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Characterization of Associative PU-algebras by the Notion of Derivations

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Abstract: In this manuscript we insert the concept of derivations in associative PU-algebras and discuss some of its important results such that we prove that for a mapping being a (Left, Right) or (Right, Left)-derivation of an associative PU-algebra then such a mapping is one-one. If a mapping is regular then it is identity. If any element of an associative PU-algebra satisfying the criteria of identity function then such a map is identity. We also prove some useful properties for a mapping being (Left, Right)-regular derivation of an associative PU-algebra and (Right, Left)-regular derivation of an associative PU-algebra. Moreover we prove that if a mapping is regular (Left, Right)-derivation of an associative PU-algebra then its Kernel is a subalgebra. We have no doubt that the research along this line can be kept up, and indeed, some results in this manuscript have already made up a foundation for further exploration concerning the further progression of PU-algebras. These definitions and main results can be similarly extended to some other algebraic systems such as BCH-algebras, Hilbert algebras, BF-algebras, J-algebras, WS-algebras, CI-algebras, SU-algebras, BCL-algebras, BP-algebras and BO-algebras, Z- algebras and so forth. The main purpose of our future work is to investigate the fuzzy derivations ideals in PU-algebras, which may have a lot of applications in different branches of theoretical physics and computer science.

Keywords: PU-Algebras, (Left, Right)-derivations of PU-algebras, (Right, Left)-derivations of PU-algebras, Regular Derivations of PU-algebras.

1. Introduction

In 1966, Y. Imai and K. Isèki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras [1-3]. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. Neggers et al. [4] introduced a notions, called Q-algebras, which is a generalization of BCH / BCI / BCK-algebras and generalized some theorems discussed in BCI-algebras. Megalai and Tamilarasi [5] laid down the foundation of a notion, called TM-algebra. Moreover, Mostafa et al. [6] introduced a new algebraic structure called PU-algebra, which is a dual for TM-algebra and investigated severed basic properties. Moreover they derived new view of several ideals on PU-algebra and studied some properties of them. Derivation is a very interesting and important area of research in the theory of algebraic structures in mathematics. Several authors [7-11] have studied derivations in rings and near rings. Jun and Xin [12] applied the notions of derivations in ring and near-ring theory to BCI-algebras, and they also introduced a new concept called a regular Derivation in BCI-algebra. They investigated some of its properties, defined a d -derivations ideal and gave conditions for an ideal to be d-derivations. Later, Abujabal and Al-Shehri [13], defined a left derivations in BCI-algebras and investigated a regular left derivations. Zhan and Liu [14] studied f-derivations in BCI-algebras and proved some results. Muhiuddin and Al-roqi [15, 16] introduced the notions of (α, β)-derivations in a BCI-algebras and investigated related properties. They provided a condition for a (α, β) - derivations to be regular. They also introduced the concepts of a d (α, β) - invariant (α, β) - derivations and α-ideal, and then they investigated their relations. Furthermore, they obtained some results on regular (α, β) - derivations. Moreover, they studied the notions of t-derivations on BCI-algebras [17] and obtain some of its related properties. Further, they characterized the notions of p-semisimple BCI-algebras X by using the notions of t-derivations. Abujabal and Shehri in their pioneer paper [18], defined the derivations as, for a self-map, d, for any algebra X, d is a left-right derivation (briefly (l, r)-derivation) of X if it satisfies the identity $d(\mathfrak{a} * \mathfrak{b})=(d(\mathfrak{a}) * \mathfrak{b}) \wedge (\mathfrak{a} * d(\mathfrak{b}))$. For all $\mathfrak{a}, \mathfrak{b} \in X$. If d satisfies the identity $d(\mathfrak{a} * \mathfrak{b}) = (\mathfrak{a} * d(\mathfrak{b})) \wedge (d(\mathfrak{a}) * \mathfrak{b})$ for all $\mathfrak{a}, \mathfrak{b} \in X$, then d is a right-left derivation (briefly (r, l)-derivation) of X. If d is both (l, r)- derivation and (r, l)-derivation, then d is a derivation of X. The aim of the paper is to complete the studies on PU-algebra; in particular, we aim to apply the notion of derivation on associative PU-algebra and obtain some related properties. We start with definitions and propositions on PU-algebra taken from [6].

