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Kinematical Brownian motion of the harmonic oscillator in non-commutative space

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Abstract: In this work the Jacobi's second equality in the form of stochastic equation and the Wiener path integral approach are used to evaluate the probability density of harmonic oscillator in non-commutative space. Using the factorization theorem and the Mastubara formalism, the thermodynamic parameters are determined. The structure of Fokker-Planck equation remained the same even in a commutative and non-commutative space. Moreover, the non-commutative parameter is depicted for increasing value of the entropy.

Keywords: Brownian Motion, Stochastic Equation, Wiener Process, Fokker-Planck Equation, Non-Commutative Space

1. Introduction

Recently, the effect of random forces on the motion of a harmonic oscillator has been studied with much interest. Thus, considerable attention has been made in this field to understand the process of Euclidian quantum mechanics in the theory of dynamic Brownian motion [1, 2].This is explained by the close relationship between the Brownian motion and quantum mechanics [3-6].

When solving the problem of ultraviolet divergences, Heisenberg and Schrödinger conjectured that space-time coordinates may be mutually non-commutative[7].This was the starting point of several researches on the development and the applications of non-commutative space in string theory, cosmology, and more recently quantum hall effect, spin-orbit interaction and in quantum dots[7-10]. The non-commutative analogue of the random variable, random walk, Gaussian, Levis, Ito, and Markovian processes seem to be the most investigated class of stochastic process in physics when studying probability theory and has been of great concern over the last twenty years [10-15].

In the present work, with the help of the Jacobi's second equality in the form of stochastic equation in a non-commutative space, some thermodynamic parameters of a harmonic oscillator are determined using the theorem of factorization and Mastubara formalism. The organization of the work is as follows: section II presents the kinematical Brownian motion and its link with the Euclidean quantum mechanics. In Section 3, in the light of the factorization theorem and the Mastubara formalism, the entropy and the specific heat capacity are determined. We then close the work with the conclusion.

2. Kinematical Brownian Motion

Considering the following Jacobi's second equality in the form of stochastic equation [1],

$$\dot{x}_\tau + \frac{1}{m}\frac{\partial S_{cl}\left(x_\tau, \tau; x, t\right)}{\partial x_\tau} = \dot{\varphi}(\tau) \quad (1)$$

The corresponding Fokker-Planck equation associated to the stochastic equation (1) is:

$$\hbar\frac{\partial W}{\partial\tau}-\frac{\hbar}{m}\frac{\partial}{\partial x_\tau}\left(\frac{\partial S_{cl}(x_\tau,\tau;x,t)}{\partial x_\tau}W\right)=\frac{\hbar^2}{2m}\frac{\partial^2 W}{\partial x_\tau^2} \quad (2)$$

Using the factorization theorem, the solution of equation (2) yields:

$$W(x_0,0;x_\tau,\tau;x,t)=\frac{\exp\left[-\frac{1}{\hbar}S_{cl}(x_\tau,\tau;x,t)\right]}{\exp\left[-\frac{1}{\hbar}S_{cl}(x_0,0;x,t)\right]}Z(x_0,0;x_\tau,\tau) \quad (3)$$

where

$$Z(x_0,0;x_\tau,\tau)=\int_C \exp\left[-\int_0^\tau\left(V(x(s),s)-\frac{1}{2m}\frac{\partial^2 S_{cl}(x(s),s;x,t)}{\partial x_s^2}\right)ds\right]d_w x_s \quad (4)$$

is the Kac's formula with

$$x(0)=x_0\,,\,x(t)=x\,,\,x(\tau)=x_\tau$$

From Eq. (3), the solution of the Bloch equation is derived as:

$$\frac{\hbar\partial\exp\left[-\frac{1}{\hbar}S_{cl}(x_\tau,\tau;x,t)\right]}{\partial\tau}=\hat{\mathrm{H}}\exp\left[-\frac{1}{\hbar}S_{cl}(x_\tau,\tau;x,t)\right] \quad (5)$$

Which demonstrate that the Brownian motion is related to Euclidean quantum mechanics [1].

