

Automated Proof Search System for Logic of Correlated Knowledge

Haroldas Giedra¹, Romas Alonderis²

¹Institute of Computer Science, Vilnius University, Vilnius, Lithuania

²Institute of Data Science and Digital Technologies, Vilnius University, Vilnius, Lithuania

Email address:

haroldas.giedra@mif.vu.lt (H. Giedra), romas.alonderis@mif.vu.lt (R. Alonderis)

To cite this article:

Haroldas Giedra, Romas Alonderis. Automated Proof Search System for Logic of Correlated Knowledge. *American Journal of Mathematical and Computer Modelling*. Vol. 5, No. 2, 2020, pp. 29-42. doi: 10.11648/j.ajmcm.20200502.11

Received: October 4, 2019; **Accepted:** March 2, 2020; **Published:** April 14, 2020

Abstract: Logic of correlated knowledge is one of the latest development in logical systems, allowing to handle information about quantum systems. Quantum system may consist of one or more elementary particles. Associating agent to each particle, we get multi-agent system, where agents can perform observations and get results. Allowing communication between agents, correlations such as quantum entanglement can be extracted. This can not be done by traditional epistemic logic or logic of distributed knowledge. Our main scientific result is proof search system GS-LCK-PROC for logic of correlated knowledge, which lets to reason about knowledge automatically. The core of the system is the sequent calculus GS-LCK with the properties of soundness, completeness, admissibility of cut and structural rules, and invertibility of all rules. The ideas of semantic internalization are used to get such properties for the calculus. The calculus provides convenient means for backward proof search and decision procedure for logic of correlated knowledge. The procedure generates a finite model for each sequent. As a result we get termination of the proof search and decidability of logic of correlated knowledge.

Keywords: Logic of Correlated Knowledge, Sequent Calculus, Automated Proof System, Decidability, Soundness, Completeness, Admissibility of the Cut Rule

1. Introduction

Information about quantum systems can be handled using logical calculi. From historical point of view the research of the area went in two main directions [20]. The first was originated by J. von Neumann and G. Birkhoff [7], introducing the ideas of quantum logic. However some important impossibility results were obtained [1, 19]. D. Aerts, C. Randall and D. Foulis showed that quantum logic rises problems when trying to describe compound systems consisting of more than one elementary particle that can exhibit quantum entanglement. The other direction was the Mackey-Piron way [13, 14, 18] - the research on an axiomatic system that can be represented as the logic of projection operators on a generalized Hilbert space. One of the latest development in this way is quantum modal logics. The research have been done by A. Baltag and S. Smets [4, 5], F. Boge [8], V. Vilasini [22] and N. Nurgalieva [17]. Our

study covers field of logic of correlated knowledge which is part of quantum modal logics. More approaches on logic and quantum mechanics you can find in [2, 6, 9].

Logic of correlated knowledge (LCK) has been introduced by A. Baltag and S. Smets in [3]. LCK is an epistemic logic enriched by observational capabilities of agents. Applications of the epistemic logic cover fields such as distributed systems, merging of knowledge bases, robotics or network security in computer science and artificial intelligence. By adding observational capabilities to agents, logic of correlated knowledge can be applied to reason about systems where knowledge correlate between spatially distributed parts of the system. This includes any social system, quantum system, distributed information system, traffic light system or any other system where knowledge is correlated.

Quantum system may consist of one or more elementary particles. Associating agent to each particle, we get multi-agent system, where agents can perform observations and get

results. Allowing communication between agents, correlations such as quantum entanglement can be extracted. This can not be done by traditional epistemic logic or logic of distributed knowledge.

Our main scientific result is proof search system GS-LCK-PROC for logic of correlated knowledge, which lets to reason about knowledge automatically. The core of the system is the sequent calculus GS-LCK with the properties of soundness, completeness, admissibility of cut and structural rules, and invertibility of all rules. The ideas of semantic internalization, suggested by Sara Negri in [15], are used to get such properties for the calculus. The calculus provides convenient means for backward proof search and decision procedure for logic of correlated knowledge. The procedure generates a finite model for each sequent. As a result we get termination of the proof search and decidability of logic of correlated knowledge.

We start by defining syntax, semantics, and the Hilbert style proof system for logic of correlated knowledge in section 2. In section 3 we present Gentzen style sequent calculus GS-LCK and the properties of the proof system. Soundness and completeness of the GS-LCK and the properties of admissibility of weakening, contraction and cut are proved in sections 4, 5 and 6. Automated proof search system GS-LCK-PROC and decidability of logic of correlated knowledge are

$$A := p \mid o^r \mid \neg A \mid A \vee B \mid A \wedge B \mid A \rightarrow B \mid K_I A$$

Where p is any atomic proposition, $o = (o_a)_{a \in I} \in O_I$, $r \in R$, and $I \subseteq N$.

2.2. Semantics

Consider a system, composed of N components or locations. Agents can be associated to locations, where they will perform observations. States (configurations) of the system are functions $s : O_{a_1} \times \dots \times O_{a_n} \rightarrow R$ or $s_I : O_I \rightarrow R$, where $I \subseteq N$ and a set of results R is in the structure (R, Σ) together with an abstract operation $\Sigma : \mathcal{P}(R) \rightarrow R$ of composing results. The operation Σ maybe partial (defined only for some subsets $A \subseteq R$), but it is required to satisfy the condition: $\Sigma\{\Sigma A_k : k \in K\} = \Sigma(\cup_{k \in K} A_k)$ whenever $\{A_k : k \in K\}$ are pairwise disjoint. $\mathcal{P}(R)$ is a power set of R . For every joint observation $e \in O_I$, the local state s_I is defined as: $s_I((e_a)_{a \in I}) := \Sigma\{s(o) : o \in O_{a_1} \times \dots \times O_{a_n} \text{ such that } o_a = e_a \text{ for all } a \in I\}$.

If s and t are two possible states of the system and a group of agents I can make exactly the same observations in these two states, then these states are observationally equivalent to I , and it is written as $s \stackrel{I}{\sim} t$. Observational equivalence is defined as follows: *Definition 2(Observational equivalence)* Two states s and t are observationally equivalent $s \stackrel{I}{\sim} t$ iff $s_I = t_I$.

A model of logic of correlated knowledge is a multi-modal Kripke model [12], where the relations between states mean observational equivalence. It is defined as: *Definition 3 Model of logic of correlated knowledge* For a set of states S , a family of binary relations $\{\stackrel{I}{\sim}\}_{I \subseteq N} \subseteq S \times S$ and a function of

presented in the final section 7.

2. Logic of Correlated Knowledge

2.1. Syntax

Consider a set $N = \{a_1, a_2, \dots, a_n\}$ of agents. Each agent can perform its local observations. Given sets O_{a_1}, \dots, O_{a_n} of possible observations for each agent, a joint observation is a tuple of observations $o = (o_a)_{a \in N} \in O_{a_1} \times \dots \times O_{a_n}$ or $o = (o_a)_{a \in I} \in O_I$, where $O_I := \times_{a \in I} O_a$ and $I \subseteq N$. Joint observations together with results $r \in R$ make new atomic formulas o^r .

Each agent can know some information, and it is written as $K_{a_1} A$ or $K_{\{a_1\}} A$, which means that the agent a_1 knows A . A group of agents can also know some information and this is denoted by $K_{\{a_1, a_2, a_3\}} A$ or $K_I A$, where $I = \{a_1, a_2, a_3\}$. A more detailed description about the knowledge operator K is given in [10, 21].

Syntax of logic of correlated knowledge is defined as follows: *Definition 1(Syntax of logic of correlated knowledge)* The language of logic of correlated knowledge has the following syntax:

interpretations $V : S \rightarrow (P \rightarrow \{true, false\})$, where P is a set of atomic propositions, a model of logic of correlated knowledge is a multi-modal Kripke model $(S, \{\stackrel{I}{\sim}\}_{I \subseteq N}, V)$ that satisfies the following conditions:

1. For each $I \subseteq N$, $\stackrel{I}{\sim}$ is a multi-modal equivalence relation;
2. Information is monotonic: if $I \subseteq J$, then $\stackrel{J}{\sim} \subseteq \stackrel{I}{\sim}$;
3. Observability principle: if $s \stackrel{N}{\sim} s'$, then $s = s'$;
4. Vacuous information: $s \stackrel{\emptyset}{\sim} s'$ for all $s, s' \in S$.

The satisfaction relation \models for model M , state s and formulas o^r and $K_I A$ is defined as follows:

- 1) $M, s \models K_I A$ iff $M, t \models A$ for all states $t \stackrel{I}{\sim} s$.
- 2) $M, s \models o^r$ iff $s_I(o) = r$.

The formula $K_I A$ means that the group of agents I carries the information that A is the case, and o^r means that r is the result of the joint observation o .