Then, we redefine the notion of derivation in associative PU-algebra and prove that for ϕ being a (Left, Right) or (Right, Left) -derivation of an associative PU-algebra $\mathcal{Z}$ then ϕ is one-one map. If ϕ is a regular map then it is identity. If there exists an element $\mathfrak{a} \in \mathcal{Z}$ such that $\phi(\mathfrak{a})=\mathfrak{a}$ then the map ϕ is identity. We prove that If ϕ is (Left, Right) -regular derivation of $\mathcal{Z}$ then $\phi(\mathfrak{a}) = \mathfrak{a} \wedge \phi(\mathfrak{a})$ also if ϕ is (Right, Left)-regular derivation of $\mathcal{Z}$ then $\phi(\mathfrak{a})=\phi(\mathfrak{a}) \wedge \mathfrak{a}$, $\forall\, \mathfrak{a} \in \mathcal{Z}$. We prove that if ϕ is a self-map of an associative PU-Algebra $\mathcal{Z}$ then $(\mathfrak{a} * (\mathfrak{a} * \phi(\mathfrak{a}))) * \mathfrak{a}=(\phi(\mathfrak{a}) * (\phi(\mathfrak{a}) * \mathfrak{a}))* \mathfrak{a}$. We also prove that if ϕ is a regular (Right, Left)-derivation of an associative PU-algebra $\mathcal{Z}$ then $\mathrm{Ker}(\phi)=\{\mathfrak{a} \in \mathcal{Z}\colon \phi(\mathfrak{a})=0\}$ is a subalgebra of $\mathcal{Z}$.

2. Preliminaries

This section consists of some preliminary definitions and basic facts about PU-algebra which are useful in the proofs of our results. Throughout this research work we denote the PU-algebra always by $\mathcal{Z}$.

Definition 2.1: [6] PU-algebra ($\mathcal{Z}$,$*$, 0) is a class of the type (2, 0) algebras which satisfies the (P_1) and (P_2) conditions for all $\mathfrak{p}, \mathfrak{q}, \mathfrak{r} \in \mathcal{Z}$, where

(P_1) $0 * \mathfrak{a} = \mathfrak{a}$ (P_2) $(\mathfrak{a} * \mathfrak{c}) * (\mathfrak{b} * \mathfrak{c})=\mathfrak{b} * \mathfrak{a}$

While the binary relation '$\leq$' on $\mathcal{Z}$ is defined as $\mathfrak{a} \leq \mathfrak{b} \Leftrightarrow \mathfrak{b} * \mathfrak{a}=0$.

Proposition 2.2: [6] In PU-algebra ($\mathcal{Z}$,$*$, 0) the following results are true for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathcal{Z}$.

(P_3) $\mathfrak{a} * \mathfrak{a}=0$

(P_4) $(\mathfrak{a} * \mathfrak{c}) * \mathfrak{c} = \mathfrak{a}$

(P_5) $\mathfrak{a} * (\mathfrak{b} * \mathfrak{c})=\mathfrak{b} * (\mathfrak{a} * \mathfrak{c})$

(P_6) $\mathfrak{a} * (\mathfrak{b} * \mathfrak{a})=\mathfrak{b} * 0$

(P_7) The following three results are similar in ($\mathcal{Z}$,$*$, 0)

(1): $\mathfrak{b} = \mathfrak{c}$ (2): $\mathfrak{b} * \mathfrak{a} = \mathfrak{c} * \mathfrak{a}$ (3): $\mathfrak{a} * \mathfrak{b} = \mathfrak{a} * \mathfrak{c}$

(P_8) Both (left and right) cancellation properties hold in ($\mathcal{Z}$,$*$, 0).

Definition 2.3: [6] PU-algebra ($\mathcal{Z}$, $*$, 0) is said to be associative if it satisfies the condition $\mathfrak{a} * (\mathfrak{b} * \mathfrak{c})=(\mathfrak{a} * \mathfrak{b}) * \mathfrak{c}$ for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathcal{Z}$.