3. Kinematical Brownian Motion of the Harmonic Oscillator in Non-Commutative Space

In a non-commutative space the action of the harmonic oscillator is [16, 17]:

$$S_\theta(x'',T,x',0)=\frac{mw}{2\sqrt{k}\sin\left(w\sqrt{k}T\right)}\begin{bmatrix}\left(x_1'^2+x_2'^2+x_1''^2+x_2''^2\right)\cos\left(w\sqrt{k}T\right)-\\2\left(x_1'x_1''+x_2'x_2''\right)\cos\left(w\sqrt{k-1}T\right)+\\2\left(x_1'x_2''-x_2'x_1''\right)\sin\left(w\sqrt{k-1}T\right)\end{bmatrix} \quad (6)$$

where

$$k=1+\frac{1}{4}w^2m^2\theta^2$$

and

$$\theta=\begin{pmatrix}0&\theta_{12}&\theta_{13}\\-\theta_{12}&0&\theta_{23}\\-\theta_{13}&-\theta_{23}&0\end{pmatrix}$$

Here, it is important to mention that the factor θ is an anti-symmetric matrix with constant elements.

By considering equation (1) and taking into account the action given by (6), the stochastic equation of the harmonic oscillator in non-commutative space [18] can be established as follows:

$$\begin{cases}\dot{x}_{1\tau}+\frac{w}{\sqrt{k}\sin\left(w\sqrt{k}T\right)}\left[x_{1\tau}\cos\left(w\sqrt{k}T\right)-x_1''\cos\left(w\sqrt{k-1}T\right)+x_2''\sin\left(w\sqrt{k-1}T\right)\right]=\dot{\varphi}(\tau)\\ \dot{x}_{2\tau}+\frac{w}{\sqrt{k}\sin\left(w\sqrt{k}T\right)}\left[x_{2\tau}\cos\left(w\sqrt{k}T\right)-x_2''\cos\left(w\sqrt{k-1}T\right)+x_1''\sin\left(w\sqrt{k-1}T\right)\right]=\dot{\varphi}(\tau)\end{cases} \quad (7)$$

where $x_1'=x_{1\tau}$; $x_2'=x_{2\tau}$

Using the canonical transformation, the solution of Fokker-Planck equation associated to (7) is:

$$W_\theta^H(x_0,0;x_\tau,\tau;x,t)=\left[\frac{\left(2\pi\hbar\sin\left(w\sqrt{k}\tau\right)\right)^{2k}\sin\left(w\sqrt{k}(t-\tau)\right)}{(mw)^{2k}\sin\left(w\sqrt{k}t\right)}\right]^{-\frac{1}{2k}}\exp\left\{\frac{1}{2}\eta\begin{bmatrix}\xi_1^2\left(x_{10},0;x_{1\tau},\tau;x_1'',t\right)\\\xi_2^2\left(x_{20},0;x_{2\tau};\tau,x_2'',t\right)\\\chi\left(x_0,0;x_\tau,\tau;x'',t\right)\end{bmatrix}\right\} \quad (8)$$

where:

$$\xi_1\left(x_{10},0;x_{1\tau},\tau;x_1'',t\right)=x_{1\tau}-\frac{x_1''-x_{10}\cos\left(w\sqrt{k}\tau\right)}{\sin\left(w\sqrt{k}t\right)}\sin\left(w\sqrt{k}\tau\right)-x_{10}\cos\left(w\sqrt{k}t\right) \quad (9)$$

$$\xi_2\left(x_{20},0;x_{2\tau},\tau;x_2'',t\right)=x_{2\tau}-\frac{x_2''-x_{20}\cos\left(w\sqrt{k}\tau\right)}{\sin\left(w\sqrt{k}t\right)}\sin\left(w\sqrt{k}\tau\right)-x_{20}\cos\left(w\sqrt{k}t\right) \quad (10)$$