If formula A is true in any state of any model, then it is named as a valid formula.

2.3. Hilbert Style Calculus HS-LCK

Alexandru Baltag and Sonja Smets defined the Hilbert style calculus for logic of correlated knowledge in [3]. Fixing a finite set $N = \{a_1, \dots, a_n\}$ of agents, a finite result structure (R, Σ) and a tuple of finite sets $\vec{O} = (O_{a_1}, \dots, O_{a_n})$ of observations, for every set $I, J \subseteq N$, every joint observation $o \in O_I$, $O_I = \times_{a \in I} O_a$, and results $r, p \in R$, the Hilbert style calculus for logic of correlated knowledge over (R, Σ, \vec{O}) is as follows:

1. Axioms:

H1. $A \rightarrow (B \rightarrow A)$

H2. $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$

H3. $(\neg A \rightarrow \neg B) \rightarrow (B \rightarrow A)$

H4. $K_I(A \rightarrow B) \rightarrow (K_I A \rightarrow K_I B)$ (Kripke's axiom)

H5. $K_I A \rightarrow A$ (Truthfulness)

H6. $K_I A \rightarrow K_I K_I A$ (Positive introspection)

H7. $\neg K_I A \rightarrow K_I \neg K_I A$ (Negative introspection)

H8. $K_I A \rightarrow K_J A$, where $I \subseteq J$ (Monotonicity of group knowledge)

H9. $A \rightarrow K_N A$ (Observability)

H10. $\bigwedge_{o \in O_I} \bigvee_{r \in R} o^r$ (Observations always yield results)

H11. $o^r \rightarrow \neg o^p$, where $r \neq p$ (Observations have unique results)

H12. $o_I^r \rightarrow K_I o_I^r$ (Groups know the results of their joint observations)

H13. $(\bigwedge_{o \in O_I} o^{r_o} \wedge K_I A) \rightarrow K_{\emptyset}(\bigwedge_{o \in O_I} o^{r_o} \rightarrow A)$ (Group knowledge is correlated knowledge (i.e. is based on joint observations))

H14. $\bigwedge_{o \in \bar{e}} o^{r_o} \rightarrow e^{\Sigma\{r_o: o \in \bar{e}\}}$, where $e \in O_I$, $\bar{e} := \{o = (o_i)_{i \in N} \in O_{i_1} \times \dots \times O_{i_n} : o_i = e_i \text{ for all } i \in I\}$. (Result composition axiom)

2. Rules:

$$\frac{A, A \rightarrow B}{B} \text{ (Modus ponens)} \quad \frac{A}{K_I A} \text{ (} K_I \text{-necessitation)}$$

Sets I, J may be empty in axioms H4 - H8 and in rule (K_I -necessitation).

The Hilbert style calculus HS-LCK for logic of correlated knowledge is sound and complete with respect to correlation models over (R, Σ, \vec{O}) [3].

3. Gentzen Style Sequent Calculus GS-LCK

Gerhard Gentzen introduced sequent calculus in 1934 [11]. Sequents in the system GS-LCK are statements of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite, possibly empty multisets of relational atoms $s \stackrel{I}{\sim} t$ and labelled formulas $s : A$, where $s, t \in S$, $I \subseteq N$ and A is any formula in the language of logic

of correlated knowledge. The formula $s : A$ means $s \models A$, and $s \stackrel{I}{\sim} t$ is an observational equivalence or relation between the states in the model of logic of correlated knowledge.

The sequent calculus consists of axioms and rules. Applying rules to the sequents, a proof-search tree for the root sequent is constructed. If axioms are in all the leaves of the proof-search tree, then the root sequent is called as a provable sequent and Δ follows from Γ of the root sequent.

Fixing a finite set $N = \{a_1, \dots, a_n\}$ of agents, a finite result structure (R, Σ) and a tuple of finite sets $\vec{O} = (O_{a_1}, \dots, O_{a_n})$ of observations, for every set $I, J \subseteq N$, every joint observation $o \in O_I$, $O_I = \times_{a \in I} O_a$, and results $r, p \in R$, the Gentzen style sequent calculus GS-LCK for logic of correlated knowledge over (R, Σ, \vec{O}) is as follows:

1. Axioms:

1) $s : p, \Gamma \Rightarrow \Delta, s : p$.

2) $s : o^r, \Gamma \Rightarrow \Delta, s : o^r$.

3) $s : o^{r_1}, s : o^{r_2}, \Gamma \Rightarrow \Delta$, where $r_1 \neq r_2$.

2. Propositional rules:

$$\frac{\Gamma \Rightarrow \Delta, s : A}{s : \neg A, \Gamma \Rightarrow \Delta} (\neg \Rightarrow) \quad \frac{s : A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, s : \neg A} (\Rightarrow \neg)$$

$$\frac{s : A, \Gamma \Rightarrow \Delta \quad s : B, \Gamma \Rightarrow \Delta}{s : A \vee B, \Gamma \Rightarrow \Delta} (\vee \Rightarrow) \quad \frac{\Gamma \Rightarrow \Delta, s : A, s : B}{\Gamma \Rightarrow \Delta, s : A \vee B} (\Rightarrow \vee)$$

$$\frac{s : A, s : B, \Gamma \Rightarrow \Delta}{s : A \wedge B, \Gamma \Rightarrow \Delta} (\wedge \Rightarrow) \quad \frac{\Gamma \Rightarrow \Delta, s : A \quad \Gamma \Rightarrow \Delta, s : B}{\Gamma \Rightarrow \Delta, s : A \wedge B} (\Rightarrow \wedge)$$

$$\frac{\Gamma \Rightarrow \Delta, s : A \quad s : B, \Gamma \Rightarrow \Delta}{s : A \rightarrow B, \Gamma \Rightarrow \Delta} (\rightarrow \Rightarrow) \quad \frac{s : A, \Gamma \Rightarrow \Delta, s : B}{\Gamma \Rightarrow \Delta, s : A \rightarrow B} (\Rightarrow \rightarrow)$$

3. Knowledge rules:

$$\frac{t : A, s : K_I A, s \overset{I}{\sim} t, \Gamma \Rightarrow \Delta}{s : K_I A, s \overset{I}{\sim} t, \Gamma \Rightarrow \Delta} (K_I \Rightarrow) \quad \frac{s \overset{I}{\sim} t, \Gamma \Rightarrow \Delta, t : A}{\Gamma \Rightarrow \Delta, s : K_I A} (\Rightarrow K_I)$$

The rule $(K_I \Rightarrow)$ requires that $I \neq N$ and $t : A$ be not in Γ . The rule $(\Rightarrow K_I)$ requires that $I \neq N$ and t be not in the conclusion. Set I maybe an empty set in both rules.

$$\frac{s : A, s : K_N A, s \overset{N}{\sim} s, \Gamma \Rightarrow \Delta}{s : K_N A, s \overset{N}{\sim} s, \Gamma \Rightarrow \Delta} (K_N \Rightarrow) \quad \frac{s \overset{N}{\sim} s, \Gamma \Rightarrow \Delta, s : A}{\Gamma \Rightarrow \Delta, s : K_N A} (\Rightarrow K_N)$$

The rule $(K_N \Rightarrow)$ requires that $s : A$ be not in Γ . The rule $(\Rightarrow K_N)$ requires that $s : A$ be not in Δ .

4. Observational rules:

$$\frac{s \overset{I}{\sim} t, \{s : o^{r_o}\}_{o \in O_I}, \{t : o^{r_o}\}_{o \in O_I}, \Gamma \Rightarrow \Delta}{\{s : o^{r_o}\}_{o \in O_I}, \{t : o^{r_o}\}_{o \in O_I}, \Gamma \Rightarrow \Delta} (OE)$$

The rule (OE) requires that $I \neq \emptyset$ and formulas $s \overset{I}{\sim} t$, $s : o^{r_o}$ and $t : o^{r_o}$ be not in Γ , where $o \in O_I$.

$$\frac{\{s : o_I^r, \Gamma \Rightarrow \Delta\}_{r \in R}}{\Gamma \Rightarrow \Delta} (OYR)$$

The rule (OYR) requires:

- (a) $s : o_I^r$ be not in Γ for all $r \in R$ and $s : o_I^{r_1}$ be in Δ for some $r_1 \in R$.
- (b) $I \neq \emptyset$.

$$\frac{s : e_I^{\Sigma\{r_{o_N} : o_N \in \bar{e}\}}, \{s : o_N^{r_{o_N}}\}_{o_N \in \bar{e}}, \Gamma \Rightarrow \Delta}{\{s : o_N^{r_{o_N}}\}_{o_N \in \bar{e}}, \Gamma \Rightarrow \Delta} (CR)$$

The rule (CR) requires that $s : e_I^{\Sigma\{r_{o_N} : o_N \in \bar{e}\}}$ be not in Γ .