3. Main Results

Definition 3.1:- Let ($\mathcal{Z}$,$*$, 0) is an associative PU-algebra and $\phi\colon \mathcal{Z} \longrightarrow \mathcal{Z}$ is a self-map then ϕ called (Left, Right)-derivation on $\mathcal{Z}$ if $\phi(\mathfrak{a} * \mathfrak{b}) = (\phi(\mathfrak{a}) * \mathfrak{b}) \wedge (\mathfrak{a} * \phi(\mathfrak{b}))$.

Definition 3.2:- Let ($\mathcal{Z}$, $*$, 0) is an associative PU-algebra and $\phi\colon \mathcal{Z} \longrightarrow \mathcal{Z}$ is a self-map then ϕ is called (Right, Left)-derivation on $\mathcal{Z}$ if $\phi(\mathfrak{a} * \mathfrak{b})=(\mathfrak{a} * \phi(\mathfrak{b})) \wedge (\phi(\mathfrak{a}) * \mathfrak{b})$.

Definition 3.3: If ϕ is both (Left, Right)-derivation and (Right, Left)-derivation on $\mathcal{Z}$ then ϕ is called derivation on $\mathcal{Z}$.

Definition 3.4: A self-map $\phi\colon \mathcal{Z} \longrightarrow \mathcal{Z}$ on associative PU-algebra $\mathcal{Z}$ is called regular if $\phi(0) = 0$.

Example 3.5: Let the set $\mathcal{Z}=\{0, \mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$ defined by the following table.

Table 1. *Tabular arrangement of the values of the set satisfying the axioms of associative PU -algebra.*

$*$	0	$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{c}$
0	0	$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{c}$
$\mathfrak{a}$	$\mathfrak{a}$	0	$\mathfrak{c}$	$\mathfrak{b}$
$\mathfrak{b}$	$\mathfrak{b}$	$\mathfrak{c}$	0	$\mathfrak{a}$
$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{b}$	$\mathfrak{a}$	0

Is an associative PU-algebra and a map, $\phi\colon \mathcal{Z} \longrightarrow \mathcal{Z}$ defined by

$\phi(0)=\mathfrak{c}$ $\phi(\mathfrak{a})=\mathfrak{b}$ $\phi(\mathfrak{b})=\mathfrak{a}$ and $\phi(\mathfrak{c})=0$ is both (Left, Right)-derivation and (Right, Left)-derivation on $\mathcal{Z}$ and thus a derivation of $\mathcal{Z}$.

Proposition 3.6: Let ϕ be a (Left, Right)-derivation of an associative PU-algebra $\mathcal{Z}$ then

(P_9): $\phi(0)=\phi(\mathfrak{a}) * \mathfrak{a}$, $\forall \mathfrak{a} \in \mathcal{Z}$.

(P_{10}): ϕ is one-one map.

(P_{11}): If ϕ is a regular map then it is identity.

(P_{12}): If there exists an element $\mathfrak{a} \in \mathcal{Z}$ such that $\phi(\mathfrak{a})=\mathfrak{a}$ then the map ϕ is identity.

(P_{13}): If $\phi(\mathfrak{b}) * \mathfrak{a}=0$ or $\mathfrak{a} * \phi(\mathfrak{b})=0$ then $\phi(\mathfrak{b})=\mathfrak{a}$, $\forall\, \mathfrak{a}, \mathfrak{b} \in \mathcal{Z}$ i.e. ϕ is constant.

Proof (P_9):

$$\phi(0)=\phi(\mathfrak{a} * \mathfrak{a}), \because \text{by } (P_3)$$

$$=(\phi(\mathfrak{a}) * \mathfrak{a}) \wedge (\mathfrak{a} * \phi(\mathfrak{a}))$$

$$=(\mathfrak{a} * \phi(\mathfrak{a})) * [(\mathfrak{a} * \phi(\mathfrak{a})) * (\phi(\mathfrak{a}) * \mathfrak{a})]$$

$$=[(\mathfrak{a} * \phi(\mathfrak{a})) * (\mathfrak{a} * \phi(\mathfrak{a}))] * (\phi(\mathfrak{a}) * \mathfrak{a}), \because \mathcal{Z} \text{ is associative}$$

$$=0 * (\phi(\mathfrak{a}) * \mathfrak{a}), \because \text{by } (P_3)$$

$$=(\phi(\mathfrak{a}) * \mathfrak{a}), \because \text{by } (P_1)$$

Proof (P_{10}): Let $\mathfrak{a}, \mathfrak{b} \in \mathcal{Z}$ such that

$$\phi(\mathfrak{a})=\phi(\mathfrak{b}) \tag{1}$$

From (P_9) we have

$$\phi(0)=\phi(\mathfrak{a}) * \mathfrak{a}\ \forall \mathfrak{a} \in \mathcal{Z} \tag{2}$$

