

# Excel Files for Teaching About Wave Fronts and Rays for a Wave Moving in Media with Changing Index of Refraction

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**Abstract:** In teaching about waves the students are learning how to find the refracted rays by using wave fronts or finding the wave fronts by using the rays. The usual teaching is based on models where the speed of the wave is constant in one medium and changes abruptly as the wave passes from one medium to another. This paper deals with ways of calculation the wave fronts and the rays for the case of a continuous changing of the index of refraction. For this purpose, Fermat's principle is applied for multiple layers of very small thickness. Two models are presented for the speed of the waves: A model on which the wave speed depends on the square root of the depth of penetration of the wave and the other model, where the speed depends on a linear dependence. In both cases it is found that as the wave progresses it is "totally reflected". In the case of the "square root dependence" the solution is a kind of cycloid which shows this behavior. In the linear case it is found that there is a moment where the wave is reflected, which is found by the maximum of a quantity "Z". By using this quantity, the coordinates x and y can be calculated. As an application the refraction of the light in the atmosphere is calculated, where the dependence of the distance from the center of the earth is calculated and again the 2 models are applied. In this case the "square root" model gives a stronger deviation from the linear model. This helps the student to understand the change on the perceived position of the celestial bodies.

**Keywords:** Rays, Wave Front, Snell' Law, Total Reflection

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## 1. Introduction

Science is primarily concerned with the description and explanation of physical patterns in the universe, from the astronomical scale to the subatomic scale. Scientists carry out their tasks through modeling inquiry. They construct, corroborate, and deploy conceptual models each of which represents a particular pattern in certain respects and to a certain extent, and serves a particular descriptive and/or explanatory function regarding the pattern in question. Scientific models and modeling can be transposed into pedagogical tools and processes which students of all levels can efficiently take advantage of for meaningful and sustainable learning of scientific theory.

The merits of models and modeling processes are being increasingly recognized not only in scientists' work, but, most importantly, in human cognition in general and in science education in particular. [1]

For physics education simulations are of great importance.

Spreadsheets give the opportunity for integrating computational physics techniques with simulation and traditional teaching learning methods of Physics, a major improvement in the conceptual understanding of students is expected [2]. There are examples of using Excel simulations in Kinematics [3, 4] Optics is very well suited for using spreadsheets [5-7].

Teaching about wave fronts in introductory courses is usually restricted to the teaching of the refraction of periodic waves in media that have a constant speed as for example [8]. This paper was based in the ideas expressed by [9] in the tutorials of introductory physics. In Pavlos Mihas and Triantafyllos Gemousakakis's study, these ideas were expanded for teaching introductory courses by using rays and wave fronts in a medium in which the index of refraction changes [10]. The rays and wave fronts in a medium with an index of refraction that changes can help the students to understand these notions. In this paper are presented ways to use Excel files without macros to teach about waves and rays

with a changing speed.

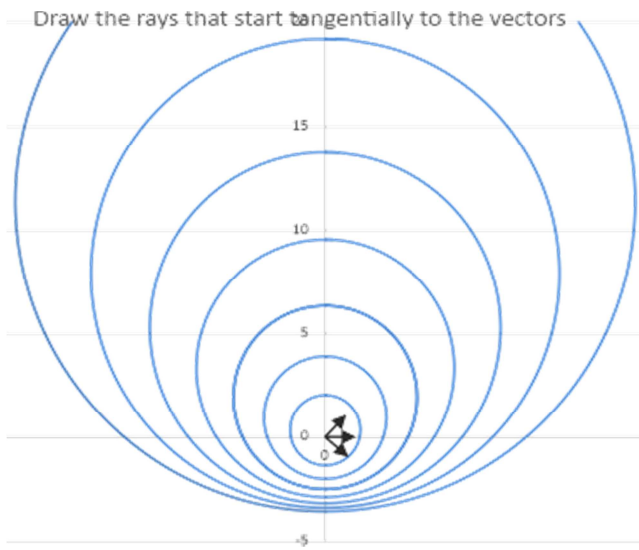


Figure 1. A question for exams: draw the rays that start along the arrows.