5. Substitution rules:

$$\frac{s : p, t : p, s \overset{N}{\sim} t, \Gamma \Rightarrow \Delta}{t : p, s \overset{N}{\sim} t, \Gamma \Rightarrow \Delta} (Sub(p) \Rightarrow) \quad \frac{s : o^r, t : o^r, s \overset{I}{\sim} t, \Gamma \Rightarrow \Delta}{t : o^r, s \overset{I}{\sim} t, \Gamma \Rightarrow \Delta} (Sub(o^r) \Rightarrow)$$

The rules $(Sub(p) \Rightarrow)$ and $(Sub(o^r) \Rightarrow)$ require that $s : p$ and $s : o^r$ be not in Γ , accordingly.

6. Relational rules:

$$\frac{s \overset{I}{\sim} s, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} (Ref) \quad \frac{s \overset{I}{\sim} t, s \overset{I}{\sim} s', s' \overset{I}{\sim} t, \Gamma \Rightarrow \Delta}{s \overset{I}{\sim} s', s' \overset{I}{\sim} t, \Gamma \Rightarrow \Delta} (Trans)$$

The rule (Ref) requires that s be in the conclusion and $s \overset{I}{\sim} s$ be not in Γ . The rule $(Trans)$ requires that $s \overset{I}{\sim} t$ be not in Γ .

$$\frac{s' \overset{I}{\sim} t, s \overset{I}{\sim} s', s \overset{I}{\sim} t, \Gamma \Rightarrow \Delta}{s \overset{I}{\sim} s', s \overset{I}{\sim} t, \Gamma \Rightarrow \Delta} (Eucl) \quad \frac{s \overset{I}{\sim} t, s \overset{J}{\sim} t, \Gamma \Rightarrow \Delta}{s \overset{J}{\sim} t, \Gamma \Rightarrow \Delta} (Mon)$$

The rule (*Mon*) stands for monotonicity and requires that $I \subseteq J$. Sets I, J may be empty. The rules (*Eucl*) and (*Mon*) require that $s' \stackrel{I}{\sim} t$ and $s \stackrel{I}{\sim} t$ be not in Γ , accordingly.

The sequent calculus GS-LCK is sound and complete with respect to correlation models over (R, Σ, \vec{O}) . It also has the beautiful properties of rule invertibility and admissibility of the cut and structural rules. It is crucial in making the automated proof system in the present paper. *Theorem 1(Properties of GS-LCK)* The sequent calculus GS-LCK has the following properties:

- 1) Invertibility of rules.
- 2) Admissibility of weakening.
- 3) Admissibility of contraction.
- 4) Admissibility of cut.
- 5) Termination.

Proofs of soundness, completeness, and the properties of GS-LCK are given in the next sections.

4. Proof of Soundness of GS-LCK

Definition 4(Extended syntax) Extended syntax of LCK is as follows:

$$A := s : A_1 \mid s \stackrel{I}{\sim} t \mid s : A_1 \vee A \mid s : A_1 \wedge A \mid s : A_1 \rightarrow A$$

$$A_1 := p \mid o^r \mid \perp \mid \top \mid \neg A_1 \mid A_1 \vee A_2 \mid A_1 \wedge A_2 \mid A_1 \rightarrow A_2 \mid K_I A_1$$

where p is any atomic proposition, $o \in O_I, I \subseteq N, r \in R$ and $s, t \in S$.

Definition 5(Extended semantics) If $s, t, v \in \mathbf{S}$ and $M \in \mathbf{M}$, then the truthfulness of the formula in the state v of the model M is defined as follows:

- 1) $v \models s : A$ iff $s \models A$.
- 2) $v \models s \stackrel{I}{\sim} t$ iff $s \stackrel{I}{\sim} t \in \mathbf{R}$.

Commas “,” in Γ of the sequent $\Gamma \Rightarrow \Delta$ mean conjunction “ \wedge ”, commas “,” in Δ - disjunction “ \vee ”. The arrow “ \Rightarrow ” stands for implication “ \rightarrow ”. *Definition 6(Formula of the sequent)* If Seq is a sequent $\Gamma \Rightarrow \Delta$, then the formula of the sequent $F(\text{Seq})$ is obtained by:

- 1) putting Γ and Δ in parentheses;
- 2) replacing empty Γ by $s : \top$;
- 3) replacing empty Δ by $s : \perp$;
- 4) replacing commas “,” by conjunction “ \wedge ” in Γ ;
- 5) replacing commas “,” by disjunction “ \vee ” in Δ ;
- 6) replacing “ \Rightarrow ” by implication “ \rightarrow ”.

Example 1 $F(\text{Seq}) := (t : A_1 \wedge s : K_I A_1 \wedge s \stackrel{I}{\sim} t \wedge t : A_2) \rightarrow (s : B_1 \vee t : B_2)$ is the formula of the sequent $\text{Seq} := t : A_1, s : K_I A_1, s \stackrel{I}{\sim} t, t : A_2 \Rightarrow s : B_1, t : B_2$.

Definition 7 (Sequent without labels and relational atoms)

If Seq is a sequent, then the sequent without labels and relational atoms of Seq is obtained removing all labels near formulas and all relational atoms from Seq .

Lemma 1 (Validity of the formula of the sequent)

If the formula of the sequent Seq is valid, then the formula of the sequent Seq without labels and relational atoms is valid, as well.

proof

Suppose we have a set of states S of a model M . For each formula of the sequent we have a tuple of its labels $(s_1, \dots, s_l) \in S \times \dots \times S$. If the formula with labels (s_1, \dots, s_l) is valid, then it is valid with substituted labels (s', \dots, s') , because $\{(s', \dots, s') : s' \in S\} \subseteq \{(s_1, \dots, s_l) : s_1, \dots, s_l \in S\}$. Having $s \models s' : A$, iff $s' \models A$, we can remove the label s' .

All relational atoms become $s' \stackrel{I}{\sim} s', I \subseteq N$. They are valid because of reflexivity in models. Applying the rules of GS-LCK they appear only in the first argument of implication of the formula of the sequent. We can remove relational atoms, because having a valid formula $(A_1 \wedge \dots \wedge A_l) \rightarrow (B_1 \vee \dots \vee B_k)$ and removing valid formula A_i from the first argument of implication, the validity is maintained.

Theorem 2 (Soundness of GS-LCK)

If sequent S is provable in GS-LCK, then the formula of the sequent S without labels and relational atoms is valid with respect to correlation models over (R, Σ, \vec{O}) .

proof

We prove the validity of all axioms and soundness of all the rules of GS-LCK:

1. Axioms:

- 1) Formula of the axiom $s : p, \Gamma \Rightarrow s : p, \Delta$ is valid, because it is true in any state of any model. The same is for the axiom $s : o^r, \Gamma \Rightarrow s : o^r, \Delta$.
- 2) Validity of the formula of the axiom $s : o^{r_1}, s : o^{r_2}, \Gamma \Rightarrow \Delta$, where $r_1 \neq r_2$, follows from the axiom “H11. $o^r \rightarrow \neg o^p$, where $r \neq p$ ”.

2. Propositional rules as in [16].

3. Knowledge rules:

1) Rule ($K_I \Rightarrow$):

$$\frac{t : A, s : K_I A, s \stackrel{I}{\sim} t, \Gamma \Rightarrow \Delta}{s : K_I A, s \stackrel{I}{\sim} t, \Gamma \Rightarrow \Delta} (K_I \Rightarrow), \quad I \neq N.$$

We prove by contraposition that, if the formula of the premise $(t : A, s : K_I A, s \stackrel{I}{\sim} t, \Gamma \Rightarrow \Delta)$ of the rule ($K_I \Rightarrow$) is valid, then the formula of the conclusion $(s : K_I A, s \stackrel{I}{\sim} t, \Gamma \Rightarrow \Delta)$ is valid, too.

The formula of the conclusion $(s : K_I A, s \stackrel{I}{\sim} t, \Gamma \Rightarrow \Delta)$ is false, when $s : K_I A, s \stackrel{I}{\sim} t$ and all formulas in Γ are true, and all formulas in Δ are false. By semantic definition of the knowledge operator K_I , formula A is true in all the states accessible from the state s by relation I . States t are accessible from the state s , because $s \stackrel{I}{\sim} t$ is true, therefore the formula $t : A$ is true. If $t : A, s : K_I A$,

$s \stackrel{I}{\sim} t$ and all formulas in Γ are true and all formulas in Δ are false, then the formula of the premise ($t : A, s : K_I A, s \stackrel{I}{\sim} t, \Gamma \Rightarrow \Delta$) is false.

2) Rule ($\Rightarrow K_I$):

$$\frac{s \stackrel{I}{\sim} t, \Gamma \Rightarrow \Delta, t : A}{\Gamma \Rightarrow \Delta, s : K_I A} (\Rightarrow K_I), \quad I \neq N$$

and t is not in the conclusion.