Also from (P_9) we have

$$\phi(0)=\phi(\mathfrak{b}) * \mathfrak{b}\ \forall\, \mathfrak{b} \in \mathcal{Z} \tag{3}$$

From (2) and (3) we get

$$\phi(\mathfrak{a}) * \mathfrak{a} = \phi(\mathfrak{b}) * \mathfrak{b} \tag{4}$$

Using the result of equation (1) in equation (4) we get

$$\phi(\mathfrak{a}) * \mathfrak{a} = \phi(\mathfrak{a}) * \mathfrak{b} \tag{5}$$

By (P_8) left cancellation law holds in $\mathcal{Z}$ therefore from (5) we get $\mathfrak{a} = \mathfrak{b}$. Hence ϕ is one to one.

Proof (P_{11}): Let ϕ is regular then we have

$$\phi(0)=0 \tag{6}$$

From (P_9) we have

$$\phi(0)=\phi(\mathfrak{a}) * \mathfrak{a}\ \forall\, \mathfrak{a} \in \mathcal{Z} \tag{7}$$

From (6) and (7) we get

$$\phi(\mathfrak{a}) * \mathfrak{a}=0\ \forall\, \mathfrak{a} \in \mathcal{Z} \tag{8}$$

Now by using (P_3) in the right hand side of equation (8) then (8) becomes

$$\phi(\mathfrak{a}) * \mathfrak{a} = \mathfrak{a} * \mathfrak{a}\ \forall\, \mathfrak{a} \in \mathcal{Z}, \because\ \mathfrak{a} * \mathfrak{a}=0 \tag{9}$$

By (P_8) right cancellation law holds in $\mathcal{Z}$ therefore (9) becomes $\phi(\mathfrak{a})=\mathfrak{a}\ \forall\, \mathfrak{a} \in \mathcal{Z}$.

Hence ϕ is the identity map.

Proof (P_{12}): Let

$$\phi(\mathfrak{a})=\mathfrak{a} \tag{10}$$

Now by proposition (P_7) equation (10) is equivalent to

$$\phi(\mathfrak{a}) * \mathfrak{a} = \mathfrak{a} * \mathfrak{a} \implies \phi(\mathfrak{a}) * \mathfrak{a}=0, \because \text{ by } (P_3) \tag{11}$$

From (P_9) we have

$$\phi(0)=\phi(\mathfrak{a}) * \mathfrak{a}\ \forall\, \mathfrak{a} \in \mathcal{Z} \tag{12}$$

So now using the result of equation (12) in the left hand side of equation (11) we get

$\phi(0)=0 \implies \phi$ is regular which by (P_{11}) ϕ is the identity map.

Proof (P_{11}): Let

$$\phi(\mathfrak{b}) * \mathfrak{a}=0 \tag{13}$$

by proposition (P_3) equation (13) becomes

$$\phi(\mathfrak{b}) * \mathfrak{a} = \mathfrak{a} * \mathfrak{a} \tag{14}$$

by (P_8) right cancellation law holds in $\mathcal{Z}$ therefore (14) becomes $\phi(\mathfrak{b})=\mathfrak{a}$.

Similarly for

$$\mathfrak{a} * \phi(\mathfrak{b})=0 \tag{15}$$

by (P_3) equation (15) becomes

$$\mathfrak{a} * \phi(\mathfrak{b})=\mathfrak{a} * \mathfrak{a} \tag{16}$$

by (P_8) left cancellation law holds in $\mathcal{Z}$ therefore (16) becomes $\phi(\mathfrak{b})=\mathfrak{a}$.

