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A Coupling Elzaki Transform and Homotopy Perturbation Method for Solving Nonlinear Fractional Heat-Like Equations

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Abstract: In this article, we have proposed a reliable combination of Elzaki transform and homotopy perturbation method (ETHPM) to solve Nonlinear Fractional Heat -Like Equations. The nonlinear terms in the equations can be handled by using homotopy perturbation method (HPM). This methods is very powerful and efficient techniques for solving different kinds of linear and nonlinear fractional differential equations. The results reveal that the combination of ELzaki transform and homotopy perturbation method (ETHPM) ismore efficient and easier to handle when is compared with existing other methods in such PDEs.

Keywords: Homotopy Decomposition Method, Nonlinear Fractional Heat-Like Equation, Elzaki Transform

I. Introduction

Nowadays there is increasing attention paid to fractional differential equations [1-3] and their applications in different research areas. Fractional differential equations (FDEs) consist of a Fractional differential with specified value of the unknown function. And represent an important tool in technology, science and economics and engineering applications included population models, control engineering electrical network analysis, gravity, medicine, etc, [9-15]. In the resent years, many researchers mainly had paid attention to studying the solution of nonlinear fractional partial differential equations by using various methods. Among these are the Variational Iteration Method (VIM) [27-28], Adomian Decomposition Method (ADM) [16-17], projected differential transform method [25], and the Differential Transform Method (ADM) [26], are the most popular ones that are used to solve differential and integral equations of integer and fractional order. The Homotopy Perturbation Method (HPM) [4-6] is a universal approach which can be used to solve both fractional ordinary differential equations FODEs as well as fractional partial differential equations FPDEs. This method, was originally proposed by He [7, 8]. The HPM is a coupling of homotopy and the perturbation method. Recently, Tarig M. Elzaki and Sailh M. Elzaki in [18-24], showed Elzaki transform, was applied to partial differential equations, ordinary differential equations, system of ordinary and partial differential equations and integral equations. Elzaki transform is a powerful tool for solving some differential equations which cannot solve by Sumudu transform. In this article, we use Elzaki transform and homotopy perturbation method together to solve Nonlinear Fractional Heat - Like Equations.

2. Basic Definitions and Notations of the Fractional Calculus

In this section, some definitions and properties of the fractional calculus that will be used in this work are presented.

Definition 1:

The Gamma function is intrinsically tied in fractional calculus. The simplest interpretation of the gamma function is simply the generalization of the fraction for all real numbers. The definition of the gamma function is given by:

$$\Gamma(\mu) = \int_0^\infty e^{-t}\, t^{\mu-1} dt, \mu > 0 \qquad (1)$$

Definition 2:

A real function$f(x), x > 0$, is said to be in the space C_μ,

$\mu \in \mathbb{R}$ if there exists a real number $p > \mu$, such that $f(x) = x^p h(x)$, where$h(x) \in [0, \infty)$ and it is said to be in space C_μ^m if $f^{(m)} \in C_\mu$, $m \in \mathbb{N}$.

Definition 3:

The Riemann-Liouville fractional integral operator of order $\alpha > 0$, of a function $f \in C_m$, $\mu \geq -1$, is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-1)^{\alpha-1} f(t) dt, \alpha > 0, x > 0 \quad (2)$$

$$J^\alpha f(x) = f(x)$$

Some Properties of the operator:

For $f \in C_m$, $\mu \geq -1$, $\alpha, \beta \geq 0$ and $\gamma > -1$

$$J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$$

$$J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x) \quad (3)$$

$$J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$$

Lemma 1:

If $m-1 < \alpha \leq m, m \in \mathbb{N}$ and $f \in C_m$, $\mu \geq -1$ then $D^\alpha J^\alpha f(x) = f(x)$ and,

$$J^\alpha D_0^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0) \frac{x^k}{k!}, x > 0 \quad (4)$$

Definition 3: (Partial Derivatives of Fractional order)

Assume now that $f(x)$ is a function of $n$variables $x_i, i = 1, \dots, n$ also of class C on $D \in \mathbb{R}_n$. As an extension of definition 2 we define partial derivative of order $\alpha$for $f(x)$ respect tox_i

$$a\partial_{\underline{x}}^\alpha f = \frac{1}{\Gamma(m-\alpha)} \int_0^{x_i} (x_i - 1)^{m-\alpha-1} \partial_{x_i}^\alpha f(x_j)\Big|_{x_j=t} dt \quad (5)$$

If it exists, where $\partial_{x_i}^\alpha$ is the usual partial derivative of integer order m.

