American Journal of Theoretical and Applied Statistics
2015; 4(1): 1-5
Published online January 13, 2015 (http://www.sciencepublishinggroup.com/j/ajtas)
doi: 10.11648/j.ajtas.20150401.11
ISSN: 2326-8999 (Print); ISSN: 2326-9006 (Online)

Order statistics from non-identical Standard type II Generalized logistic variables and applications at moments

Z. AL-Saiary

Department of Statistics, King Abdulaziz University, Sciences Faculty for Girls, Saudia Arabia

Email address:
Zalsiary@kau.edu.sa, zakeia@hotmail.com

To cite this article:
Z. AL-Saiary. Order Statistics from Non-Identical Standard Type II Generalized Logistic Variables and Applications at Moments. *American Journal of Theoretical and Applied Statistics*. Vol. 4, No. 1, 2015, pp. 1-5. doi: 10.11648/j.ajtas.20150401.11

Abstract: In this paper the moment generating function of order statistics arising from independent non-identically distributed (INID) Standard type II Generalized logistic (SGLII) variables is established. A recurrence relation for all moments of all order statistics arising from INID SGLII is computed. Special cases for moments are deduced using polygamma function. Some numerical examples are given.

Keywords: Moments, Nonidentically Order Statistics, Standard Type II Generalized Logistic distribution, Polygamma Function

1. Introduction

The moments of order statistics (o.s) of independent non-identically distributed (INID) random variables (r. v. 's) have been established in literature in three directions. The first direction is initiated (Balakrishnan, 1994), requires a basic relation between the probability density function (p.d.f) and the cumulative distribution function (c.d.f). This technique is referred to this as differential equation technique (DET). It enables one to compute all the single and product moments of all order statistics in a simple recursive manner and the derivation of the moments depends mainly on integration by parts. The second technique is established by Abdelkader and Barakat will be referred to this as (BAT). It is an easier manner to evaluate the moments of (INID. o.s) but can only be applied to distributions with (c.d.f) in the form: $F(x)=1-\lambda(x)$ and we cannot evaluate the product moments. The third direction was developed by Jamjoom and Alsaiary (2011) which is the moment generating function technique. In this paper we compute the moments of order statistics arising from independent non-identically distributed Standard type II generalized logistic random variables using the moment generating function technique.

The subject on nonidentically order statistics is discussed widely in the literature in (David and Nagaraja, 2003). (Vaughan and Venables, 1972) denoted the joint p.d.f and marginal p.d.f of order statistics of (INID) random variables by means of permanent. Applications of the previous two methods are also found in the literature for several continuous distributions. The (DET) was used by (Balakrishnan, 1994) to derive recurrence relations satisfied be single and product moments of (INID) order statistics for the Exponential and right truncated distributions. (Childs and Balakrishnan, 2006) applied (DET) to derive the moments of (INID) order statistics for Logistic random variables. (Mohie Elidin et al., 2007) applied this method to derive the moments of (INID) order statistics for several distributions. The first applied of (BAT) was by (Barakat and Abdelkader, 2000) to weibull distribution and then the method was generalized by (Barakat and Abdelkader, 2003) and applied to Erlang, Positive Exponential, Pareto, and Laplace distributions. (Abdelkader, 2004) and (Abdelkader, 2008) used a closed expression for the survival function of Gamma and Beta distributions to compute the moments of (INID) (o.s) using this technique. (Jamjoom, 2006) applied it to Burr type XII random variables.

