American Journal of Physics and Applications
2015; 3(2): 60-62
Published online March 23, 2015 (http://www.sciencepublishinggroup.com/j/ajpa)
doi: 10.11648/j.ajpa.20150302.17
ISSN: 2330-4286 (Print); ISSN: 2330-4308 (Online)

Linear Momentum Conservation in the Motion of Electric Charges

Farrin Payandeh

Department of Physics, Payame Noor University (PNU), Tehran, Iran

Email address:
payandehfarrin92@gmail.com

To cite this article:
Farrin Payandeh. Linear Momentum Conservation in the Motion of Electric Charges. *American Journal of Physics and Applications.* Vol. 3, No. 2, 2015, pp. 60-62. doi: 10.11648/j.ajpa.20150302.17

Abstract: In this letter I will discuss the linear momentum conservation for an electric charge which is moving in a magnetic field. This will enrich the knowledge of undergraduate physics students, about the important concept of conservation of linear momentum, in classical electrodynamics.

Keywords: Conservation Laws, Lagrangian Mechanics, Electric Charges

1. Introduction

In Newtonian mechanics, the law of conservation of momentum can be derived from the law of action and reaction, which states that every force has a reciprocating equal and opposite force. Under some circumstances one moving charged particle can exert a force on another without any return force [1-6]. Moreover, Maxwell's equations, the foundation of classical electrodynamics, are Lorentz-invariant. Nevertheless, the combined momentum of the particles and the electromagnetic field is conserved.

Two electric charges q_1 and q_2 corresponding to masses m_1 and m_2 are supposed to be moving with velocities $\vec{v}_1$ and $\vec{v}_2$ in 3-dimensional space. As it is known from electromagnetism, reciprocal electric and magnetic forces are exerted on these two charges. According to this situation, the linear momentum conservation for particles in Coulomb potentials has been investigated and solved [7-11]. However in magnetic fields, this concept still appears to be obscure, since despite of the fact that the exerted magnetic forces on the charge are equal, they are not aligned in a same direction.

To obtain the conservation of linear momentum in this situation, it is sufficient to apply the Lagrangian mechanics. Assume G to be a function of t_i, p_i and q_i such that

$$G = G\left(p_i, t_i, q_i\right). \tag{1}$$

So the time derivative of this function becomes

$$\frac{\mathrm{d}G}{\mathrm{d}t} = \frac{\partial G}{\partial t} + \sum_i \left(\frac{\partial G}{\partial p_i}\frac{\partial p_i}{\partial t} + \frac{\partial G}{\partial q_i}\frac{\partial q_i}{\partial t} \right). \tag{2}$$

The Hamilton equations imply that

$$\begin{aligned} \frac{\partial H}{\partial q_i} &= -\frac{\mathrm{d}p_i}{\mathrm{d}t}, \\ \frac{\partial H}{\partial p_i} &= \frac{\mathrm{d}q_i}{\mathrm{d}t}. \end{aligned} \tag{3}$$

According to (2) we have

$$\frac{\mathrm{d}G}{\mathrm{d}t} = \frac{\partial G}{\partial t} + \sum_i \left(\frac{\partial G}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial G}{\partial p_i}\frac{\partial H}{\partial q_i} \right). \tag{4}$$

Let us notate

$$\left\{G, H\right\} \equiv \frac{\partial G}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial G}{\partial p_i}\frac{\partial H}{\partial q_i},$$

which is the Poisson's bracket. If G is not an explicit function of time, then

$$\frac{\mathrm{d}G}{\mathrm{d}t} = \left\{G, H\right\}. \tag{5}$$

In the case of a vanishing Poisson's bracket, G is a constant of motion in this physical system. Now to see what really happens in this physical process, in the next section, we deal with the conservation of the linear momentum for these charges by writing the usual Hamiltonian of two moving electric charges in a magnetic field.

2. Obtaining the Conservation of Momentum

Let us write the Hamiltonian for a two-particle system, consisting of the electric charges q_1 and q_2. As we know, the Hamiltonian of a charged particle in a magnetic field is $H=\frac{1}{2m}\left(\vec p+\frac{e}{c}\vec A\right)^2$, which for a two-particle system becomes

$$H=\frac{1}{2m_1}\left(\vec p_1+\frac{q_1}{c}\vec A_{21}\right)^2+\frac{1}{2m_2}\left(\vec p_2+\frac{q_2}{c}\vec A_{12}\right)^2. \qquad (6)$$