$$\chi(x_0,0;x_\tau,\tau;x,t)=\begin{bmatrix}(x_{10}x_{1\tau}+x_{20}x_{2\tau})\left(\cos\left(w\sqrt{k-1}\tau\right)-1\right)+\left(x_{1\tau}x_1''+x_{2\tau}x_2''\right)\cdot \\ \left(1-\cos w\sqrt{k-1}(t-\tau)\right)+ \\ \cos w\sqrt{k}\tau\left(\cos w\sqrt{k-1}t-1\right)\left(x_{10}x_1''+x_{20}x_2''\right)\end{bmatrix}\frac{\sin\left(w\sqrt{k-1}\tau\right)}{\sin\left(w\sqrt{k}t\right)}+$$

$$\left(1-\cos w\sqrt{k-1}t\right)\left(x_{10}x_1''+x_{20}x_2''\right)\left(\frac{\sin w\sqrt{k-1}\tau}{\sin w\sqrt{k}t}\right)^2+ \tag{11}$$

$$\sin w\sqrt{k-1}t\sin w\sqrt{k}(t-\tau)\left[\left(x_{10}x_{2\tau}-x_{20}x_{1\tau}\right)\cdot\right.$$
$$\left.\sin w\sqrt{k}t-\left(x_{10}x_{20}-x_{20}x_1''\right)\sin w\sqrt{k}\tau\right]+$$
$$\left(x_{1\tau}x_2''-x_{2\tau}x_1''\right)\sin w\sqrt{k-1}(t-\tau)\sin w\sqrt{k}\tau\sin w\sqrt{k}t$$

with

$$\eta=\frac{-mw\sin\left(w\sqrt{k}t\right)}{\hbar\sqrt{k}\sin\left(w\sqrt{k}\tau\right)\sin\left(w\sqrt{k}(t-\tau)\right)} \tag{12}$$

For $\theta=0$, equation (8) becomes

$$W_\theta^H(x_0,0;x_\tau,\tau;x,t)=\left[\frac{(2\pi\hbar\sin(w\tau))\sin(w(t-\tau))}{mw\sin(wt)}\right]^{\frac{-1}{2}}\exp\left\{\frac{-\mathrm{mw}\cos(\mathrm{wt})}{2\hbar\sin(\mathrm{w}\tau)\sin(\mathrm{w}(\mathrm{t}-\tau))}\left[\xi_1^2\left(\mathrm{x}_{10},0;\mathrm{x}_{1\tau},\tau;\mathrm{x}_1''\right)+\xi_2^2\left(\mathrm{x}_{20},0;\mathrm{x}_{2\tau},\tau;\mathrm{x}_2''\right)\right]\right\} \tag{13}$$

Equation (13) is the probability density of the harmonic oscillator in two dimensional spaces [1]. With the help of the factorization theorem, the solution of (8) is expressed as follows:

$$W_\theta^H=\frac{\exp\left\{-\frac{1}{\hbar}S_\theta^{H1}\left(x_\tau,\tau;x'',t\right)\right\}}{\exp\left\{-\frac{1}{\hbar}S_\theta^{H1}\left(x_0,0;x'',t\right)\right\}}Z_\theta(x_0,0;x_\tau,\tau) \tag{14}$$

where

$$Z_\theta(x_0,0,x_\tau,\tau)=\int_0\exp\left\{-\frac{1}{\hbar}\int_0^t V(x_\tau,\tau)dw(x_\tau)\right\}$$

and

$$V(x_\tau,\tau)=\frac{2}{\hbar}\frac{\partial S_\theta^{H1}\left(x_\tau,\tau;x'',t\right)}{\partial\tau}-\frac{1}{2m\hbar}\left[\frac{\partial S_\theta^{H1}\left(x_\tau,\tau;x'',t\right)}{\partial x_{1\tau}}+\frac{\partial S_\theta^{H2}\left(x_\tau,\tau;x'',t\right)}{\partial x_{1\tau}}+\frac{\partial S_\theta^{H1}\left(x_\tau,\tau;x''\right)}{\partial x_{2\tau}}+\frac{\partial S_\theta^{H2}\left(x_\tau,\tau,x'',t\right)}{\partial x_{2\tau}}\right]$$
$$+\frac{m}{\hbar}\left[\frac{\partial S_\theta^{H2}\left(x_\tau,\tau;x'',t\right)}{\partial x_{1\tau}}\left(\dot{x}_{1\tau}+\frac{1}{m}\frac{\partial S_\theta^{H1}\left(x_\tau,\tau;x'',t\right)}{\partial x_{1\tau}}\right)+\frac{\partial S_\theta^{H2}\left(x_\tau,\tau;x'',t\right)}{\partial x_{2\tau}}\left(\dot{x}_{2\tau}+\frac{1}{m}\frac{\partial S_\theta^{H1}\left(x_\tau,\tau;x,t\right)}{\partial x_{2\tau}}\right)\right]+$$
$$\frac{\hbar^2}{2m}\left[\left(\frac{\partial S_\theta^{H1}\left(x_\tau,\tau;x'',t\right)}{x_{1\tau}}\right)^2+\left(\frac{\partial S_\theta^{H1}\left(x_\tau,\tau;x'',t\right)}{x_{2\tau}}\right)^2\right]$$