2. Nonidentical Order Statistics from Standard Type II Generalized Logistic Distribution

Let $X_1, X_2, \ldots, X_n$ be independent random variables having cumulative distribution functions $F_1(x), F_2(x), \ldots, F_n(x)$ and probability density functions $f_1(x), f_2(x), \ldots, f_n(x)$, respectively. Let $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ denote the order statistics obtained by arranging the n $X_i\,s$ in increasing order of magnitude. Then the p.d.f and the c.d.f of the rth order statistic $X_{r:n}$ $(1\le r \le n \le)$ can be written as:

$$f_{r:n}(x)=\frac{1}{(r-1)!\,(n-r)!}\sum_{p}\prod_{a=1}^{r-1}F_{i_a}(x)\,f_{i_r}(x)\prod_{c=r+1}^{n}\left\{1-F_{i_c}(x)\right\} \quad (1)$$

Where $\sum_{p}$ denotes the summation over all n! permutations $(i_1,\ i_2,\ \dots,\ i_n)$ of $(1,2,\dots n)$. (Bapat and Beg, 1989) put it in the form of permanent as:

$$f_{r:n}(x)=\frac{1}{(r-1)!\,(n-r)!}\,per\left[\underbrace{F(x)}_{r-1}\quad \underbrace{f(x)}_{1}\quad \underbrace{\{1-F(x)\}}_{n-r}\right] \quad (2)$$

$$F_{(r)}(x)=\sum_{j=r}^{n}\sum_{p_j}\prod_{a=1}^{j}F_{i_a}(x)\prod_{a=j+1}^{n}\left[1-F_{i_a}(x)\right] \quad (3)$$

$\sum_{p_j}$ is all permutations of $(i_1,i_2,\cdots,i_n)$ for $(1,\cdots,n)$ which satisify $i_1<i_2<\cdots<i_j$ and $i_{j+1}<i_{j+2}<\cdots<i_n$. And using permanent as

$$F_{(r)}(x)=\sum_{i=r}^{n}\frac{1}{i!\,(n-i)!}\,per\begin{bmatrix}F_1(x) & 1-F_1(x)\\ \vdots & \\ \underbrace{F_n(x)}_{i} & \underbrace{1-F_n(x)}_{n-i}\end{bmatrix},\ -\infty<x<\infty \quad (4)$$

The p.d.f. and the c.d.f. of SGL II distribution are given as

$$f(x)=\frac{\alpha e^{-\alpha x}}{\left(1+e^{-x}\right)^{\alpha+1}},-\infty\le x\le\infty,\alpha>0 \quad (5)$$

$$F(x)=1-\left[\frac{e^{-\alpha x}}{\left(1+e^{-x}\right)^{\alpha}}\right],-\infty\le x\le\infty,\alpha>0 \quad (6)$$

Now we can obtain the p.d.f. and the c.d.f. of the rth INID o.s. of SGLII distribution from eq (2) and (4) using (Permanent) and Mathematica Program as

$$f_{1:3}(x)=\frac{(\alpha_1+\alpha_2+\alpha_3)(1+e^{x})^{-(\alpha_1+\alpha_2+\alpha_3)}}{1+e^{-x}}$$

$$F_{1:3}(x)=1-(1+e^{x})^{-(\alpha_1+\alpha_2+\alpha_3)}$$

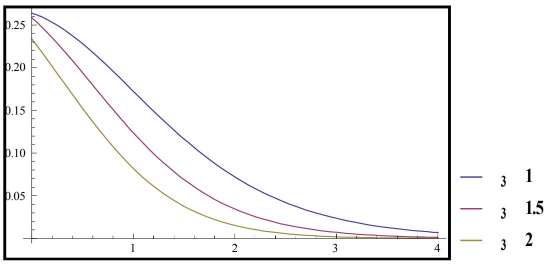

Figure 1. *Graph of p.d.f. of first(inid) o.s. from SGL II distribution for selected* $\alpha_1,\alpha_2,\alpha_3$

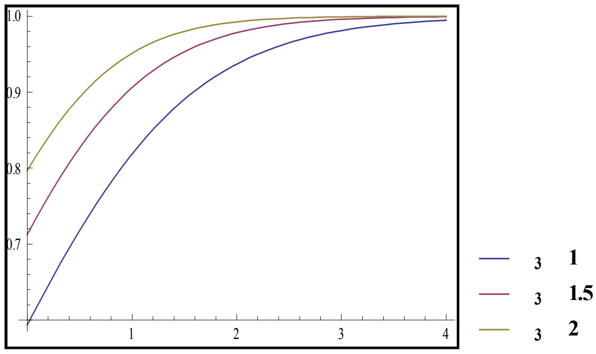

Figure 2. *Graph of c.d.f. of first (inid) o.s. from SGL II distribution for selected* $\alpha_1,\alpha_2;,\alpha_3$