To obtain the vector potentials $\vec A_{12}$ and $\vec A_{21}$, let us note that $\vec B_{12}$, i.e. the magnetic field felt by q_1 which is produced by q_2 is (see figure 1)

$$\vec B_{12}=\frac{\vec v_2}{c}\times\frac{\vec E_2}{c}=\frac{q_2}{4\pi\varepsilon_0 c^2}\vec v_2\times\frac{\vec r_1-\vec r_2}{|\vec r_1-\vec r_2|^3}=\frac{q_2}{4\pi\varepsilon_0 c^2}\vec v_2\times\vec\nabla_1\frac{1}{|\vec r_2-\vec r_1|}, \qquad (7)$$

or

$$\vec B_{12}=\vec\nabla_1\times\left(\frac{q_2}{4\pi\varepsilon_0 c^2}\frac{\vec v_2}{|\vec r_2-\vec r_1|}\right)=\vec\nabla_1\times\vec A_{12}. \qquad (8)$$

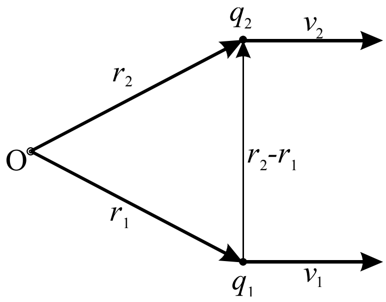

Figure 1. *The vector difference between to moving charges q_1 and q_2, respectively of velocities v_1 and v_2.*

Therefore according to the relation between the magnetic field and vector potential, $\vec B=\vec\nabla\times\vec A$, one obtains

$$\vec A_{12}=\frac{q_2}{4\pi\varepsilon_0 c^2}\frac{\vec v_2}{|\vec r_2-\vec r_1|}, \qquad (9)$$

and similarly

$$\vec A_{21}=\frac{q_1}{4\pi\varepsilon_0 c^2}\frac{\vec v_1}{|\vec r_2-\vec r_1|}. \qquad (10)$$

Substituting (9) and (10) in (6) we have

$$H=\frac{1}{2m_1}\left(\vec p_1+\frac{q_1q_2}{4\pi\varepsilon_0 c^3}\frac{\vec v_2}{|\vec r_2-\vec r_1|}\right)^2+\frac{1}{2m_2}\left(\vec p_2+\frac{q_1q_2}{4\pi\varepsilon_0 c^3}\frac{\vec v_1}{|\vec r_2-\vec r_1|}\right)^2. \qquad (11)$$

Now we interpolate this relation in the Poisson's bracket $\{\vec p,H\}$, where $\vec p=\vec p_1+\vec p_2$. We get

$$\{\vec p,H\}=\{\vec p_1+\vec p_2,H\}=\frac{\partial(\vec p_1+\vec p_2)}{\partial q_i}\frac{\partial H}{\partial p_i}-\frac{\partial H}{\partial q_i}\frac{\partial(\vec p_1+\vec p_2)}{\partial p_i}=-\frac{\partial H}{\partial q_i}\frac{\partial(\vec p)}{\partial p_i}=-\frac{\partial H}{\partial q_i}\hat e_i, \qquad (12)$$

in which $\hat e_i$ is a unit vector along $\vec p_i$ and the index i indicates six independent components of $\vec p_1$ and $\vec p_2$; $p_{1x}, p_{1y}, p_{1z}, p_{2x}, p_{2y}$ and p_{2z} .)

As it is seen, the Poisson's bracket (12) contains only the derivatives of Hamiltonian with respect to the generalized coordinates. Therefore one can omit the terms $\frac{p_1^2}{2m_1}$ and $\frac{p_2^2}{2m_2}$ in (11), i.e.

$$H=\frac{q_1q_2}{4\pi\varepsilon_0 c^3}\frac{\vec p_1}{2m_1}.\frac{\vec v_2}{|\vec r_2-\vec r_1|}+\frac{1}{2m_1}\left(\frac{q_1q_2}{4\pi\varepsilon_0 c^3}\right)^2\frac{\vec v_2^{\,2}}{|\vec r_2-\vec r_1|}+\frac{q_1q_2}{4\pi\varepsilon_0 c^3}\frac{\vec p_2}{2m_2}.\frac{\vec v_1}{|\vec r_2-\vec r_1|}+\frac{1}{2m_2}\left(\frac{q_1q_2}{4\pi\varepsilon_0 c^3}\right)^2\frac{\vec v_1^{\,2}}{|\vec r_2-\vec r_1|}. \qquad (13)$$