3. Fundamental Facts of the Elzaki Transformation Method

A new transform called the Elzaki transform defined for function of exponential order we consider functions in the set **A**, defined by:

$$A = \{f(t) : \exists\, M, k_1, k_2 > 0, |f(t)| < Me^{\frac{|t|}{k_j}}, if t \in (-1)^j \times [0, \infty) \quad (6)$$

For a given function in the set, the constant M must be finite number, k_1, k_2 may be finite or infinite. The Elzaki transform which is defined by the integral equation

$$E[f(t)] = T(v) = v \int_0^\infty f(t) e^{\frac{-t}{v}} dt, t \geq 0, \mathrm{k}_1 \leq v \leq \mathrm{k}_2 \quad (7)$$

The following results can be obtained from the definition and simple calculations

1) $E[t^n] = n!\, v^{n+2}$

2)$E[f'(t)] = \frac{T(v)}{v} - vf(0)$

3) $E[f''(t)] = \frac{T(v)}{v^2} - f(0) - vf'(0)$

4) $E[f^{(n)}(t)] = \frac{T(v)}{v^n} - \sum_{k=0}^{n-1} v^{2-n+k} f^{(k)}(0)$

Theorem 1:

If $T(v)$ is Elzaki transform of (t), one can take into consideration the Elzaki transform of the Riemann-Liouville derivative as follow:

$$E[D^\alpha f(t)] = v^{-\alpha} \left[T(v) - \sum_{k=1}^{n} v^{\alpha-k+2} \left[D^{\alpha-k} f(0)\right] \right]; \quad -1 < n-1 \leq \alpha < n \quad (8)$$

Proof: Let us take Laplace transformation of

$$f'(t) = \frac{d}{dt} f(t)$$

$$L[D^\alpha f(t)] = S^\alpha T(s) - \sum_{k=0}^{n-1} s^k [D^{\alpha-k-1} f(0)]$$

$$= s^\alpha T(s) - \sum_{k=0}^{n} s^{k-1} [D^{\alpha-k} f(0)] = s^\alpha T(s) - \sum_{k=1}^{n} s^{k-2} [D^{\alpha-k} f(0)]$$

$$= s^\alpha T(s) - \frac{1}{s^{-k+2}} \sum_{k=1}^{n} [D^{\alpha-k} f(0)] = s^\alpha T(s) - \sum_{k=0}^{n} \frac{1}{s^{\alpha-k+2-\alpha}} [D^{\alpha-k} f(0)]$$

$$= s^\alpha T(s) - \sum_{k=1}^{n} s^\alpha \frac{1}{s^{\alpha-k+2}} [D^{\alpha-k} f(0)]$$

$$L[D^\alpha f(t)] = s^\alpha \left[T(s) - \sum_{k=1}^{n} \left(\frac{1}{s}\right)^{\alpha-k+2} [D^{\alpha-k} f(0)] \right]$$

Therefore, when we substitute $\frac{1}{v}$ for s, we get the Elzaki transformation of fractional order of $f(t)$ as follows:

$$E[D^\alpha f(t)] = v^{-\alpha} [T(v) - \sum_{k=1}^{n} v^{\alpha-k+2} [D^{\alpha-k} f(0)]] \quad (9)$$

Definition 4:

The Elzaki transform of the Caputo fractional derivative by using Theorem 1 is defined as follows:

$$E[D_t^\alpha f(t)] = v^{-\alpha} E[f(t)] - \sum_{k=0}^{m-1} v^{2-\alpha+k} f^{(k)}(0), \quad (10)$$

where $m-1 < \alpha < m$

4. Basic Idea of HPETM

To illustrate the basic idea of this method, we consider a general form of nonlinear non homogeneous partial differential equation as the follow:

$$D_t^\alpha u(x,t) = L(u(x,t)) + N(u(x,t)) + f(x,t), \alpha > 0 \quad (11)$$

with the following initial conditions

$$D_0^k u(x,0) = g_k, k = 0, \ldots, n-1, D_0^n u(x,0) = 0 \text{ and} n = [\alpha] \quad (12)$$

Where D_t^α denotes without loss of generality the Caputo fraction derivative operator, f is a known function, N is the general nonlinear fractional differential operator and L represents a linear fractional differential operator.