3. The Moment Generating Function of the r^th^ Nonidentically (o.s) from Standard Type II Generalized Logistic Distribution

We know that the distribution function of Standard type II Generalized Logistic distribution can be written as:

$$F_i(x)=1-\left[\frac{e^{-\alpha_i x}}{\left(1+e^{-x}\right)^{\alpha_i}}\right],-\infty\le x\le\infty,\alpha_i>0,i=1,2,\cdots,n$$

See (Balakrishnan & Hossain 2007).

To derive the moments of INID o.s. arising from (6), we need the following theorem .

3.1. Theorem

Let $X_1,X_2,\cdots,X_n$ be independent non identically distributed r.v,.s. The moment generating function of the r^th^ order statistics, $M_{r:n}(t)$, for $1\le r\le n$ and $k=1,2,\dots$ is given by:

$$M_{r:n}(t)=t\sum_{j=n-r+1}^{n}(-1)^{j-(n-r+1)}\binom{j-1}{n-r}I_j \quad (7)$$

Where:

$$I_j=\sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum\int_0^{\infty}e^{tx}\prod_{b=1}^{j}[1-F_{i_b}(x)]dx,\ j=1,2,\cdots,n \quad (8)$$

where $(i_1,i_2,\cdots,i_n)$ are a permutations of (1, 2, …, n) for which $i_1\le i_2<\cdots<i_n$.

Proof: See (Jamjoom and Al-Saiary 2011)

3.2. Theorem

For $1\le r\le n$ and k = 1, 2,…

$$I_j=\sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum t\beta\left(t,\sum_{b=1}^{j}\alpha_{i_b}-t\right) \quad (9)$$

Proof:

On a pplying theorem 2.1 and using (6), we get:

$$I_j = \sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum t\int_{-\infty}^{\infty} e^{tx}\prod_{t=1}^{j}\left[\frac{e^{-\alpha_i x}}{\left(1+e^{-x}\right)^{\alpha_i}}\right]dx,$$

$$= \sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum t\int_{-\infty}^{\infty} e^{tx}\left[\frac{e^{-\sum_{b=1}^{j}\alpha_{i_b}x}}{\left(1+e^{-x}\right)^{\sum_{b=1}^{j}\alpha_{i_b}}}\right]dx,$$

$$= \sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum t\int_{-\infty}^{\infty}\left[\frac{e^{-x(\sum_{b=1}^{j}\alpha_{i_b}-t)}}{\left(1+e^{-x}\right)^{\sum_{b=1}^{j}\alpha_{i_b}}}\right]dx,$$

Substituting :

$$y = e^{-x}$$
$$\therefore\ x = \ln\frac{1}{y}$$
$$dx = -\frac{1}{y}dy$$

$$\therefore I_j = \sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum\int_{0}^{\infty}\frac{y^{(\sum_{b=1}^{j}\alpha_{i_b}-t)-1}}{\left[1+y\right]^{\sum_{b=1}^{j}\alpha_{i_b}}}dy,$$

By using the known relation:

$$\beta(p,q)=\int_{0}^{\infty}\frac{x^{p-1}}{(1+x)^{p+q}}dx;\ where\ p=(\sum_{b=1}^{j}\alpha_{i_b}-t),q=t$$

we will get relation (9).