$$S_\theta^{H1}\left(x_\tau,\tau;x'',t\right)=\frac{mw}{2\sqrt{k}\sin\left(w\sqrt{k}T\right)}\begin{bmatrix}\left(x_{1\tau}^2+x_{2\tau}^2+x_1''^2+x_2''^2\right)\cos\left(w\sqrt{k}T\right)+ \\ 2\left(x_{1\tau}x_1''+x_{2\tau}x_{2''}\right)\cos\left(w\sqrt{k-1}T\right)\end{bmatrix} \tag{15}$$

$$S_\theta^{H2}\left(x_\tau,\tau;x'',t\right)=\frac{mw}{\sqrt{k}\sin\left(w\sqrt{k}T\right)}\left[\left(x_{1\tau}x_2''-x_{2\tau}x_1''\right)\sin\left(w\sqrt{k-1}T\right)\right] \tag{16}$$

From (14), and with the help of the transformation leading to transition from Euclidian quantum mechanics to statistical physics, the solution of Bloch equation yields:

$$Y_\theta^H(x_\tau,x,\hbar\lambda)=\frac{mw}{2\pi i\hbar\sqrt{k}\,\sinh\left(w\hbar\sqrt{k}\,\lambda\right)}\exp\left\{-\frac{mw}{2i\sqrt{k}\hbar\sinh w\sqrt{k}\,(t-\tau)}\left[\begin{array}{l}\left(x_{1\tau}^2+x_{2\tau}^2+x_1^{"2}+x_2^{"2}\right)\cosh\left(w\hbar\sqrt{k}\,\lambda\right)\\-2\left(x_{1\tau}x_1^{"}+x_{2\tau}x_2^{"}\right)\cosh\left(w\hbar\sqrt{k-1}\lambda\right)\end{array}\right]\right\} \quad (17)$$

Then the partition function of the harmonic oscillator in non-commutative space is derived[19, 20]

$$Z_\theta^H=\frac{1}{2\left[\cosh\left(\hbar w\sqrt{k}\,\lambda\right)-\cosh\left(w\hbar\sqrt{k-1}\lambda\right)\right]} \quad (18)$$

The entropy, the internal energy and the specific heat capacity are respectively expressed by the following relations:

$$S_\theta^H(T)=\frac{w\hbar\lambda}{k_BT\sigma}-\ln 2\sigma \quad (19)$$

$$E_\theta^H(T)=-\frac{w\hbar\lambda}{k_B\sigma} \quad (20)$$

$$C_{\theta V}^H=\frac{w\hbar}{k_B(T\sigma)^2}\left[\frac{2w\hbar\sqrt{k(k-1)}}{k_B}a-\frac{w\hbar}{k_B}\left(\sqrt{k(k-1)}+k\right)b\right] \quad (21)$$

where:

$$\sigma=\cosh\left(\frac{w\hbar\sqrt{k}}{k_BT}\right)-\cosh\left(\frac{w\hbar\sqrt{k-1}}{k_BT}\right) \quad (22)$$

$$\lambda=\sqrt{k}\,\sinh\left(\frac{w\hbar\sqrt{k}}{k_BT}\right)-\sqrt{k-1}\,\sinh\left(\frac{w\hbar\sqrt{k-1}}{k_BT}\right) \quad (23)$$

$$a=\cosh\left(\frac{w\hbar\sqrt{k}}{k_BT}\right)\cosh\left(\frac{w\hbar\sqrt{k-1}}{k_BT}\right)-1 \quad (24)$$

$$b=\sinh\left(\frac{w\hbar\sqrt{k}}{k_BT}\right)\sinh\left(\frac{w\hbar\sqrt{k-1}}{k_BT}\right) \quad (25)$$