Proposition 3.7:- Let ϕ be a (Right, Left)-derivation of an associative PU-algebra $\mathcal{Z}$ then

(P_{14}): $\phi(0)=\mathfrak{a} * \phi(\mathfrak{a}), \forall\, \mathfrak{a} \in \mathcal{Z}$

(P_{15}): ϕ is one-one map.

(P_{16}): If ϕ is a regular map then it is identity.

(P_{17}): If there exists an element $\mathfrak{a} \in \mathcal{Z}$ such that $\phi(\mathfrak{a})=\mathfrak{a}$ then the map

ϕ is identity.

(P_{18}): If $\phi(\mathfrak{b}) * \mathfrak{a}=0$ or $\mathfrak{a} * \phi(\mathfrak{b})=0$ then $\phi(\mathfrak{b})=\mathfrak{a}, \forall\, \mathfrak{a}, \mathfrak{b} \in \mathcal{Z}$ i.e. ϕ is constant.

Proof (P_{14}):

$$\phi(0)=\phi(\mathfrak{a} * \mathfrak{a}), \because \text{ by } (P_3)$$

$$=(\mathfrak{a} * \phi(\mathfrak{a})) \wedge (\phi(\mathfrak{a}) * \mathfrak{a})$$

$$=(\phi(\mathfrak{a}) * \mathfrak{a}) * [(\phi(\mathfrak{a}) * \mathfrak{a}) * (\mathfrak{a} * \phi(\mathfrak{a}))]$$

$$=[(\phi(\mathfrak{a}) * \mathfrak{a}) * (\phi(\mathfrak{a}) * \mathfrak{a})] * (\mathfrak{a} * \phi(\mathfrak{a})), \because\ \mathcal{Z} \text{ is associative}$$

$$=0 * (\mathfrak{a} * \phi(\mathfrak{a})), \because \text{ by } (P_3)$$

$$=\mathfrak{a} * \phi(\mathfrak{a}), \because \text{ by } (P_1)$$

Proof (P_{15}): Let $\mathfrak{a}, \mathfrak{b} \in \mathcal{Z}$ such that

$$\phi(\mathfrak{a})=\phi(\mathfrak{b}) \tag{17}$$

From (P_{14}) we have

$$\phi(0)=\mathfrak{a} * \phi(\mathfrak{a})\ \forall\, \mathfrak{a} \in \mathcal{Z} \tag{18}$$

Also from (P_{14}) we have

$$\phi(0)=\mathfrak{b} * \phi(\mathfrak{b})\ \forall\, \mathfrak{b} \in \mathcal{Z} \tag{19}$$

From (17) and (18) we get

$$\mathfrak{a} * \phi(\mathfrak{a})=\mathfrak{b} * \phi(\mathfrak{b}) \tag{20}$$

Using (17) in the right hand side of equation (20) we get

$$\mathfrak{a} * \phi(\mathfrak{a})=\mathfrak{b} * \phi(\mathfrak{a}) \tag{21}$$

By (P_8) right cancellation law holds in $\mathcal{Z}$ therefore from (21) we get $\mathfrak{a} = \mathfrak{b}$ that is ϕ is one to one.

Proof (P_{16}): Let ϕ is regular then we have

$$\phi(0)=0 \tag{22}$$

From (P_{14}) we have

$$\phi(0)=\mathfrak{a} * \phi(\mathfrak{a})\ \forall\, \mathfrak{a} \in \mathcal{Z} \tag{23}$$

From (22) and (23) we get

$$\mathfrak{a} * \phi(\mathfrak{a})=0\ \forall\, \mathfrak{a} \in \mathcal{Z} \tag{24}$$

Now by using (P_3) in the right hand side of equation (24) then (24) becomes

$$\mathfrak{a} * \phi(\mathfrak{a})=\mathfrak{a} * \mathfrak{a}\ \forall\, \mathfrak{a} \in \mathcal{Z} \because\ \mathfrak{a} * \mathfrak{a}=0 \tag{25}$$

By (P_8) left cancellation law holds in $\mathcal{Z}$ therefore (iv) becomes $\phi(\mathfrak{a})=\mathfrak{a}, \forall\, \mathfrak{a} \in \mathcal{Z}$.

Hence ϕ is the identity map.