The formula of conclusion ($\Gamma \Rightarrow \Delta, s : K_I A$) is false, when all formulas in Γ are true and all formulas in Δ and $s : K_I A$ are false. If the formula $s : K_I A$ is false, then there exists a state t accessible from state s by relation I , where A is false. If $s \stackrel{I}{\sim} t$ and all formulas in Γ are true and all formulas in Δ and $t : A$ are false, then the formula of the premise ($s \stackrel{I}{\sim} t, \Gamma \Rightarrow \Delta, t : A$) is false.

The label t cannot be in the conclusion, because we can get situations, where the formula of the premise ($s \stackrel{I}{\sim} t, \Gamma \Rightarrow \Delta, t : A$) is valid and the formula of the conclusion ($\Gamma \Rightarrow \Delta, s : K_I A$) is not. An example:

$$\frac{s \stackrel{I}{\sim} t, t : A \Rightarrow t : A}{t : A \Rightarrow s : K_I A} (\Rightarrow K_I)$$

3) The validity of the rules ($K_N \Rightarrow$) and ($\Rightarrow K_N$) is proved in the same way.

4. Observational rules:

1) Rule (OYR):

$$\frac{\{s : o^r, \Gamma \Rightarrow \Delta\}_{r \in R}}{\Gamma \Rightarrow \Delta} (OYR)$$

If R is a set of results, and o is a joint observation, then there exists a result $r \in R$ that o^r is true. If there exists r that o^r is true and all formulas in Γ are true and all formulas in Δ are false, then one formula of premises ($\{s : o^r, \Gamma \Rightarrow \Delta\}_{r \in R}$) is false.

$\langle h = 1 \rangle$

If the sequent ($\Pi_{atomic}, \Pi_{atomic}, \Gamma \Rightarrow \Delta, \Lambda_{atomic}, \Lambda_{atomic}$) is an axiom, then the sequent ($\Pi_{atomic}, \Gamma \Rightarrow \Delta, \Lambda_{atomic}$) is an axiom too.

$\langle h > 1 \rangle$

1. The rule ($K_I \Rightarrow$) was applied in the last step of the proof of the sequent.

1) One or two formulas of the principal pair is in Π_{atomic} .

$$\frac{t : A, s : K_I A, s \stackrel{I}{\sim} t, s \stackrel{I}{\sim} t, \Pi'_{atomic}, \Pi'_{atomic}, \Gamma' \Rightarrow \Delta, \Lambda_{atomic}, \Lambda_{atomic}}{s : K_I A, s \stackrel{I}{\sim} t, s \stackrel{I}{\sim} t, \Pi'_{atomic}, \Pi'_{atomic}, \Gamma' \Rightarrow \Delta, \Lambda_{atomic}, \Lambda_{atomic}} (K_I \Rightarrow)$$

The height of the proof of the premise of application of the rule ($K_I \Rightarrow$) reduced to $\langle h - 1 \rangle$. By the induction hypothesis the sequent ($t : A, s : K_I A, s \stackrel{I}{\sim} t, \Pi'_{atomic}, \Gamma' \Rightarrow \Delta, \Lambda_{atomic}$) is provable with the height h' , where $h' \leq h - 1$. The sequent of the lemma is proved by applying the rule ($K_I \Rightarrow$):

$$\frac{t : A, s : K_I A, s \stackrel{I}{\sim} t, \Pi'_{atomic}, \Gamma' \Rightarrow \Delta, \Lambda_{atomic}}{s : K_I A, s \stackrel{I}{\sim} t, \Pi'_{atomic}, \Gamma' \Rightarrow \Delta, \Lambda_{atomic}} (K_I \Rightarrow)$$

2) Rule (CR):

$$\frac{s : e^{\Sigma\{r_o : o \in \bar{e}\}}, \{s : o^{r_o}\}_{o \in \bar{e}}, \Gamma \Rightarrow \Delta}{\{s : o^{r_o}\}_{o \in \bar{e}}, \Gamma \Rightarrow \Delta} (CR)$$

The contraposition is proved by the axiom "H14. $\bigwedge_{o \in \bar{e}} o^{r_o} \rightarrow e^{\Sigma\{r_o : o \in \bar{e}\}}$ ".

3) The soundness of rules (OE), ($Sub(p) \Rightarrow$) and ($Sub(o^r) \Rightarrow$) is proved in the same way.

5. Relational rules:

1) Rule (Mon):

$$\frac{s \stackrel{I}{\sim} t, s \stackrel{J}{\sim} t, \Gamma \Rightarrow \Delta}{s \stackrel{J}{\sim} t, \Gamma \Rightarrow \Delta} (Mon)$$

The contraposition follows from condition to models of LCK: 2. If $I \subseteq J$ then $\stackrel{J}{\sim} \subseteq \stackrel{I}{\sim}$.

2) The validity of rules (Ref), ($Trans$) and ($Eucl$) is proved in the same way.

We have proved the validity of all axioms and soundness of all the rules of GS-LCK. The statement of the theorem follows from lemma 4.

5. Proof of the Properties of GS-LCK

lemma2 (Admissibility of contraction with atomic formulas)

If a sequent ($\Pi_{atomic}, \Pi_{atomic}, \Gamma \Rightarrow \Delta, \Lambda_{atomic}, \Lambda_{atomic}$) is provable in GS-LCK, then the sequent ($\Pi_{atomic}, \Gamma \Rightarrow \Delta, \Lambda_{atomic}$) is also provable with the same bound of the height of the proof in GS-LCK. Γ, Δ are any multisets of formulas. $\Pi_{atomic}, \Lambda_{atomic}$ are any multisets of atomic formulas $s : p, s : o^r, s \stackrel{I}{\sim} t$.

Proof

Lemma 5 is proved by induction on the height $\langle h \rangle$ of the proof of the sequent ($\Pi_{atomic}, \Pi_{atomic}, \Gamma \Rightarrow \Delta, \Lambda_{atomic}, \Lambda_{atomic}$).

Other cases are proved in a similar way.

2) Any formula of the principal pair is not in Π_{atomic} .

$$\frac{t : A, s : K_I A, s \overset{I}{\sim} t, \Pi_{atomic}, \Pi_{atomic}, \Gamma' \Rightarrow \Delta, \Lambda_{atomic}, \Lambda_{atomic}}{s : K_I A, s \overset{I}{\sim} t, \Pi_{atomic}, \Pi_{atomic}, \Gamma' \Rightarrow \Delta, \Lambda_{atomic}, \Lambda_{atomic}} (K_I \Rightarrow)$$

By the induction hypothesis the sequent $(t : A, s : K_I A, s \overset{I}{\sim} t, \Pi_{atomic}, \Gamma' \Rightarrow \Delta, \Lambda_{atomic})$ is provable with the height h' , where $h' \leq h - 1$. The sequent of the lemma is proved by applying the rule $(K_I \Rightarrow)$:

$$\frac{t : A, s : K_I A, s \overset{I}{\sim} t, \Pi_{atomic}, \Gamma' \Rightarrow \Delta, \Lambda_{atomic}}{s : K_I A, s \overset{I}{\sim} t, \Pi_{atomic}, \Gamma' \Rightarrow \Delta, \Lambda_{atomic}} (K_I \Rightarrow)$$

2. The cases of the remaining rules are considered similarly.

Lemma 3 (Substitution)

If a sequent $(\Gamma \Rightarrow \Delta)$ is provable in GS-LCK, then the sequent $(\Gamma(t/s) \Rightarrow \Delta(t/s))$ is also provable with the same bound of the height of the proof in GS-LCK.

proof

Lemma is proved by induction on the height $\langle h \rangle$ of the proof of the sequent $(\Gamma \Rightarrow \Delta)$.

$\langle h = 1 \rangle$

If the sequent $(\Gamma \Rightarrow \Delta)$ is an axiom, then the sequent $(\Gamma(t/s) \Rightarrow \Delta(t/s))$ is an axiom as well.

$\langle h > 1 \rangle$

1. The rule $(\Rightarrow K_I)$ was applied in the last step of the proof of the sequent.

$$\frac{s \overset{I}{\sim} t, \Gamma \Rightarrow \Delta, t : A}{\Gamma \Rightarrow \Delta, s : K_I A} (\Rightarrow K_I)$$

1) Substitution (l/z) .

By the induction hypothesis the sequent $(s \overset{I}{\sim} t, \Gamma(l/z) \Rightarrow \Delta(l/z), t : A)$ is provable with the height h' , where $h' \leq h - 1$. The sequent of the lemma is proved by applying the rule $(\Rightarrow K_I)$:

$$\frac{s \overset{I}{\sim} t, \Gamma(l/z) \Rightarrow \Delta(l/z), t : A}{\Gamma(l/z) \Rightarrow \Delta(l/z), s : K_I A} (\Rightarrow K_I)$$

2) Substitution (l/t) .