4. The Moments of the r^{th} Non Identically (o.s) from Standard type II Generalized Logistic Distribution

From (7) and (9) the k^{th} moment of the r^{th} o.s. can be written as

$$\mu^{(k)}{}_{r:n}=M^{(k)}_{r:n}(0)=\sum_{j=n-r+1}^{n}(-1)^{j-(n-r+1)}\binom{j-1}{n-r}\sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum\times$$
$$\sum_{l=0}^{k}\binom{k}{l}\left[\frac{\Gamma^{[i]}(t+1)\,\Gamma^{[k-i]}(\sum_{b=1}^{j}\alpha_{i_b}-t)}{\Gamma(\sum_{b=1}^{j}\alpha_{i_b})}\right]_{t=0} \quad (10)$$

where $\Gamma^{[i]}(\,.\,)$ is the ith derivative of the gamma function. the first two moments can be given by the following relationships

$$\mu^{(1)}{}_{r:n}=\sum_{j=n-r+1}^{n}(-1)^{j-(n-r+1)}\binom{j-1}{n-r}\times$$
$$\sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum\left[\psi(1)-\psi(\sum_{b=1}^{j}\alpha_{i_b})\right]$$
$$\mu^{(2)}{}_{r:n}=\sum_{j=n-r+1}^{n}(-1)^{j-(n-r+1)}\binom{j-1}{n-r}\sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum\times \quad (11)$$
$$\left[\psi^2(1)+\psi'(1)-2\,\psi(1)\psi(\sum_{b=1}^{j}\alpha_{i_b})+\psi^2(\sum_{b=1}^{j}\alpha_{i_b})+\psi'(\sum_{b=1}^{j}\alpha_{i_b})\right]$$

Where $\psi(z)$ is PolyGamma [z] the logarithmic derivative of the Gamma function and $\mu^{(k)}{}_{r:n}$ is the k^{th} moment of the r^{th} o.s. and $M^{(k)}_{r:n}=\frac{d^k}{dt^k}M_{r:n}(t)$.

Corollary 1. For the case of a sample of n IID r.v.,s arising from SGL II distribution, the moment generating function of the r^{th} o.s. $(1\le r\le n)$ is given by

$$M_{r:n}(t)=t\sum_{j=n-r+1}^{n}(-1)^{j-(n-r+1)}\binom{j-1}{n-r}\binom{n}{j}\beta(t,j\alpha-t) \quad (12)$$

and the k^{th} moment becomes

$$\mu^{k}_{r:n}=\sum_{j=n-r+1}^{n}(-1)^{j-(n-r+1)}\binom{j-1}{n-r}$$
$$\times\binom{n}{j}\sum_{l=0}^{k}\binom{k}{l}\left[\frac{\Gamma^{[l]}(t+1)\,\Gamma^{[k-l]}(j\alpha-t)}{\Gamma(j\alpha)}\right]_{t=0} \quad (13)$$

Special cases: from (11):

$$\mu^{(1)}{}_{r:n}=\sum_{j=n-r+1}^{n}(-1)^{j-(n-r+1)}\binom{j-1}{n-r}\binom{n}{j}[\psi(1)-\psi(j\alpha)]$$
$$\mu^{(2)}{}_{r:n}=\sum_{j=n-r+1}^{n}(-1)^{j-(n-r+1)}\binom{j-1}{n-r}\binom{n}{j}\left[\psi^2(1)+\psi'(1)-2\,\psi(1)\psi(j\alpha)+\psi^2(j\alpha)+\psi'(j\alpha)\right] \quad (14)$$

5. Numerical Applications

The following examples are compute when k = 1 .

Example (1):

Let n=2 and α =1, 2, 3, 4,5 and 5 in (13). Table 1 shows the results:

Table 1. *The moments $\mu_{2:2}$ arising from IID SGL type II r.v.,s.*

α	1	2	3	4	5
$\mu_{2:2}$	1	-0.166667	-0.716667	-1.07381	-1.3377

For example when $\alpha = 3$, ,

$$M_{2:2} = t\,I_1 - t\,I_2\,,$$

$$I_1(k) = \beta(t,\alpha-t) = \frac{\Gamma(t).\Gamma(\alpha-t)}{\Gamma(\alpha)}$$

$$I_2 = \beta(t,2\alpha-t) = \frac{\Gamma(t).\Gamma(2\alpha-t)}{\Gamma(2\alpha)}$$