Substituting $\vec p_1=m_1\vec v_1$ and $\vec p_2=m_2\vec v_2$ in above expression we get:

$$H=\frac{2q_1q_2}{4\pi\varepsilon_0 c^3}\frac{\vec v_1.\vec v_2}{|\vec r_2-\vec r_1|}+\frac{1}{2m_1}\left(\frac{q_1q_2}{4\pi\varepsilon_0 c^3}\right)^2\frac{\vec v_2^{\,2}}{|\vec r_2-\vec r_1|}+\frac{1}{2m_2}\left(\frac{q_1q_2}{4\pi\varepsilon_0 c^3}\right)^2\frac{\vec v_1^{\,2}}{|\vec r_2-\vec r_1|}. \qquad (14)$$

Now in order to simplify our results, we notate

$$\frac{2q_1q_2}{4\pi\varepsilon_0c^3}\vec{v}_1.\vec{v}_2 \equiv k_1,$$
$$\frac{1}{2m_1}\left(\frac{q_1q_2}{4\pi\varepsilon_0c^3}\right)^2\vec{v}_2^{\,2} \equiv k_2, \tag{15}$$
$$\frac{1}{2m_2}\left(\frac{q_1q_2}{4\pi\varepsilon_0c^3}\right)^2\vec{v}_1^{\,2} \equiv k_2'.$$

Consequently (14) becomes

$$H = \frac{k_1}{|\vec{r}_2-\vec{r}_1|} + \frac{k_2}{|\vec{r}_2-\vec{r}_1|^2} + \frac{k_2'}{|\vec{r}_2-\vec{r}_1|^2}. \tag{16}$$

3. Discussion and Conclusion

An interesting point in the above Hamiltonian is its symmetry with respect to exchanges between $\vec{r}_1$ and $\vec{r}_2$. Now expanding the Poisson's bracket (12) we get

$$\{\vec{p}_1+\vec{p}_2, H\} = -\frac{\partial H}{\partial q_i}\hat{e}_i = -\frac{\partial H}{\partial x_1}\hat{e}_1 - \frac{\partial H}{\partial y_1}\hat{e}_2 - \frac{\partial H}{\partial z_1}\hat{e}_3 \\ \frac{\partial H}{\partial x_2}\hat{e}_1 - \frac{\partial H}{\partial y_2}\hat{e}_2 - \frac{\partial H}{\partial z_2}\hat{e}_3. \tag{17}$$

If the above differentiations are done with respect to the indexes 1 and 2, then one observes that $\frac{\partial H}{\partial x_1} = -\frac{\partial H}{\partial x_2}$, …. This is of course observable from the equation of symmetry. Hence we have

$$\{\vec{p}_1+\vec{p}_2, H\} = 0,$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}(\vec{p}_1+\vec{p}_2) = 0 \qquad \Rightarrow \qquad \vec{p}_1+\vec{p}_2 = \text{const.},$$

which means that for a two-electron system (or for two charges q_1 and q_2 in general), the conservation of linear momentum is retained, despite the fact that they are not subjected to centripetal forces. Therefore once can observe that, when no central force is applied on charges, the linear momentum is still conserved. This interesting conclusion extends the usual domain of linear momentum conservation and this is what we were looking into in this paper.

References

[1] Herbert Goldstein: Classical mechanics (2nd ed.), Reading, Mass.: Addison-Wesley Pub. Co., ISBN 0201029189 (1980)

[2] John David Jackson: Classical electrodynamics (2nd ed.), New York: Wiley, ISBN 047143132X (1975)

[3] D. Halliday, R. Resnick, J. Walker: Fundamentals of Physics, John Wiley and Sons; 7th edition (June 14, 2004), ISBN-13: 978-0471216438

[4] A. Eden: The search for Christian Doppler, Springer-Verlag, Wien (1992)

[5] E. Adrian: Doppler Effect, NCSA (1995)

[6] B. Ballot: Annalen der Physik und Chemie 11: 321–351 (1845)

[7] Fizeau: Acoustique et optique, Lecture, Société Philomathique de Paris, 29 December , (1848)

[8] J. S. Russell: Report of the Eighteenth Meeting of the British Association for the Advancement of Science, London, 18, 7, 37–38 (1849)

[9] J. Rosen, L.Q. Gothard: Encyclopedia of Physical Science. Infobase Publishing. p. 155 (2009)

[10] K.R. Symon: Mechanics (3rd edition), Addison-Wesley Pub. Co., ISBN 978-0201073928 (1971)

[11] S.T. Thomton: Classical Mechanics of Particles and Systems (5th edition), Cengage Learning, ISBN 978-0534408961 (2003)