There is no label t in the sequent $\Gamma \Rightarrow \Delta, s : K_I A$ because of the requirement of the application of the rule $(\Rightarrow K_I)$ that t is a new label.

3) Substitution (l/s) and $l \neq t$.

By the induction hypothesis the sequent $(l \overset{I}{\sim} t, \Gamma(l/s) \Rightarrow \Delta(l/s), t : A)$ is provable with the height h' , where $h' \leq h - 1$. The sequent of the lemma is proved by applying the rule $(\Rightarrow K_I)$:

$$\frac{l \overset{I}{\sim} t, \Gamma(l/s) \Rightarrow \Delta(l/s), t : A}{\Gamma(l/s) \Rightarrow \Delta(l/s), l : K_I A} (\Rightarrow K_I)$$

4) Substitution (l/s) and $l = t$.

By the induction hypothesis with substitution (w/t) , the sequent $(s \overset{I}{\sim} w, \Gamma \Rightarrow \Delta, w : A)$ is provable with the height h' , where $h' \leq h - 1$. The label w is a new label absent in the sequent. By the induction hypothesis with substitution (l/s) , the sequent $(l \overset{I}{\sim} w, \Gamma(l/s) \Rightarrow \Delta(l/s), w : A)$ is provable with the height h'' , where $h'' \leq h - 1$. The sequent of the lemma is proved by applying the rule $(\Rightarrow K_I)$:

$$\frac{l \overset{I}{\sim} w, \Gamma(l/s) \Rightarrow \Delta(l/s), w : A}{\Gamma(l/s) \Rightarrow \Delta(l/s), l : K_I A} (\Rightarrow K_I)$$

2. The rule (Ref) was applied in the last step of the proof of the sequent.

$$\frac{s \overset{I}{\sim} s, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} (Ref)$$

1) Substitution (s/t) , and relational atom $s \overset{I}{\sim} t$ is in Γ .

By the induction hypothesis, the sequent $(s \overset{I}{\sim} s, s \overset{I}{\sim} s, \Gamma(s/t) \Rightarrow \Delta(s/t))$ is provable with the height h' , where $h' \leq h - 1$. The sequent of the lemma is proved by applying Lemma 5.

2) Other substitutions are considered in a similar way.

3. The cases of the remaining rules are considered similarly.

Theorem 3 (Admissibility of weakening)

If a sequent $(\Gamma \Rightarrow \Delta)$ is provable in GS-LCK, then a sequent $(\Pi, \Gamma \Rightarrow \Delta, \Lambda)$ is provable with the same bound of the height of the proof in GS-LCK, too. $\Pi, \Gamma, \Delta, \Lambda$ are any multisets of formulas.

proof

Theorem is proved by induction on the height $\langle h \rangle$ of the proof of the sequent $(\Gamma \Rightarrow \Delta)$.

$\langle h = 1 \rangle$

If the sequent $(\Gamma \Rightarrow \Delta)$ is an axiom, then the sequent $(\Pi, \Gamma \Rightarrow \Delta, \Lambda)$ is an axiom, as well.

$\langle h > 1 \rangle$

- 1) The rule $(\Rightarrow K_I)$ was applied in the last step of the proof of the sequent.

$$\frac{s \overset{I}{\sim} t, \Gamma \Rightarrow \Delta, t : A}{\Gamma \Rightarrow \Delta, s : K_I A} (\Rightarrow K_I)$$

- a. A new label t for the application of the rule $(\Rightarrow K_I)$ is in Π or Λ .

By Lemma , the sequent $(s \overset{I}{\sim} t, \Gamma \Rightarrow \Delta, t : A)$ with substitution (l/t) is provable. By the induction hypothesis, the sequent $(s \overset{I}{\sim} l, \Pi, \Gamma \Rightarrow \Delta, \Lambda, l : A)$ is provable with the height h' , where $h' \leq h - 1$. Here l is a new label, absent in Π, Γ, Δ and Λ . The sequent of the theorem is proved by applying the rule $(\Rightarrow K_I)$:

$$\frac{s \overset{I}{\sim} l, \Pi, \Gamma \Rightarrow \Delta, \Lambda, l : A}{\Pi, \Gamma \Rightarrow \Delta, \Lambda, s : K_I A} (\Rightarrow K_I)$$

- b. The new label t for application of the rule $(\Rightarrow K_I)$ is absent in Π or Λ .

By the induction hypothesis, the sequent $(s \overset{I}{\sim} t, \Pi, \Gamma \Rightarrow \Delta, \Lambda, t : A)$ is provable with the height

- 1) The formula $s \overset{I}{\sim} t$ is the principal formula.

- a. The rule $(Sub(o^r) \Rightarrow)$ was applied in the last step of the proof of the sequent.

$$\frac{s : o^r, s : K_I A, s \overset{I}{\sim} t, t : o^r, \Gamma' \Rightarrow \Delta}{s : K_I A, s \overset{I}{\sim} t, t : o^r, \Gamma' \Rightarrow \Delta} (Sub(o^r) \Rightarrow)$$

By the induction hypothesis, the sequent $(t : A, s : o^r, s : K_I A, s \overset{I}{\sim} t, t : o^r, \Gamma' \Rightarrow \Delta)$ is provable with the height h' , where $h' \leq h - 1$. The sequent of the premise of the rule $(K_I \Rightarrow)$ is proved by applying the rule $(Sub(o^r) \Rightarrow)$:

$$\frac{t : A, s : o^r, s : K_I A, s \overset{I}{\sim} t, t : o^r, \Gamma' \Rightarrow \Delta}{t : A, s : K_I A, s \overset{I}{\sim} t, t : o^r, \Gamma' \Rightarrow \Delta} (Sub(o^r) \Rightarrow)$$

- b. For rules $(K_I \Rightarrow)$, $(Trans)$, $(Eucl)$, (Mon) in a similar way.

- 2) The case where the formula $s : K_I A$ is the principal formula and the case where formulas $s \overset{I}{\sim} t$ and $s : K_I A$ both are not principal formulas are considered similarly.

Invertibility of the remaining rules is proved in a similar way.

Theorem 5 (Admissibility of contraction)

If a sequent $(\Pi, \Pi, \Gamma \Rightarrow \Delta, \Lambda, \Lambda)$ is provable in GS-LCK, then sequent $(\Pi, \Gamma \Rightarrow \Delta, \Lambda)$ is provable with the same bound of the height of the proof in GS-LCK, too. $\Pi, \Gamma, \Delta, \Lambda$ are any multisets of formulas.

proof

Theorem is proved by induction on the ordered tuple pair $\langle c, h \rangle$, where c is the sum of complexity of all the formulas in Π and Λ , and h is the height of the proof of the sequent $(\Pi, \Pi, \Gamma \Rightarrow \Delta, \Lambda, \Lambda)$.

h' , where $h' \leq h - 1$. The sequent of the theorem is proved by applying the rule $(\Rightarrow K_I)$:

$$\frac{s \overset{I}{\sim} t, \Pi, \Gamma \Rightarrow \Delta, \Lambda, t : A}{\Pi, \Gamma \Rightarrow \Delta, \Lambda, s : K_I A} (\Rightarrow K_I)$$

- 2) The cases of the remaining rules are considered similarly.

Theorem 4 (Invertibility of rules)

All the rules of GS-LCK are invertible with the same bound of the height of the proof.

proof

Theorem is proved for each rule separately.

The rule $(K_I \Rightarrow)$

$$\frac{t : A, s : K_I A, s \overset{I}{\sim} t, \Gamma \Rightarrow \Delta}{s : K_I A, s \overset{I}{\sim} t, \Gamma \Rightarrow \Delta} (K_I \Rightarrow)$$

Invertibility is proved by induction on the height $\langle h \rangle$ of the proof of the sequent of the conclusion of the rule $(K_I \Rightarrow)$.

$\langle h = 1 \rangle$

If the sequent $(s : K_I A, s \overset{I}{\sim} t, \Gamma \Rightarrow \Delta)$ is an axiom, then the sequent $(t : A, s : K_I A, s \overset{I}{\sim} t, \Gamma \Rightarrow \Delta)$ is an axiom, too.

$\langle h > 1 \rangle$

$\langle c \geq 1, h = 1 \rangle$

If the sequent $(\Pi, \Pi, \Gamma \Rightarrow \Delta, \Lambda, \Lambda)$ is an axiom, then the sequent $(\Pi, \Gamma \Rightarrow \Delta, \Lambda)$ is an axiom, too.

$\langle c \geq 1, h > 1 \rangle$

- 1) The rule $(\neg \Rightarrow)$ was applied in the last step of the proof of the sequent.