One of the first solutions of this problem it was done by Jean Bernoulli [11, 12]. He was interested in solving the problem of “brachistochrone.” If we imagine a curve on a coordinate system with the x-axis horizontal, and the y-axis vertically downward. When the material point sliding down the curve passes a certain point (x, y) with a certain velocity  $v$ . We have the relation:  $\frac{v^2}{2} = gy$  which was familiar to Bernoulli. So,  $v = (2gy)^{\frac{1}{2}}$ . Bernoulli solved the problem by using Fermat’s principle. He proposed that the speed of the wave changes with square root of  $y$ . and then finds the curve from A to B as a cycloid. This was a solution of the problem of the brachistochrone. As Polya points out the mechanical problem admits an optical reinterpretation. The medium in

which the light is transmitted is an optically inhomogeneous medium. This medium is stratified, has strata of different index of refraction; the velocity of light in the horizontal layer at the depth  $y$  is  $(2gy)^{\frac{1}{2}}$ .

In this paper we compare models of propagation of waves. Model comparison is at the heart of all scientific methodologies [13]. Progress is made in science by constructing many models (possibly of different complexities), testing them against measurements, and determining which of them explain the data the best.

## 2. Models of Refraction

Two models are presented for the speed of the wave:

A) A model with the velocity being equal to  $\sqrt{A + \beta y}$  where  $A$  and  $\beta$  constants, and b) A model with the velocity being equal to  $A + \beta y$

We then apply the method to a ray coming from the moon. The movement of light in the atmosphere was dealt first time by Newton (Nauenberg 2017) who dealt on the movement of the particles of light. We apply methods similar to the ones who are used in the movement of light, but we have to solve for the (r,  $\theta$ ) coordinates. The speed is given by a square root.

If we consider a medium in which the speed of light (or more generally of a wave) depends on the y-coordinate as  $v = \sqrt{A + \beta y}$  then Snell’s law is given by the constancy of  $\frac{\sin \alpha}{v} = \Gamma$  where  $\Gamma = \frac{\sin \alpha_0}{v_0}$  where  $\alpha_0$  is the initial angle of incidence. The angle  $\alpha$  is related to the angle of inclination  $\theta$  by  $\alpha = \pi/2 - \theta$  or  $\sin \alpha = \cos \theta$  since  $\tan \theta = \frac{dy}{dx}$   $\sin \alpha = \frac{1}{\sqrt{1 + (\frac{dy}{dx})^2}}$  or  $\frac{1}{\sqrt{1 + (\frac{dy}{dx})^2}} = \Gamma^2 \cdot (A + \beta y)$ .

$$x = \int_0^x dy \sqrt{\frac{\Gamma^2(A + \beta y)}{E - \beta \Gamma^2 y}} \quad E = 1 - \beta \Gamma^2 A \quad (1) \quad x = \frac{1}{\beta \Gamma^2} \left[ \varepsilon - \frac{\sin 2\varepsilon}{2} \right]_{\varepsilon_0}$$

$$\text{where } \varepsilon = \arcsin(\Gamma(A + \beta y)^{\frac{1}{2}})$$

For the time  $t$  we get: For the time we ge

$$t = \int \frac{ds}{v} = \int dy \frac{\sqrt{1 + (\frac{dx}{dy})^2}}{\sqrt{A + \beta y}} = \int dy \frac{1}{\sqrt{(E - \Gamma^2 \beta y)(A + \beta y)}} = \frac{2}{\sqrt{\Gamma^2 \beta^2}} \arctan\left(\sqrt{\frac{\Gamma^2 \beta (y + A)}{E - \Gamma^2 \beta y + E}}\right) \quad (2)$$