$$M_{2:2}(t) = \frac{t\Gamma(t).\Gamma(\alpha-t)}{\Gamma(\alpha)} - \frac{t\Gamma(t).\Gamma(2\alpha-t)}{\Gamma(2\alpha)} = \frac{\Gamma(t+1).\Gamma(\alpha-t)}{\Gamma(\alpha)} - \frac{t\Gamma(t+1).\Gamma(2\alpha-t)}{\Gamma(2\alpha)}$$

$$\mu_{2:2}(t) = M_{2:2}^{`}(0) = -0.716667$$

Example (2): Set n=2, and $\alpha_1 = 1(5)1$, $\alpha_2 = 1(1)5$ in Eq (11), we obtain results in Table 2:

$$\mu^{(1)}{}_{2:2} = \sum_{1=i_1}^{2}\left[\psi(1)-\psi(\alpha_{i_1})\right] - \sum_{1\le i_1<i_2\le 2}\left[\psi(1)-\psi(\sum_{b=1}^{j}\alpha_{i_b})\right] = \psi(1)-\psi(\alpha_1)-\psi(\alpha_2)+\psi(\alpha_1+\alpha_2)$$

$$\mu^{(1)}{}_{2:2} = \sum_{1=i_1}^{2}\left[\psi(1)-\psi(\alpha_{i_1})\right] - \sum_{1\le i_1<i_2\le 2}\left[\psi(1)-\psi(\sum_{b=1}^{j}\alpha_{i_b})\right] = \psi(1)-\psi(\alpha_1)-\psi(\alpha_2)+\psi(\alpha_1+\alpha_2)$$

Table 2. *The mean $\mu_{r:n}$ of all order statistics from INID SGL type II r.v.,s.*

α_1 \ α_2	1	2	3	4	5
1	1	0.5	0.333333	0.25	0.2
2	0.5	-0.16667	-0.41667	-0.55	-0.63333
3	0.333333	-0.41667	-0.71667	-0.88333	-0.99048
4	0.25	-0.55	-0.88333	-1.07381	-1.19881
5	0.2	-0.63333	-0.99048	-1.19881	-1.3377

Example (3):

Let $X_1, X_2, X_3 \sim SGL\ typ\ II(\alpha_1=0.5, \alpha_2=1, \alpha_3=1.5)$ Theorem (2.1) gives

$$M_{2:3}(t) = t\sum_{j=2}^{3}(-1)^j\binom{j-1}{1}I_j = tI_2 - 2tI_3$$

From (10) we get:

$$M_{2:3}(t) = t\sum_{1\le i_1<i_2\le 3}\cdots\sum\beta(t,\sum_{b=1}^{2}\alpha_{i_b}-t) - 2t\sum_{1\le i_1<i_2<i_3\le 3}\cdots\sum\beta(t,\sum_{b=1}^{3}\alpha_{i_b}-t)$$
$$= t\beta(t,\alpha_1+\alpha_2-t)+t\beta(t,\alpha_1+\alpha_3-t)+t\beta(t,\alpha_2+\alpha_3-t) - 2t\,\beta(t,\alpha_1+\alpha_2+\alpha_3-t)$$

$$M_{2:3}(t) = \frac{t\Gamma(t)\Gamma(\alpha_1+\alpha_2-t)}{\Gamma(\alpha_1+\varepsilon_2)} + \frac{t\Gamma(t)\Gamma(\alpha_1+\alpha_3-t)}{\Gamma(\alpha_1+\alpha_3)} + \frac{t\Gamma(t)\Gamma(\alpha_2+\alpha_3-t)}{\Gamma(\alpha_2+\alpha_3)} - \frac{2t\Gamma(t)\Gamma(\alpha_1+\alpha_2+\alpha_3-t)}{\Gamma(\alpha_1+\alpha_2+\alpha_3)}$$

$$M_{2:3}(t) = \frac{\Gamma(t+1)\Gamma(\alpha_1+\alpha_2-t)}{\Gamma(\alpha_1+\alpha_2)} + \frac{\Gamma(t+1)\Gamma(\alpha_1+\alpha_3-t)}{\Gamma(\alpha_1+\alpha_3)} + \frac{\Gamma(t+1)\Gamma(\alpha_2+\alpha_3-t)}{\Gamma(\alpha_2+\alpha_3)} - \frac{2\Gamma(t+1)\Gamma(\alpha_1+\alpha_2+\alpha_3-t)}{\Gamma(\alpha_1+\alpha_2+\alpha_3)}$$