- a.) The principal formula is in Π .

$$\frac{s : \neg A, \Pi', \Pi', \Gamma \Rightarrow \Delta, \Lambda, \Lambda, s : A}{s : \neg A, s : \neg A, \Pi', \Pi', \Gamma \Rightarrow \Delta, \Lambda, \Lambda} (\neg \Rightarrow)$$

By invertibility of the rule $(\neg \Rightarrow)$, the sequent

$(\Pi', \Pi', \Gamma \Rightarrow \Delta, \Lambda, \Lambda, s : A, s : A)$ is provable. The value of the ordered tuple pair has reduced to $\langle c - 1, h \rangle$. By the induction hypothesis, the sequent $(\Pi', \Gamma \Rightarrow \Delta, \Lambda, s : A)$ is provable with the height h' , where $h' \leq h - 1$. The sequent of the theorem is proved by applying the rule $(\neg \Rightarrow)$:

$$\frac{\Pi', \Gamma \Rightarrow \Delta, \Lambda, s : A}{s : \neg A, \Pi', \Gamma \Rightarrow \Delta, \Lambda} (\neg \Rightarrow)$$

b. The principal formula is absent in Π .

$$\frac{\Pi, \Pi, \Gamma \Rightarrow \Delta, \Lambda, \Lambda, s : A}{s : \neg A, \Pi, \Pi, \Gamma \Rightarrow \Delta, \Lambda, \Lambda} (\neg \Rightarrow)$$

By the induction hypothesis, the sequent $(\Pi, \Gamma \Rightarrow \Delta, \Lambda, s : A)$ is provable with the height h' , where $h' \leq h - 1$. The sequent of the theorem is proved by applying the rule $(\neg \Rightarrow)$:

$$\frac{\Pi, \Gamma \Rightarrow \Delta, \Lambda, s : A}{s : \neg A, \Pi, \Gamma \Rightarrow \Delta, \Lambda} (\neg \Rightarrow)$$

$\langle c \geq 1, h > 2 \rangle$

1) Formula F is not principal in the sequent $(\Gamma \Rightarrow \Delta, F)$.

a. The rule $(Sub(o^r) \Rightarrow)$ was applied in the last step of the proof of the sequent $(\Gamma \Rightarrow \Delta, F)$.

$$\frac{s : o^r, t : o^r, s \overset{N}{\sim} t, \Gamma \Rightarrow \Delta, F}{t : o^r, s \overset{N}{\sim} t, \Gamma \Rightarrow \Delta, F} (Sub(o^r) \Rightarrow)$$

By the induction hypothesis, the sequent $(s : o^r, t : o^r, s \overset{N}{\sim} t, \Pi, \Gamma \Rightarrow \Delta, \Lambda)$ is provable. The sequent of the theorem is proved by applying the rule $(Sub(o^r) \Rightarrow)$:

$$\frac{s : o^r, t : o^r, s \overset{N}{\sim} t, \Pi, \Gamma \Rightarrow \Delta, \Lambda}{t : o^r, s \overset{N}{\sim} t, \Pi, \Gamma \Rightarrow \Delta, \Lambda} (Sub(o^r) \Rightarrow)$$

b. For applications of other rules in a similar way.

2) Formula F is not principal in the sequent $(F, \Pi \Rightarrow \Lambda)$. The case is considered in a similar way.

3) Formula F is principal in both sequents $(\Gamma \Rightarrow \Delta, F)$ and $(F, \Pi \Rightarrow \Lambda)$.

a. The sequent $(\Gamma \Rightarrow \Delta, F)$ is an axiom and the rule (OE) was applied in the last step of the proof of the sequent $(F, \Pi \Rightarrow \Lambda)$.

$$\frac{s : o_1^{r_{o_1}}, \Gamma \Rightarrow \Delta, s : o_1^{r_{o_1}}}{s \overset{I}{\sim} t, s : o_1^{r_{o_1}}, \{s : o^{r_o}\}_{o \in \{O_I \setminus o_1\}}, \{t : o^{r_o}\}_{o \in O_I}, \Pi \Rightarrow \Lambda}{s : o_1^{r_{o_1}}, \{s : o^{r_o}\}_{o \in \{O_I \setminus o_1\}}, \{t : o^{r_o}\}_{o \in O_I}, \Pi \Rightarrow \Lambda} (OE)$$

By the induction hypothesis, the sequent $(s : o_1^{r_{o_1}}, s \overset{I}{\sim} t, \{s : o^{r_o}\}_{o \in \{O_I \setminus o_1\}}, \{t : o^{r_o}\}_{o \in O_I}, \Pi, \Gamma \Rightarrow \Delta, \Lambda)$ is provable. The sequent of the theorem is proved by applying the rule (OE) :

$$\frac{s : o_1^{r_{o_1}}, s \overset{I}{\sim} t, \{s : o^{r_o}\}_{o \in \{O_I \setminus o_1\}}, \{t : o^{r_o}\}_{o \in O_I}, \Pi, \Gamma \Rightarrow \Delta, \Lambda}{s : o_1^{r_{o_1}}, \{s : o^{r_o}\}_{o \in \{O_I \setminus o_1\}}, \{t : o^{r_o}\}_{o \in O_I}, \Pi, \Gamma \Rightarrow \Delta, \Lambda} (OE)$$

b. The cases of the remaining rules are considered similarly.

2) The cases of the remaining rules are considered similarly.

Theorem 4 (Admissibility of cut)

If sequents $(\Gamma \Rightarrow \Delta, F)$ and $(F, \Pi \Rightarrow \Lambda)$ are provable in GS-LCK, then sequent $(\Pi, \Gamma \Rightarrow \Delta, \Lambda)$ is provable in GS-LCK too. F is any formula and $\Pi, \Gamma, \Delta, \Lambda$ are any multisets of formulas.

proof

Theorem is proved by induction on the ordered tuple pair $\langle c, h \rangle$, where c is the complexity of formula F , and h is the sum of heights of the proof of the sequents $(\Gamma \Rightarrow \Delta, F)$ and $(F, \Pi \Rightarrow \Lambda)$.

$\langle c \geq 1, h = 2 \rangle$

The sequents $(\Gamma \Rightarrow \Delta, F)$ and $(F, \Pi \Rightarrow \Lambda)$ are the axioms. If formula F is not principal in one at least of the sequents, then $(\Pi, \Gamma \Rightarrow \Delta, \Lambda)$ is an axiom. If formula F is principal in both sequents, then F should be in Γ and Δ or only in Γ (the case where the axiom is of type $s : o^{r_1}, s : o^{r_2}, \Gamma \Rightarrow \Delta$). Therefore the sequent $(\Pi, \Gamma \Rightarrow \Delta, \Lambda)$ is also an axiom.

6. Proof of Completeness of GS-LCK

Theorem 7 (Completeness of GS-LCK)

If formula A is valid with respect to correlation models over (R, Σ, \vec{O}) , then sequent $(\Rightarrow s : A)$ is provable in GS-LCK.

proof

The Hilbert style proof system HS-LCK for logic of correlated knowledge is complete. Showing the provability of all valid formulas of HS-LCK in GS-LCK, the completeness of GS-LCK is proved. Theorem 6 is proved by induction on the number of steps $\langle NSteps \rangle$, used to prove formula A in HS-LCK.

$\langle NSteps = 1 \rangle$

Formula A is an axiom of calculus HS-LCK.

1) The axiom "H4. $K_I(A \rightarrow B) \rightarrow (K_I A \rightarrow K_I B)$ ", was used.

$$\frac{\frac{\frac{t : A, \dots \Rightarrow t : B, t : A \quad t : B, t : A, \dots \Rightarrow t : B}{t : A \rightarrow B, t : A, s \overset{I}{\sim} t, s : K_I(A \rightarrow B), s : K_I A \Rightarrow t : B} (\Rightarrow \Rightarrow)}{t : A, s \overset{I}{\sim} t, s : K_I(A \rightarrow B), s : K_I A \Rightarrow t : B} (K_I \Rightarrow)}{s \overset{I}{\sim} t, s : K_I(A \rightarrow B), s : K_I A \Rightarrow t : B} (\Rightarrow K_I)}{\frac{s : K_I(A \rightarrow B), s : K_I A \Rightarrow s : K_I B}{s : K_I(A \rightarrow B) \Rightarrow s : K_I A \rightarrow K_I B} (\Rightarrow \rightarrow)}{\Rightarrow s : K_I(A \rightarrow B) \rightarrow (K_I A \rightarrow K_I B)} (\Rightarrow \rightarrow)$$

2) The axiom "H8. $K_I A \rightarrow K_J A$, when $I \subseteq J$ ", was used.