So solving for  $y$  we get an expression of  $y$  as a function of  $t = \frac{2}{\beta \Gamma} \arctan \sqrt{\frac{\Gamma^2 \beta (y + A)}{E - \Gamma^2 \beta y + E}} - Q$  where  $Q = \frac{2}{\beta \Gamma} \arctan \sqrt{\frac{\Gamma^2 A}{E}} = \frac{2a_0}{|\beta \Gamma|}$  so

$$\tan\left(\frac{(t+Q)\beta \Gamma}{2}\right)^2 = M = \frac{\Gamma^2 \beta (y + A)}{E - \Gamma^2 \beta y + E}$$

$$\text{so } y = \frac{\varepsilon \sin^2 \varepsilon - \Gamma^2 A \cos^2 \varepsilon}{\beta \Gamma^2} \text{ where } \varepsilon = \frac{(t+Q)|\beta \Gamma|}{2} = \frac{(t|\beta \Gamma| + 2a_0)}{2}$$

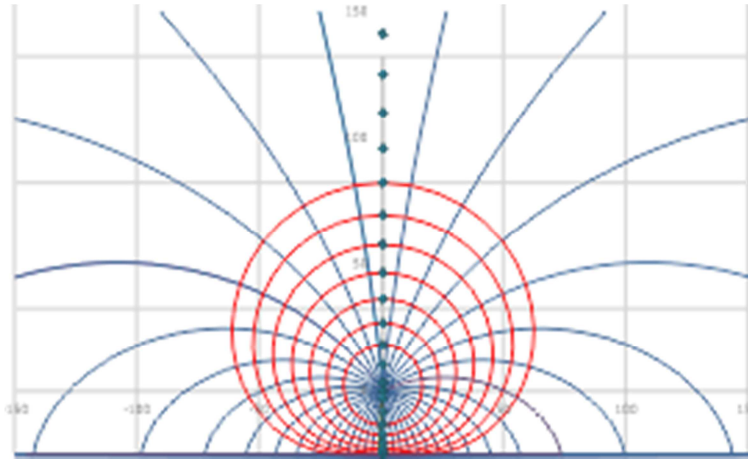
$$y = \frac{\sin^2 \varepsilon - \Gamma^2 A}{\beta \Gamma^2} = \frac{\sin^2 \varepsilon - \sin^2 a_0}{\beta \Gamma^2} \quad (3)$$

$$x = \frac{1}{\beta \Gamma^2} (\sin(2\varepsilon) - \varepsilon - \sin(2a_0) + a_0) \quad (4)$$

for  $\sin \varepsilon = 1$   $y_{\max} = \frac{1-\Gamma^2 A}{B\Gamma^2} = A^2 \frac{\cos^2 a_0}{B \sin^2 a_0} = \frac{A^2}{B} \cot^2 a_0$  so  $\varepsilon = \frac{\pi}{2} = \frac{(t+Q)|B\Gamma|}{2}$  so

$$t(y_{\max}) = \frac{\pi}{|B\Gamma|} - Q = \frac{\pi}{|B\Gamma|} - \frac{a_0}{|B\Gamma|} = 2 \frac{\frac{\pi}{2} - a_0}{B\Gamma} \text{ for } t = 2t(y_{\max}) = \frac{1}{|B\Gamma|} (2\pi - 2a_0) \varepsilon = \pi - a_0 \text{ so } y = 0$$

$$\text{for } t=0 \quad x - x_0 = \frac{2}{B\Gamma^2} \left( \Gamma \sqrt{AE} - \frac{q}{1+q^2} \right) = \frac{2}{B\Gamma^2} (\Gamma \sqrt{AE} - \Gamma \sqrt{AE}) = 0$$