Proof (P_{17}): Let

$$\phi(\mathfrak{a})=\mathfrak{a} \tag{26}$$

Now by proposition (P_7) equation (26) is equivalent to

$$\mathfrak{a} * \phi(\mathfrak{a})=\mathfrak{a} * \mathfrak{a} \implies \mathfrak{a} * \phi(\mathfrak{a})=0, \because \text{ by } (P_3) \tag{27}$$

From (P_{14}) we have

$$\phi(0)=\mathfrak{a} * \phi(\mathfrak{a})\ \forall\, \mathfrak{a} \in \mathcal{Z} \tag{28}$$

so now using (28) in (27) we get $\phi(0)=0 \implies \phi$ is regular which by (P_{16}) ϕ is the identity map.

Proof (P_{18}):

$$\text{Let } \phi(\mathfrak{b}) * \mathfrak{a}=0 \tag{29}$$

by proposition (P_3) the right hand side of equation (29) becomes

$$\phi(\mathfrak{b}) * \mathfrak{a} = \mathfrak{a} * \mathfrak{a} \tag{30}$$

by (P_8) right cancellation law holds in $\mathcal{Z}$ therefore (30) becomes $\phi(\mathfrak{b})=\mathfrak{a}$

Similarly

$$\mathfrak{a} * \phi(\mathfrak{b})=0 \tag{31}$$

by (P_3) equation (31) becomes,

$$\mathfrak{a} * \phi(\mathfrak{b})=\mathfrak{a} * \mathfrak{a} \tag{32}$$

by (P_8) left cancellation law holds in $\mathcal{Z}$ therefore (32) becomes $\phi(\mathfrak{b})=\mathfrak{a}$.

Theorem 3.8: Let $\mathcal{Z}$ is an associative PU-algebra

(P_a): If ϕ is (Left, Right) -regular derivation of $\mathcal{Z}$ then $\phi(\mathfrak{a})=\mathfrak{a} \wedge \phi(\mathfrak{a})\ \forall\ \mathfrak{a} \in \mathcal{Z}$.

(P_b): If ϕ is (Right, Left) -regular derivation of $\mathcal{Z}$ then $\phi(\mathfrak{a})=\phi(\mathfrak{a}) \wedge \mathfrak{a}\ \forall\ \mathfrak{a} \in \mathcal{Z}$.

Proof (P_a): Since ϕ is regular therefore we have

$$\phi(0)=0 \tag{33}$$

Now consider for some $\mathfrak{a} \in \mathcal{Z}$ we have

$$\phi(\mathfrak{a})=\phi(0 * \mathfrak{a}),\ \because \text{by } (P_1)$$

$$=(\phi(0) * \mathfrak{a}) \wedge (0 * \phi(\mathfrak{a})), \because \text{by definition 3.1}$$

$$=(0 * \mathfrak{a}) \wedge (0 * \phi(\mathfrak{a})),\ \because \text{by using (33)}$$

$$=\mathfrak{a} \wedge \phi(\mathfrak{a}),\ \because \text{by } (P_1)$$

Proof (P_b): Since ϕ is regular therefore we have

$$\phi(0)=0 \tag{34}$$

Now consider for some $\in \mathcal{Z}$, $\phi(\mathfrak{a})=\phi(0 * \mathfrak{a})$, $\because$ by (P_1)

$=(0 * \phi(\mathfrak{a})) \wedge (\phi(0) * \mathfrak{a})$ $\because$ by definition 3.1

$=(0 * \phi(\mathfrak{a})) \wedge (0 * \mathfrak{a})$ $\because$ by using (34)

$$=\phi(\mathfrak{a}) \wedge \mathfrak{a}\ \because \text{by } (P_1)$$

Theorem 3.9: Let ϕ is a self-map of an associative PU-Algebra $\mathcal{Z}$ then $(\mathfrak{a} * (\mathfrak{a} * \phi(\mathfrak{a}))) * \mathfrak{a}=(\phi(\mathfrak{a}) * (\phi(\mathfrak{a}) * \mathfrak{a})) * \mathfrak{a}$.