$$\frac{\frac{t : A, s \overset{I}{\sim} t, s \overset{J}{\sim} t, s : K_I A \Rightarrow t : A}{s \overset{I}{\sim} t, s \overset{J}{\sim} t, s : K_I A \Rightarrow t : A} (K_I \Rightarrow)}{s \overset{J}{\sim} t, s : K_I A \Rightarrow t : A} (\Rightarrow K_J)}{\frac{s : K_I A \Rightarrow s : K_J A}{\Rightarrow s : K_I A \rightarrow K_J A} (\Rightarrow \rightarrow)}$$

3) The axiom "H12. $o_I^r \rightarrow K_I o_I^r$ ", was used.

$$\frac{\frac{t : o_I^r, s \overset{I}{\sim} t, s : o_I^r \Rightarrow t : o_I^r}{s \overset{I}{\sim} t, s : o_I^r \Rightarrow t : o_I^r} (Sub(o^r) \Rightarrow)}{s : o_I^r \Rightarrow s : K_I o_I^r} (\Rightarrow K_I)}{\Rightarrow s : o_I^r \rightarrow K_I o_I^r} (\Rightarrow \rightarrow)$$

4) The axiom "H13. $(\bigwedge_{o \in O_I} o^{r_o} \wedge K_I A) \rightarrow K_\emptyset(\bigwedge_{o \in O_I} o^{r_o} \rightarrow A)$, when $I \subset N$ ", was used.

$$\frac{\frac{\frac{t : A, s \overset{I}{\sim} t, t : \bigwedge_{o \in O_I} o^{r_o}, s \overset{\emptyset}{\sim} t, s : \bigwedge_{o \in O_I} o^{r_o}, s : K_I A \Rightarrow t : A}{s \overset{I}{\sim} t, t : \bigwedge_{o \in O_I} o^{r_o}, s \overset{\emptyset}{\sim} t, s : \bigwedge_{o \in O_I} o^{r_o}, s : K_I A \Rightarrow t : A} (K_I \Rightarrow)}{t : \bigwedge_{o \in O_I} o^{r_o}, s \overset{\emptyset}{\sim} t, s : \bigwedge_{o \in O_I} o^{r_o}, s : K_I A \Rightarrow t : A} (OE)}{s \overset{\emptyset}{\sim} t, s : \bigwedge_{o \in O_I} o^{r_o}, s : K_I A \Rightarrow t : \bigwedge_{o \in O_I} o^{r_o} \rightarrow A} (\Rightarrow \rightarrow)}{\frac{s \overset{\emptyset}{\sim} t, s : \bigwedge_{o \in O_I} o^{r_o} \wedge K_I A \Rightarrow t : \bigwedge_{o \in O_I} o^{r_o} \rightarrow A}{s : \bigwedge_{o \in O_I} o^{r_o} \wedge K_I A \Rightarrow s : K_\emptyset(\bigwedge_{o \in O_I} o^{r_o} \rightarrow A)} (\Rightarrow K_\emptyset)}{\Rightarrow s : (\bigwedge_{o \in O_I} o^{r_o} \wedge K_I A) \rightarrow K_\emptyset(\bigwedge_{o \in O_I} o^{r_o} \rightarrow A)} (\Rightarrow \rightarrow)}$$

5) The remaining axioms are considered in a similar way.

< *NSteps* > 1 >

One of the rules (*Modus ponens*) or (*K_I – necessitation*) of calculus HS-LCK was applied in the last step of the proof of the formula.

6) The rule (*Modus ponens*) was applied.

$$\frac{A, A \rightarrow B}{B} \text{ (Modus ponens)}$$

By the induction hypothesis, sequents ($\Rightarrow s : A$) and ($\Rightarrow s : A \rightarrow B$) are provable in GS-LCK. By invertibility of the rule ($\Rightarrow \rightarrow$), the sequent ($s : A \Rightarrow s : B$) is provable. The sequent ($\Rightarrow s : B$) of the theorem is proved by applying Theorem "Admissibility of cut".

7) The rule (*K_I – necessitation*) was applied.

$$\frac{A}{K_I A} \text{ (K}_I \text{ – necessitation)}$$

By the induction hypothesis, the sequent ($\Rightarrow s : A$) is provable in GS-LCK. By Lemma "Substitution", the sequent ($\Rightarrow t : A$) is provable. By Theorem "Admissibility of weakening", the sequent ($s \overset{I}{\sim} t \Rightarrow t : A$) is provable. The sequent of the theorem is proved by applying the rule ($\Rightarrow K_I$):

$$\frac{s \overset{I}{\sim} t \Rightarrow t : A}{\Rightarrow s : K_I A} (\Rightarrow K_I)$$

7. Automated Proof Search System GS-LCK-PROC

proof search system for LCK. GS-LCK-PROC is defined as procedure, which uses rules and axioms of sequent calculus GS-LCK. Principal formulas of the applications of the rules (*K_I ⇒*), (*K_N ⇒*) and ($\Rightarrow K_I$), and the chains of new appeared relational atoms of applications of the rule ($\Rightarrow K_I$) are saved in tables *TableLK* and *TableRK*.

Having sound and complete sequent calculus GS-LCK for logic of correlated knowledge we can model automated

Definition 8 (Table TableLK) Table TableLK of the principal pairs of the applications of the rules (*K_I ⇒*) and (*K_N ⇒*):

TableLK	
Main formula	Relational atom

Example 2 Example of TableLK:

TableLK	
Main formula	Relational atom
$s : K_I A$	$s \overset{I}{\sim} t$
$l : K_I B$	$l \overset{I}{\sim} z$

Definition 9 (Table TableRK) Table TableRK of the principal formulas and chains of new appeared relational atoms of the applications of the rule ($\Rightarrow K_I$):

TableRK			
Main formula	Chain of the relational atoms	Length of chain	Max

where Max is the maximum length of the chain, defined by $n(K_I) + 1$. Formula $n(K_I)$ denotes the number of negative occurrences of knowledge operator K_I in a sequent.

Example 2 Example of TableRK:

TableRK			
Main formula	Chain of the relational atoms	Length of chain	Max
$s, s_1, s_2, w_1 : K_I A$	$s \overset{I}{\sim} s_1, s_1 \overset{I}{\sim} s_2, s_2 \overset{I}{\sim} s_3$	3	5
	$s \overset{I}{\sim} t_1$	1	5
	$s \overset{I}{\sim} w_1, w_1 \overset{I}{\sim} w_2$	2	5
$z, z_1 : K_J B$	$z \overset{J}{\sim} z_1, z_1 \overset{J}{\sim} z_2$	2	7

Definition 10(Procedure of the proof search) Procedure GS-LCK-PROC of the proof search in the sequent calculus GS-LCK:

Initialisation:

- 1) Define set N of agents, tuple of sets $\vec{O} = (O_{a_1}, \dots, O_{a_n})$ of possible observations and result structure (R, Σ) .
- 2) Initialise the tables *TableLK* and *TableRK* by setting Max values to $(n(K_I) + 1)$, the length of the chain to 0 and the other cells leaving empty.
- 3) Set Output = False.

PROCEDURE GS-LCK-PROC (Sequent, TableLK, TableRK, Output)

BEGIN

1. Check if the sequent is the axiom. If the sequent is the axiom, set $Output = True$ and go to step Finish.
2. If possible, apply any of the rules $(\neg \Rightarrow), (\Rightarrow \neg), (\Rightarrow \vee), (\wedge \Rightarrow), (\Rightarrow \rightarrow)$ and go to step 1.
3. If possible, apply any of the rules $(\vee \Rightarrow), (\Rightarrow \wedge)$ or $(\rightarrow \Rightarrow)$ and call procedure GS-LCK-PROC() for the premises of the application:
Output1 = False;
Output2 = False;
GS-LCK-PROC(Premise1, TableLK, TableRK, Output1);
GS-LCK-PROC(Premise2, TableLK, TableRK, Output2);
IF (Output1 == True) AND (Output2 == True)
THEN Set Output = True and go to Finish;
ELSE Set Output = False and go to Finish;
4. If possible to apply any of the rules $(K_I \Rightarrow)$ or $(K_N \Rightarrow)$, check if the principal pair is absent in the table TableLK. If it is absent, apply rule $(K_I \Rightarrow)$ or $(K_N \Rightarrow)$, add principal pair to TableLK and go to step 1.
5. If possible to apply rule $(\Rightarrow K_I)$, check if the principal formula is absent in the table TableRK and the length of the chain is lower than Max. If the principal formula is absent and the length of the chain is lower than Max, apply rule $(\Rightarrow K_I)$, add principal formula and new relational atom to TableRK, increment the length of the chain by 1, and go to step 1.
6. If possible, apply rule (OYR) and call procedure GS-LCK-PROC() for the premises of the application:

For each k set Output(k) = False and call GS-LCK-PROC(Premise(k), TableLK, TableRK, Output(k)), where k is the index of the premise;

IF (for each k Output(k) == True)

THEN Set Output = True and go to Finish;

ELSE Set Output = False and go to Finish;

7. If possible, apply any of the rules $(\Rightarrow K_N), (OE), (CR), (Sub(p) \Rightarrow), (Sub(o^r) \Rightarrow), (Ref), (Trans), (Eucl)$ or (Mon) and go to step 1.