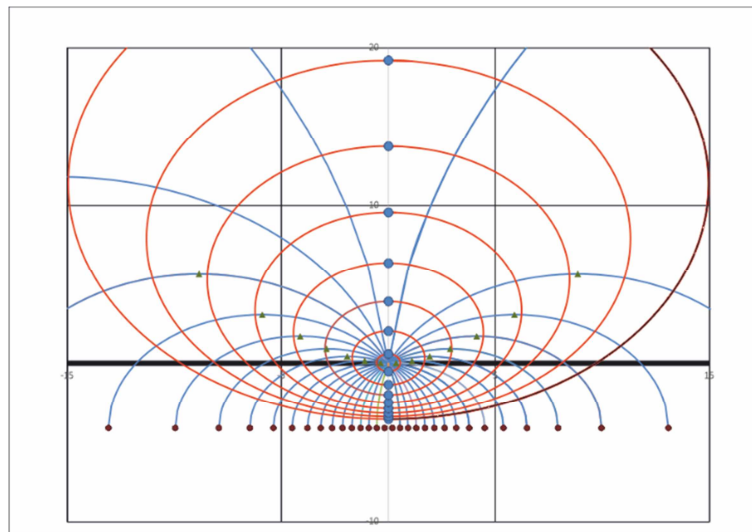
**Figure 2.** Case when the speed of the wave is proportional to  $(A+by)^{1/2}$ .

For  $a_0=0$  then the movement is perpendicular to the x-axis and then  $x=0$  and we find

$$t = \int \frac{ds}{v} = \int \frac{dy}{\sqrt{A+By}} = \frac{2}{B} (\sqrt{A+By} - \sqrt{A}) \quad \frac{(Bt + \sqrt{A})^2}{4} = A + By \text{ or } y = \frac{Bt^2}{4} + \sqrt{A} t \quad \frac{dy}{dt} = \frac{Bt}{2} + \sqrt{A} \quad (5)$$

B) If the speed is given by a linear case  $v = A + By$

Then the integration is proceeding in some respects easier:



**Figure 3.** Rays and wave fronts if the speed is given by a linear case.

From Snell's law and  $\Gamma = \frac{\sin a_0}{v_0}$  we get  $\frac{1}{v^2 \left( 1 + \left( \frac{dy}{dx} \right)^2 \right)} = \Gamma^2$  or  $\frac{1}{(A+By)^2} = \Gamma^2 \left( 1 + \left( \frac{dy}{dx} \right)^2 \right) \left( \frac{dy}{dx} \right)^2 = \frac{1}{\Gamma^2 (A+By)^2} - 1 =$

$$\frac{1 - \Gamma^2 (A+By)^2}{\Gamma^2 (A+By)^2}$$

$$\left( \frac{dx}{dy} \right)^2 + 1 = \frac{1}{1 - \Gamma^2 (A+By)^2} \quad (6)$$

By putting  $\Gamma(A+By)=z$  we get:

$$x - x_0 = \int_{y_0}^y dy \frac{\Gamma(A+By)}{\sqrt{1-\Gamma^2(A+By)^2}} = \frac{1}{2\Gamma B} \int_{z_0}^z \frac{dz^2}{\sqrt{1-z^2}} = -\frac{1}{\Gamma B} \left( \sqrt{1-z^2} - \sqrt{1-z_0^2} \right) \quad (7)$$

$$\sqrt{1-z^2} = -(B\Gamma x - \cos a_0) \text{ or}$$

$$1 - \Gamma^2(A+By)^2 = (B\Gamma x - \cos a_0)^2 \Gamma(A+By) = \sqrt{1 - (B\Gamma x - \cos a_0)^2} \text{ so}$$

$$y = \frac{\sqrt{1 - (B\Gamma x - \cos a_0)^2}}{|\sin a_0| \Gamma B} - \frac{A}{B} \text{ for } x = \frac{A \cdot \cot a_0}{B} \frac{dy}{dx} = 0 \quad y_m = \frac{A}{B} \frac{1}{|\sin a_0|} - \frac{A}{B} \quad (8)$$

For  $a_0 < \pi/2$  or  $a_0 > 3\pi/2$  that is the rays are moving downwards ( $y < 0$ ) the lowest point is *reflection*  $x = -1/(\beta\Gamma)(1 - \cos a_0) = -A/B \cot(a_0/2)$  and  $y = -\frac{A}{B}$

The time  $t$  is given by the relation:

$$t = \int \frac{ds}{v} = \int dy \sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \int \frac{dy}{(A+By)\sqrt{1-\Gamma^2(A+By)^2}}$$