Proof: Since by (theorem 3.8 (P_b)) we have,

$$\phi(\mathfrak{a})=\phi(\mathfrak{a}) \wedge \mathfrak{a} = \mathfrak{a} * (\mathfrak{a} * \phi(\mathfrak{a})) \tag{35}$$

By (P_7) equation (35) is equivalent to

$$\phi(\mathfrak{a}) * \mathfrak{a}=(\mathfrak{a} * (\mathfrak{a} * \phi(\mathfrak{a}))) * \mathfrak{a} \tag{36}$$

on the other hand from (theorem 3.8 (P_a)) we have

$$\phi(\mathfrak{a})=\mathfrak{a} \wedge \phi(\mathfrak{a})=\phi(\mathfrak{a}) * (\phi(\mathfrak{a}) * \mathfrak{a}) \tag{37}$$

Similarly by (P_7) equation (37) is equivalent to

$$\phi(\mathfrak{a}) * \mathfrak{a}=(\phi(\mathfrak{a}) * (\phi(\mathfrak{a}) * \mathfrak{a})) * \mathfrak{a} \tag{38}$$

from (36) and (38) we get

$$(\mathfrak{a} * (\mathfrak{a} * \phi(\mathfrak{a}))) * \mathfrak{a}=(\phi(\mathfrak{a}) * (\phi(\mathfrak{a}) * \mathfrak{a})) * \mathfrak{a}.$$

Theorem 3.10: If ϕ is a derivation on an associative PU-algebra $\mathcal{Z}$ then $\forall\ \mathfrak{a} \in \mathcal{Z}$

$$(P_c): \phi(\mathfrak{a} * \phi(\mathfrak{a}))=0$$

$$(P_d): \phi(\phi(\mathfrak{a}) * \mathfrak{a})=0$$

Proof (P_c): Let ϕ is a (Left, Right)-derivation on $\mathcal{Z}$ then

$$\phi(\mathfrak{a} * \phi(\mathfrak{a}))=(\phi(\mathfrak{a}) * \phi(\mathfrak{a})) \wedge (\mathfrak{a} * \phi(\phi(\mathfrak{a})))=0 \wedge (\mathfrak{a} * \phi(\phi(\mathfrak{a}))),\ \because \text{by using } (P_3)$$

$$\Longrightarrow \phi(\mathfrak{a} * \phi(\mathfrak{a}))=(\mathfrak{a} * \phi(\phi(\mathfrak{a}))) * [(\mathfrak{a} * \phi(\phi(\mathfrak{a}))) * 0] \tag{39}$$

As $\mathcal{Z}$ is an associative PU-algebra therefore we can write equation (39) as

$$\phi(\mathfrak{a} * \phi(\mathfrak{a}))=[\mathfrak{a} * \phi(\phi(\mathfrak{a}))) * (\mathfrak{a} * \phi(\phi(\mathfrak{a}))] * 0=0 * 0=0,$$

$\because$ by using (P_3)

Proof (P_d): Let ϕ is a (Right, Left)-derivation on $\mathcal{Z}$ then

$$\phi(\phi(\mathfrak{a}) * \mathfrak{a})=(\phi(\mathfrak{a}) * \phi(\mathfrak{a})) \wedge (\phi(\phi(\mathfrak{a})) * \mathfrak{a})=0 \wedge (\phi(\phi(\mathfrak{a})) * \mathfrak{a}),\ \because \text{by using } (P_3)$$

$$\Longrightarrow \phi(\phi(\mathfrak{a}) * \mathfrak{a})=(\phi(\phi(\mathfrak{a})) * \mathfrak{a}) * [(\phi(\phi(\mathfrak{a})) * \mathfrak{a}) * 0] \tag{40}$$

As $\mathcal{Z}$ is an associative PU-algebra therefore (39) can be written as

$$\phi(\phi(\mathfrak{a})) * \mathfrak{a})=[(\phi(\phi(\mathfrak{a})) * \mathfrak{a}) * (\phi(\phi(\mathfrak{a})) * \mathfrak{a})] * 0=0 * 0=0, \because \text{by using } (P_3)$$

Theorem 3.11: Let ϕ is a regular (Left, Right)-derivation of an associative PU-algebra $\mathcal{Z}$ then the following results hold in $\mathcal{Z}$.

