differential Equations via Extended (G'/G) – Expansion Method, Inter. J. Math. Trends and Tech. (IJMTT) – Vol. 36, No. 1-Aug (2016), 60-71.

[7] O. P. Agrawal, A formulation and numerical scheme for fractional optimal control problems. J. Vib. Control 14 (2008), no. 9-10, 1291-1299.

[8] O. P. Agrawal, O. Defterli and D. Baleanu, Dumitru Fractional optimal control problems with several state and control variables. J. Vib. Control 16 (2010), no. 13, 1967-1976.

[9] G. S. F. Frederico and D. F. M. Torres, Fractional optimal control in the sense of Coputo and the fractional Noether's theorem. Int. Math. Forum 3 (2008), no. 9-12, 479-493.

[10] G. S. F. Frederico and D. F. M. Torres. Fractional conservation laws in optimal control theory, Nonlinear Dynam. 53 (2008), no. 3, 215-222.

[11] C. Tricaud and Y. Chen. An approximate method for numerically solving fractional order optimal control problems of general form. Comput. Math. Appl. 59 (2010), no. 5, 1644-1655.

[12] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo. Theory and Applications of Fractional Differential Equations. Elsevier, North-Holland Mathematics Studies, (2006), p: 204.

[13] S. G. Samko, A. A. Kibas O. I. Marichev. Fractional Integrals and Derivatives. Gordon and Breach Science Publishers, (2004).

[14] C. Tricaud and Y. Chen. Time Optimal Control of Systems with Fractional Dynamics, Int. J. Differ. Equ. Appl., Volume 2010 (2010). Article ID 461048.

[15] Z. D. Jelicic and N. Petrovacki, Optimality conditions and a solution scheme for fractional optimal control problems, Struct. Multidiscipline. Optimum. 38 (2009), no. 6, 571–581.

[16] Mahmoud M. El-Borai, Khairia El-Said El-Nadi, On the fractional optimal control, Int. J. of Appl. Math. and Mech. (2008), 13-18.

[17] Pontryagin LS, Baltyanskii VG, Gamkrelidze RV, and Mischenko EF (1962). The mathematical theory of optimal process. Inter science publishers Inc, New York.