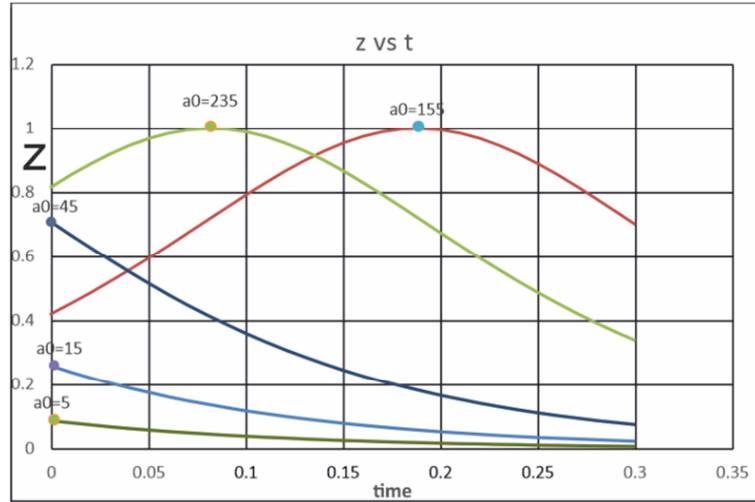


Figure 4. Time dependence of  $Z$ . Only the parts for  $t > 0$  concern the time.

$$t = \int \frac{dz}{Bz\sqrt{1-z^2}} = -\frac{1}{B} \ln\left(\frac{1+\sqrt{1-z^2}}{z}\right) \text{ so, by integrating between the limits } z_0 \text{ and } z \text{ we put } \frac{1+\sqrt{1-z_0^2}}{z_0} = \frac{1+2\cos^2(\frac{a_0}{2})-1}{\pm \sin z_0} = |\cot(\frac{a_0}{2})| = k$$

$$t = \int \frac{dz}{Bz\sqrt{1-z^2}} = -\frac{1}{B} \ln\left(\frac{1+\sqrt{1-z^2}}{z}\right) + \frac{1}{B} \ln k = \frac{1}{B} \ln\left(\frac{kz}{1+\sqrt{1-z^2}}\right) \quad (9)$$

By solving for  $z$  we get

$$\exp(Bt) = \frac{zk}{1+\sqrt{1-z^2}} \quad \exp(Bt) (\sqrt{1-z^2}) = zk - \exp(Bt) \quad (10)$$

$$z = \frac{2ke^{Bt}}{(k^2 + e^{2Bt})}$$

IF  $z=1$   $t_m = \left| \frac{\ln(k)}{B} \right|$  which is the point where  $\frac{dy}{dx} = 0$ . we get for  $z=1$  the maximum value of  $y$ .

$$\text{For } y < 0 \quad z = \frac{2ke^{-Bt}}{(k^2 + e^{-2Bt})} = \frac{2e^{Bt_m}e^{-Bt}}{(e^{2Bt_m} + e^{-2Bt})} = \frac{2e^{-B(t+t_m)}}{(1 + e^{-2B\xi})} = \frac{2e^{-B\xi}}{(1 + e^{-2B\xi})} \quad \xi = t + t_m$$

$$\text{For } y > 0 \quad z = \frac{2ke^{Bt}}{(k^2 + e^{2Bt})} = \frac{2e^{Bt_m}e^{Bt}}{(e^{2Bt_m} + e^{2Bt})} = \frac{2e^{B(t-t_m)}}{(1 + e^{2B\xi})} = \frac{2e^{-B\xi}}{(1 + e^{-2B\xi})} \quad \xi = t - t_m$$

In the first case we have a simple calculation for  $x$ , but on the second we need to distinguish the  $x$  for  $t \leq t_m$  and  $t > t_m$ .