$$(P_e): \phi(\mathfrak{a})=\mathfrak{a}$$

$$(P_f): \phi(\mathfrak{a}) * \mathfrak{b} = \mathfrak{a} * \phi(\mathfrak{b})\ \forall\ \mathfrak{a}, \mathfrak{b} \in \mathcal{Z}$$

$$(P_g): \phi(\mathfrak{b} * \mathfrak{a})=\phi(\mathfrak{a}) * \mathfrak{b} = \phi(\mathfrak{a}) * \phi(\mathfrak{b})=\mathfrak{a} * \phi(\mathfrak{b})$$

$$(P_h): \mathrm{Ker}(\phi)=\{\mathfrak{a} \in \mathcal{Z}: \phi(\mathfrak{a})=0\} \text{ is a subalgebra of } \mathcal{Z}.$$

Proof (P_e): $\phi(\mathfrak{a})=\phi(0 * \mathfrak{a})$,

$$\because \text{by using } (P_1)$$

$$=(\phi(0) * \mathfrak{a}) \wedge (0 * \phi(\mathfrak{a}))$$

$$=(0 * \mathfrak{a}) \wedge (0 * \phi(\mathfrak{a})), \because \phi \text{ is regular}$$

$$=\mathfrak{a} \wedge \phi(\mathfrak{a}),\ \because \text{by using } (P_1)$$

$$=\phi(\mathfrak{a}) * (\phi(\mathfrak{a}) * \mathfrak{a})$$

$$=(\phi(\mathfrak{a}) * \phi(\mathfrak{a})) * \mathfrak{a},\ \because \mathcal{Z} \text{ is associative}$$

$$=0 * \mathfrak{a} = \mathfrak{a}\ \because \text{by using } (P_1)$$

Hence $\phi(\mathfrak{a})=\mathfrak{a}$.

Proof (P_f): As ϕ is a regular (Left, Right)-derivation of an associative PU-algebra $\mathcal{Z}$ then by (P_e) we have

$$\phi(a)=a, \forall a \in \mathcal{Z} \quad (41)$$

And

$$\phi(b)=b, \forall b \in \mathcal{Z} \quad (42)$$

By using the results of equations (41) and (42) we get $\phi(a) * b = a * b = a * \phi(b)$.

Proof (P_g): As ϕ is a regular (Left, Right)-derivation of an associative PU-algebra $\mathcal{Z}$ then by (P_e) we have

$$\phi(a)=a, \forall a \in \mathcal{Z} \quad (43)$$

Therefore for any $a, b \in \mathcal{Z}$,

$$\text{We have } \phi(a * b)=a * b \because \text{by using } (P_e) \quad (44)$$

$$\Longrightarrow \phi(a * b)=\phi(a) * \phi(b), \because \text{by equation (43)} \quad (45)$$

$$\Longrightarrow \phi(a * b)=\phi(a) * b, \because \text{by equation (43)} \quad (46)$$

$$\text{And } \phi(a * b)=a * \phi(b) \because \text{by equation (43)} \quad (47)$$

The equations (45), (46) and (47) imply that

$$\phi(b * a)=\phi(a) * b = \phi(a) * \phi(b)=a * \phi(b)$$

$$\text{Proof } (P_h)\text{: Let } a, b \in \mathrm{Ker}(\phi) \text{ then } \phi(a)=0 \quad (48)$$

And

$$\phi(b)=0 \quad (49)$$

As ϕ is a regular derivation therefore from (P_g) we have,

$$\phi(a * b)=\phi(a) * \phi(b) \quad (50)$$

Using (48) and (49) in the right hand side of equation (50) we get $\phi(a * b)=0$

$\Longrightarrow a * b \in \mathrm{Ker}(\phi) \Longrightarrow \mathrm{Ker}(\phi)$ is a subalgebra of $\mathcal{Z}$.

4. Conclusion

We see that derivations with special properties play a central role in the investigation of the structure of an algebraic system.

The forthcoming study of derivations in PU-algebras may be the following topics are worth to be taken into account.

To describe left derivations in PU-algebras and investigate a regular left derivations by using this concept.

To introduce the concept of f-derivations, t-derivations, t-bi-derivations and (α, β)-derivations in PU-algebras.

To refer this concept to some other algebraic structures.

To consider the results of this concept to some possible applications in information systems and computer sciences.

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