Taking Elzaki transform on both sides of equation (11), to get:

$$E[D_t^\alpha u(x,t)] = E[L(u(x,t))] + E[N(u(x,t))] + E[f(x,t)] \quad (13)$$

Using the differentiation property of Elzaki transform and above initial conditions, we have:

$$E[u(x,t)] = v^\alpha E[L(u(x,t))] + v^\alpha E[N(u(x,t))] + g(x,t) \quad (14)$$

Operating with the Elzaki inverse on both sides of equation (14) gives:

$$u(x,t) = G(x,t) + E^{-1}\left[v^\alpha E[L(u(x,t))] + v^\alpha E[N(u(x,t))]\right] \quad (15)$$

Where $G(x,t)$ represents the term arising from the known function $f(x,t)$ and the initial condition.

Now, we apply the homotopy perturbation method

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \quad (16)$$

And the nonlinear term can be decomposed as:

$$Nu(x,t) = \sum_{n=0}^{\infty} p^n H_n(u) \quad (17)$$

Where$H_n(u)$ are He's polynomial and given by:

$$H_n(u_0, u_1, u_2 \ldots u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}[N(\sum_{i=0}^{\infty} p^i u_i(x,t))]_{p=0}, n = 0,1,2, \ldots \quad (18)$$

Substituting equations. (16) and (17) in equation (15) we get:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = G(x,t) + p\left[E^{-1}[v^\alpha E[L(\sum_{n=0}^{\infty} p^n u_n(x,t))] + v^\alpha E[N(\sum_{n=0}^{\infty} p^n u_n(x,t))]]\right] \quad (19)$$

Which is the coupling of the Elzaki transform and the homotopy perturbation method using He's polynomials. Comparing the coefficient of like powers of p , thefollowingapproximations are obtained:

$$p^0 : u_0(x,t) = G(x,t),$$

$$p^1 : u_1(x,t) = E^{-1}[v^\alpha E[L(u_0(x,t)) + H_0(u)]],$$

$$p^2 : u_2(x,t) = E^{-1}[v^\alpha E[L(u_1(x,t)) + H_1(u)]],$$

$$p^3 : u_3(x,t) = E^{-1}[v^\alpha E[L(u_2(x,t)) + H_2(u)]],$$

$$p^n : u_n(x,t) = E^{-1}[v^\alpha E[L(u_{n-1}(x,t)) + H_{n-1}(u)]], \quad (20)$$

Then the solution is;

$$u(x,t) = \lim_{p\to 1} u_n(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots \quad (21)$$

The above series solution generally converges very rapidly.

5. Applications

In this section we solve some examples of nonlinear fractional heat- like equation:

Example 5.1:

Let consider the following one dimensional fractional heat- like equation:

$$D_t^\alpha u(x,t) = \frac{1}{2}x^2 u_{xx}(x,t), 0 < x < 1, 0 < \alpha \le 1, t > 0 \quad (22)$$

with initial condition

$$u(x,0) = x^2 \quad (23)$$

Applying the Elzaki transform of both sides of Eq. (22),

$$E[D_t^\alpha u(x,t)] = E\left[\frac{1}{2}x^2 u_{xx}(x,t)\right] \quad (24)$$

Using the differential property of Elzaki transform Eq.(24) can be written as:

$$v^{-\alpha}\left(E[u(x,t)] - v^2 u(x,0)\right) = E\left[\frac{1}{2}x^2 u_{xx}(x,t)\right] \quad (25)$$

Using initial condition (23), Eq. (25) can be written as:

$$E[u(x,t)] = v^2 x^2 + v^\alpha E\left[\frac{1}{2}x^2 u_{xx}(x,t)\right] \quad (26)$$

The inverse Elzaki transform implies that:

$$u(x,t) = x^2 + E^{-1}\left[v^\alpha E\left[\frac{1}{2}x^2 u_{xx}(x,t)\right]\right] \quad (27)$$

Now, we apply the homotopy perturbation method, we get:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = x^2 + pE^{-1}\left[v^\alpha E\left[\frac{1}{2}x^2(\sum_{n=0}^{\infty} p^n u_n(x,t))_{xx}\right]\right] \quad (28)$$

Comparing the coefficient of like powers of p , the following approximations are obtained;

$$p^0 : u_0(x,t) = x^2$$

$$p^1 : u_1(x,t) = E^{-1}\left[v^\alpha E\left[\frac{1}{2}x^2 u_0(x,t)_{xx}\right]\right] = E^{-1}[v^\alpha E[x^2]] = E^{-1}[x^2 v^{\alpha+2}] = \frac{x^2 t^\alpha}{\alpha!} = \frac{x^2 t^\alpha}{\Gamma(\alpha+1)},$$

$$p^2 : u_2(x,t) = E^{-1}\left[v^\alpha E\left[\frac{1}{2}x^2 u_1(x,t)_{xx}\right]\right] = E^{-1}\left[v^\alpha E\left[\frac{x^2 t^\alpha}{\Gamma(\alpha+1)}\right]\right] = E^{-1}\left[\frac{(v^{2\alpha+2})x^2}{\Gamma(\alpha+1)}\right] = \frac{x^2 t^{2\alpha}}{\Gamma(2\alpha+1)},$$