For the time  $t \leq t_m$   $x = -\frac{1}{\Gamma B} * (\text{sign}(\sin(a_0)) * \sqrt{1-z^2} - \cos(a_0))$  we get for  $z=1$

For the time  $t > t_m$   $x = 2x_m + \frac{1}{(Br)} ((\text{sign}(\sin(a_0)) * \sqrt{1 - z^2} - \cos(a_0))$  this is due to the symmetry of  $z$  so that  $x(t_m + \xi) = x_m + (x_m - x(t_m - \xi)) = 2x_m - x(t_m - \xi)$

$$Z(-\xi) = \frac{2 e^{-B\xi}}{(1+e^{-2B\xi})} = Z(\xi)$$

$$\text{for } x=0 \ y = \frac{Al}{Vl} * (e^{(Vl*t)} - 1) \ dy/dt = Al * e^{Vl*t} \text{ for } y>0 \text{ or } y = \frac{Al}{Vl} * (e^{(-Vl*t)} - 1) \quad (11)$$

To calculate the wave fronts in excel we find for  $-180 < a_0 < 180$  the  $\sin(a_0)$  and  $\Gamma = \frac{\sin(a_0)}{A}$   $E = 1 - \frac{\Gamma^2}{A}$   
 $z_0 = \sin(a_0)$   $k = \text{abs}\left(\cot\left(\frac{a_0}{2}\right)\right)$   $t_m = t_{\text{reflection}} = |\ln(k) / \beta|$  so for  $k < 1$  there is no reflection or if  $a_0 < 2 \arccot(1) = \frac{\pi}{2}$  or  
 $a_0 > \frac{3\pi}{2}$  then there is a reflection time and a maximum

$$z = \frac{2 * EXP(\beta * |t - t_{reflection}|)}{1 + EXP(2 * \beta * |t - t_{reflection}|)} x_{reflection} = \frac{A}{\beta} cot(a_0)$$

If it is negative or  $t < t_{\text{reflection}}$   $x = -1/(\beta\Gamma)(\text{SIGN}(\cos\alpha_0) * \sqrt{1 - z^2} - \cos\alpha_0)$

If it is positive and then if  $t > t_{\text{reflection}}$  then

$$x = 2 * x_{reflection} + \frac{1}{\beta * \Gamma} * (SIGN(\cos \alpha_0) * \sqrt{1 - z^2}) - \cos \alpha_0 \quad (12)$$

For a small time  $d\tau$  then we calculate the change of  $r$ :

$$dr = -(c + br)\cos\alpha \, d\tau, -d\tau = \frac{dr}{(c+br)\cos\alpha} = \frac{d(c+br)}{b(c+br)\sqrt{1-K^2}(c+br)^2} = \frac{dz}{bz\sqrt{1-K^2}z^2} = \frac{dKz}{bKz\sqrt{1-K^2}z^2} = \frac{\cos\omega \, d\omega}{b \sin\omega \cos\omega} \text{ so the integral}$$

$$\text{with } z=l \text{ for } \tau=0 \text{ is } -b\tau = \ln\left(\frac{1}{\sin\omega} - \frac{\cos\omega}{\sin\omega}\right) = \ln\left(\frac{1-\sqrt{1-K^2}z^2}{Kz}\right) - \ln\left(\frac{1-\sqrt{1-K^2}}{K}\right) = \ln\left(\frac{1-\sqrt{1-K^2}z^2}{z(1-\sqrt{1-K^2})}\right)$$

$$SO \, e^{-b\tau} = \frac{1-\sqrt{1-K^2}z^2}{z(1-\sqrt{1-K^2})}$$

$$z = \frac{2(1-\sqrt{1-K^2})e^{-b\tau}}{\left((2-K^2-2\sqrt{1-K^2})e^{-2b\tau}+K^2\right)} = \frac{2(1-\cos\alpha)e^{-\beta t}}{(1-\cos\alpha)^2 e^{-2\beta t} + \sin^2 \alpha} \quad (13)$$

$$\text{For } \tau=0 \, z = 2 \frac{1-\sqrt{1-K^2}}{(2-2\sqrt{1-K^2})} = 1 \text{ then for } r = \frac{\frac{2(1-\sqrt{1-K^2})e^{-b\tau}}{\left((2-K^2-2\sqrt{1-K^2})e^{-2b\tau}+K^2\right)} - C}{b} = \frac{\frac{2(1-\cos\alpha)e^{-b\tau}}{\left((1-\cos\alpha)^2 e^{-2b\tau} + \sin^2 \alpha\right)} - C}{b}$$