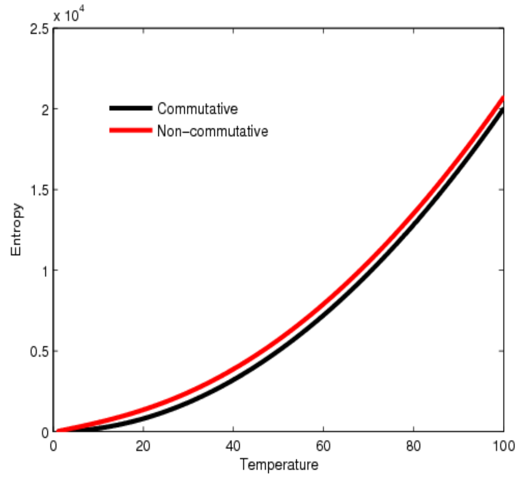

Figure 1. *Entropy versus temperature in commutative and non-commutative space.*

The curves plotted respectively in Figs. 1 to 3 show the influence of temperature on the behavior of the thermodynamic parameters when considering simultaneously the commutative and the non-commutative space. It is found that the non-commutative parameter is depicted for increasing value of the entropy [21, 22] and the internal energy.

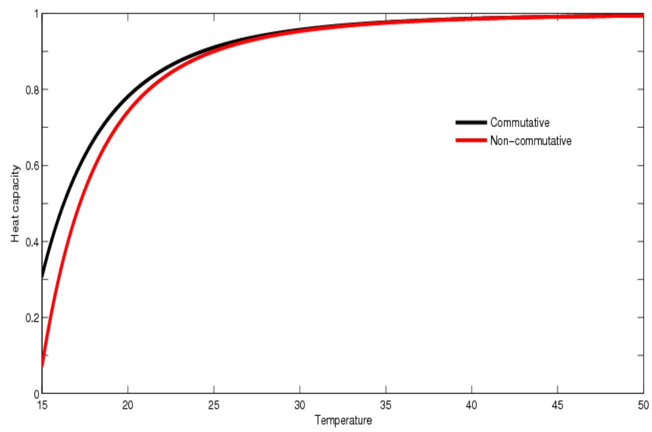

Figure 2. *Heat capacity versus temperature in commutative and non-commutative space.*

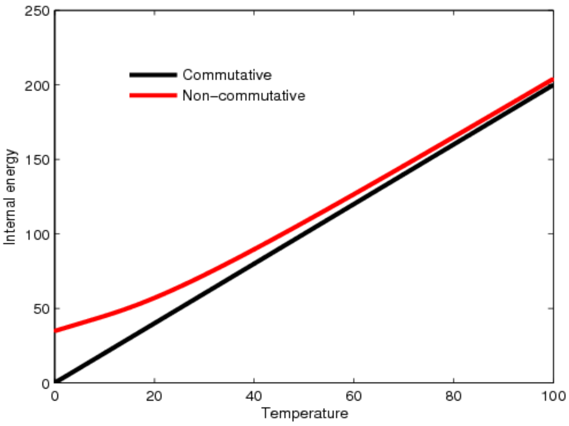

Figure 3. *Internal energy versus temperature in commutative and non-commutative space.*

4. Conclusion

In this work, we present the Jacobi's second equality in the form of stochastic equation of harmonic oscillator in non-commutative space. The results show that there is no modification on the structure of the Fokker-Planck equation; i.e. the factorization theorem is conserved in commutative and non-commutative space. Moreover, the non-commutative parameter is depicted for increasing value of the entropy and internal energy of the system.

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