8. Finish.

END

Procedure GS-LCK-PROC gets the sequent, *TableLK*, *TableRK*, starting *Output* and returns "True", if the sequent is provable. Otherwise - "False", if it is not provable. Procedure is constructed in such a way, that it produces proofs, where number of applications of the knowledge rules of sequent calculus GS-LCK is finite. Also number of applications of other rules are bounded by requirements to rules and finite initial sets of agents, observations and results, which allows procedure to perform terminating proof search.

Lemma 4 (Permutation of the rule $(K_I \Rightarrow)$)

Rule $(K_I \Rightarrow)$ permutes down with respect to all rules of GS-LCK, except rules $(\Rightarrow K_I)$ and (OE) . Rule $(K_I \Rightarrow)$ permutes down with rules $(\Rightarrow K_I)$ and (OE) in case the principal atom of $(K_I \Rightarrow)$ is not active in it.

proof

The Lemma 6 is proved in the same way as the Lemma 6.3. in [15].

Lemma 5 (Number of applications of the rule $(K_I \Rightarrow)$)

If a sequent S is provable in GS-LCK, then there exists the proof of S such that rule $(K_I \Rightarrow)$ is applied no more than once on the same pair of principal formulas on any branch.

proof

The Lemma 6 is proved by induction on the number N of pairs of applications of rule $(K_I \Rightarrow)$ on the same branch with the same principal pair.

$\langle N = 0 \rangle$ The proof of the lemma is obtained.

$\langle N > 0 \rangle$

We diminish the inductive parameter in the same way as in the proof of Corollary 6.5. in [15], using Lemma 4. QED

Lemma 6 (Number of applications of the rule $(\Rightarrow K_I)$)

If a sequent S is provable in GS-LCK, then there exists the proof of S such that for each formula $s : K_I A$ in its positive part there are at most $n(K_I)$ applications of $(\Rightarrow K_I)$ iterated

on a chain of accessible worlds $s \stackrel{I}{\sim} s_1, s_1 \stackrel{I}{\sim} s_2, \dots$, with principal formula $s_i : K_I A$. The latter proof is called regular.

proof

The Lemma 6 is proved by induction on the number N of series of applications of rule $(\Rightarrow K_I)$, which make the initial proof non-regular.

$\langle N = 0 \rangle$ The proof of the lemma is obtained.

$\langle N > 0 \rangle$

We diminish the inductive parameter in the same way as in the proof of Proposition 6.9. in [15]. QED

Theorem 8(Termination of GS-LCK-PROC)

The procedure GS-LCK-PROC performs terminating proof search for each formula over (R, Σ, \vec{O}) .

proof

From construction of the procedure GS-LCK-PROC follows that the number of applications of the rules $(K_I \Rightarrow)$ and $(\Rightarrow K_I)$ is finite.

All the propositional rules reduce the complexity of the root sequent. Since the sets $N, (R, \Sigma), \vec{O}$ and the number of applications of the rules $(K_I \Rightarrow), (\Rightarrow K_I)$ are finite, and the requirements are imposed on the rules, the number of applications of the rules $(K_N \Rightarrow), (\Rightarrow K_N), (OE), (OYR), (CR), (Sub(p) \Rightarrow), (Sub(o^r) \Rightarrow), (Ref), (Trans), (Eucl)$ and (Mon) is also finite.

According to finite number of applications of all rules, the procedure GS-LCK-PROC performs the terminating proof search for any sequent. QED

Theorem 9 (Soundness and completeness of GS-LCK-PROC)

The procedure GS-LCK-PROC is sound and complete over (R, Σ, \vec{O}) .

proof

From construction of the procedure GS-LCK-PROC follows that if procedure returns "True" for a sequent S , then S is provable in GS-LCK. If procedure returns "False", then sequent S is not provable in GS-LCK, according to Lemma 6 and Lemma 6. QED

Theorem 10 (Decidability of LCK)

Logic LCK is decidable.

proof

From Theorem 6 and Theorem 6 follows that GS-LCK-PROC is a decision procedure for logic LCK. QED

8. Conclusions

Sequent calculus GS-LCK has properties of soundness, completeness, admissibility of cut and structural rules, and invertibility of all rules. Procedure GS-LCK-PROC performs automated terminating proof search for logic of correlated knowledge and also has properties of soundness and completeness.

Using GS-LCK-PROC, the validity of the formula of any sequent can be determined and inferences can be checked if they follow from some knowledge base. Modelling the knowledge of distributed systems in the logic of correlated knowledge, questions about the systems can be answered

automatically. Also soundness, completeness and termination of GS-LCK-PROC show that GS-LCK-PROC is a decision procedure for logic of correlated knowledge and LCK is decidable logic, which means asking questions about the system we will always get the answer.

Logic of correlated knowledge expands the range of the applications of family of epistemic logics and captures deeper knowledge of the group of agents in the distributed systems. GS-LCK-PROC allows to reason about correlated knowledge automatically, without human interaction in the reasoning process.

References

- [1] D. Aerts. Description of compound physical systems and logical interaction of physical systems. *Current Issues on Quantum Logic*, 8: 381C405, 1981.
- [2] A. Araujo and M. Finger. A formal system for quantum communication environments. *VIII - Brazilian National Meeting for Artificial Intelligence*, pages 1C11, 2011.
- [3] A. Baltag and S. Smets. Correlated knowledge: an epistemic-logic view on quantum entanglement. *International Journal of Theoretical Physics*, 49 (12): 3005C3021, 2010.
- [4] A. Baltag and S. Smets. Logics of informational interactions. *Journal of Philosophical Logic*, 44: 595C607, 2015.
- [5] A. Baltag and S. Smets. Modeling correlated information change: from conditional beliefs to quantum conditionals. *Soft computing*, 21 (6): 1523C1535, 2017.
- [6] G. Battilotti. Characterization of quantum states in predicative logic. *Int. J. Theor. Phys.*, 50: 3669C3681, 2011.
- [7] G. Birkhoff and J. von Neumann. The logic of quantum mechanics. *Annals of Mathematics*, 37: 823C843, 1936.
- [8] F. Boge. Quantum information vs. epistemic logic: An analysis of the frauchiger-renner theorem. *Foundations of Physics*, 49 (10): 1143C1165, 2019.
- [9] B. Coecke, C. Heunen, and A. Kissinger. *Compositional quantum logic. Computation, Logic, Games, and Quantum Foundations*, pages 21C36, 2013.
- [10] R. Fagin, J. Y. Halpern, and M. Y. Vardi. What can machines know? on the properties of knowledge in distributed systems. *Journal of the ACM*, 39 (2): 328C376, 1992.
- [11] G. Gentzen. Untersuchungenuberdaslogischeschliesen. i. *Mathematische Zeitschrift*, 39 (2): 176C210, 1934.
- [12] S. Kripke. Semantical analysis of modal logic i. normal propositional calculi. *Zeitschrift fur mathematische Logik und Grundlagen der Mathematik*, 9: 67C96, 1963.

- [13] G. Mackey. Quantum mechanics and hilbert space. *American Mathematical Monthly*, 64 (2): 45C57, 1957.
- [14] G. Mackey. The mathematical foundations of quantum mechanics. *W.A. Benjamin, NY*, 1963.
- [15] S. Negri. Proof analysis in modal logic. *Journal of Philosophical Logic*, 34 (5): 507C544, 2005.
- [16] S. Negri and J. von Plato. *Structural Proof Theory*. Cambridge University Press, 2001.
- [17] N. Nurgalieva and L. del Rio. Inadequacy of modal logic in quantum settings. *EPTCS 287*, pages 267C297, 2019.
- [18] C. Piron. Foundations of quantum physics. *W.A. Benjamin Inc., Massachusetts*, 1976.
- [19] C. Randall and D. Foulis. Tensor products of quantum logics do not exist. *Notices Amer. Math. Soc.*, 26 (6), 1979.
- [20] S. Smets. Logic and quantum physics. *Journal of the Indian Council of Philosophical Research Special Issue*, XXVII(2), 2010.
- [21] W. van der Hoek and J.-J. Ch. Meyer. A complete epistemic logic for multiple agents combining distributed and common knowledge. *Epistemic Logic and the Theory of Games and Decisions*, pages 35C68, 1997.
- [22] V. Vilasini, N. Nurgalieva, and L. del Rio. Multi-agent paradoxes beyond quantum theory. *New Journal of Physics*, 21 (11), 2019.