So this relation gives as the radius  $r$  as a function of time  $\tau$ . In excel we proceed from a radius  $r_1$  to a radius  $r_2$  in time  $d\tau$  and then using geometry we find where the ray that starts at the circle with radius  $r_1$  meets the circle at radius  $r_2$ . This method permits us to construct the ray.

$$\text{since } c_{out} = 1 \text{ we get: } r(0) = \frac{1-C}{b} = \frac{1 - \frac{c_{out}}{n_{atm}} + r_g}{\frac{c_{out}}{n_{atm}}} = \frac{1 - \frac{1}{n_{atm}} + r_g}{\frac{1}{n_{atm}}} = r_g + h_{atm}$$

$$d\theta = \frac{(c+br)d\tau K(c+br)}{rK^2} = \frac{z^2 d\tau}{rK}$$

### 3.2. Speed Velocity Depends on the Square Root of the Distance

We proceed in a similar way as in case A:

$$dr = -\sqrt{(c+br)}\cos\alpha \, d\tau, -d\tau = \frac{dr}{\sqrt{(c+br)}\cos\alpha} = \frac{d(c+br)}{b\sqrt{(c+br)}\sqrt{1-K}(c+br)} = \frac{dz}{bz\sqrt{1-K}z} \, \tau = \int_1^z \frac{dz}{bz\sqrt{1-K}z}$$

$$= \frac{2}{\sqrt{K}} \left( \arctan\left(\frac{Kz}{1-Kz}\right) - \arctan\left(\frac{K}{1-K}\right) \right) = \frac{2}{\sqrt{K}} \arctan\left(\frac{\frac{Kz}{1-Kz} - \frac{K}{1-K}}{\frac{K^2 z}{(1-Kz)(1-K)}}\right) = \frac{2}{\sqrt{K}} \arctan\left(\frac{K(z-1)}{1-Kz+2K^2 z}\right)$$

$$\tan\left(\frac{\sqrt{K}\tau}{2}\right) = \frac{K(z-1)}{zK(K-1)} \, zK(K-1)\tan\psi = Kz - K$$

$$z((K-1)\tan\psi - 1) = -1 \, z = -\frac{1}{(K-1)\tan\psi - 1} \text{ where } \psi = \frac{\sqrt{K}\tau}{2}$$

$$r = \frac{1}{\frac{(K-1)\tan\psi - 1}{b_s}} - C_s$$

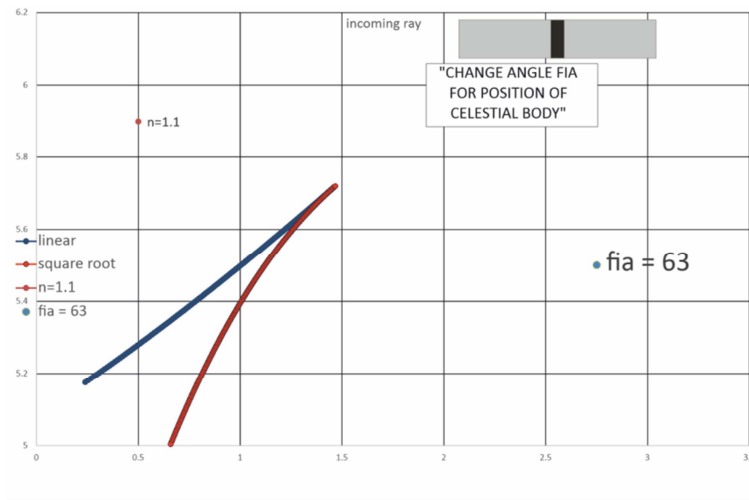


Figure 6. The deviation of the ray for 2 models.

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