$$\mu_{2:3}(t) = M_{2:3}^{`}(0) = 0.105922.$$

$$\mu_{3:3} = \psi(1)-\sum_{i=1}^{3}\psi(\alpha_i)+\psi(\alpha_1+\alpha_2)+\psi(\alpha_1+\alpha_3)+\psi(\alpha_2+\alpha_3)-\psi(\sum_{i=1}^{3}\alpha_i)$$
$$= \psi(2)+\psi(2.5)-\psi(3)-\psi(0.5)$$
$$= 2.16667$$

Table 3. *The mean $\mu_{r:n}$ of all order statistics from INID SGL type II r.v.,s.*

n \ r	1	2	3	4	5
2	-2.08333	-0.716667			
3	-1.50000	0.105922	2.16667		
4	-2.45	-1.17106	-0.248413	0.975397	
5	-2.82897	-1.62612	-0.854759	-3069.11	1.02362

6. Conclusion

The moment generating function technique is strongly recommended for derivation of recurrence relations of moments of order statistics from inid r.v.'s for any other continuous distributions with cdf in the form: F(x) = 1-$e^{\lambda(x)}$. Comparison between relative techniques available in the literature is also recommended for future study.

Acknowledgement

I would like to thank King Abdulaziz University for encouraging faculty staff to participate in scientific researches and motivate them to publish in scientific journals, which resulted in writing this paper.

References

[1] ABDELKADER, Y. (2004) Computing the moments of order statistics from nonidentically distributed Gamma variables with applications Int J Math Game Theo Algebra, 14.

[2] ABDELKADER, Y. (2008) Computing the moments of order statistics from independent nonidentically distributed Beta random variables. Stat Pap, 49, 136-149.

[3] BALAKRISHNAN, N. (1994) Order statistics from non-identically exponential random variables and some applications. Comput. in Statistics Data-Anal, 18, 203-253.

[4] BALAKRISHNAN, N. & HOSSAIN, A. (2007) Inference for the type II generalized logistic distribution under progressive type II censoring. Journal of statistical computation and simulation, 77, 1013-1031.

[5] BAPAT, R. B. & BEG, M. I. (1989) Order statistics from non-identically distributed variables and permanents. Sankh˜ya, A, 51, 79-93.

[6] BARAKAT, H. & ABDELKADER, Y. (2003) Computing the moments of order statistics from nonidentical random variables Stat Meth Appl, 13, 15-26.

[7] BARAKAT, H. M. & ABDELKADER, Y. H. (2000) Computing the moments of order statistics from nonidentically distributed Weibull variables. J. Comp. Appl. Math, 117, 85-90.

[8] CHILDS, A. & BALAKRISHNAN, N. (2006) Relations for order statistics from non-identical logistic random variables and assessment of the effect of multiple outliers on bias of linear estimators. J. Statist. Plann. Inference, 136, 2227-2253.

[9] DAVID, H. A. & NAGARAJA, H. N. (2003) Order Statistics, Third Edition. Wiley, New York.

[10] JAMJOOM, A. A. (2006) Computing the moments of order statistics from independent nonidentically distributed Burr type XII random variables. journal of Mathematics and Statistics, 2, 432-438.

[11] JAMJOOM, A. & AL-SAIARY. A. (2011) Moment Generating Function Technique for Moments of Order Statistics from Nonidentically Distributed Random variables. International journal of Statistics and Systems, 6, 177-188.

[12] MOHIE ELIDIN, M., MAHMOUD, M., MOSHREF, M. & MOHMED, M. (2007) On independent and nonidentical order statistics and associated inference. Department of mathematics. Cairo, Ai-Azhar University.

[13] VAUGHAN, R. J. & VENABLES, W. N. (1972) Permanent expressions for order statistics densities. J. R. Statist. Soc, 34, 08 - 10.