Proceeding in a similar manner, we have:

$$p^3 : u_3(x,t) = E^{-1}\left[v^\alpha E\left[\frac{1}{2}x^2 u_2(x,t)_{xx}\right]\right] = \frac{x^2 t^{3\alpha}}{\Gamma(3\alpha+1)},$$

$$p^n : u_n(x,t) = E^{-1}\left[v^\alpha E\left[\frac{1}{2}x^2 u_n(x,t)_{xx}\right]\right] = \frac{x^2 t^{n\alpha}}{\Gamma(n\alpha+1)},$$

Therefore the series solution $u(x,t)$ is given by:

$$u(x,t) = x^2\left(1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots + \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} + \cdots\right) \quad (29)$$

This equivalent to the exact solution in closed form:

$$u(x,t) = x^2 E_\alpha(t^\alpha) \quad (30)$$

Where $E_\alpha(t^\alpha)$ is the Mittag-Leffler function

Example 5.2:

Consider the following tow - dimensional fractional heat like equation:

$$D_t^\alpha u = u_{xx} + u_{yy}, 0 < x, y < 2\pi, 0 < \alpha \le 2, t > 0 \quad (31)$$

With the initial conditions

$$u(x,y,0) = \sin x \sin y \quad (32)$$

Applying the Elzaki transform of both sides of Eq. (31),

$$E[D_t^\alpha u(x,y,t)] = E\left[u_{xx} + u_{yy}\right] \quad (33)$$

Using the differential property of Elzaki transform Eq.(33) can be written as:

$$v^{-\alpha}\left(E[u(x,y,t)] - v^2 u(x,y,0)\right) = E\left[u(x,y,t)_{xx} + u(x,y,t)_{yy}\right] \quad (34)$$

Using initial condition (32), Eq. (34) can be written as:

$$E[u(x,y,t)] = v^2 \sin x \sin y + v^\alpha E\left[u(x,y,t)_{xx} + u(x,y,t)_{yy}\right] \quad (35)$$

The inverse Elzaki transform implies that:

$$u(x,y,t) = \sin x \sin y + E^{-1}\left[v^\alpha E\left[u(x,y,t)_{xx} + u(x,y,t)_{yy}\right]\right] \quad (36)$$

Now, we apply the homotopy perturbation method, we get:

$$\sum_{n=0}^{\infty} p^n u_n(x,y,t) = \sin x \sin y + pE^{-1}\left[v^\alpha E\left[\left(\sum_{n=0}^{\infty} p^n u_n(x,y,t)\right)_{xx} + \left(\sum_{n=0}^{\infty} p^n u_n(x,y,t)\right)_{yy}\right]\right] \quad (37)$$

Comparing the coefficient of like powers of p, the following approximations are obtained;

$$p^0 : u_0(x,y,t) = \sin x \sin y$$

$$p^1 : u_1(x,y,t) = E^{-1}\left[v^\alpha E\left[u_{0_{xx}} + u_{0_{yy}}\right]\right] = \frac{-2 \sin x \sin y\, t^\alpha}{\Gamma(\alpha+1)}$$

$$p^2 : u_2(x,y,t) = E^{-1}\left[v^\alpha E\left[u_{1_{xx}} + u_{1_{yy}}\right]\right] = \frac{4 \sin x \sin y\, t^{2\alpha}}{\Gamma(2\alpha+1)}$$

Proceeding in a similar manner, we have:

$$p^3 : u_3(x,y,t) = \frac{-8 \sin x \sin y t^{3\alpha}}{\Gamma(3\alpha+1)},$$

$$p^n : u_n(x,y,t) = \frac{(-2)^n \sin x \sin y t^{n\alpha}}{\Gamma(n\alpha+1)},$$

Therefore the series solution $u(x,t)$ is given by:

$$u(x,y,t) = \sin x \sin y\left(1 - \frac{(2t^\alpha)}{\Gamma(\alpha+1)} + \frac{(2t^\alpha)^2}{\Gamma(2\alpha+1)} - \frac{(2t^\alpha)^3}{\Gamma(3\alpha+1)} + \cdots + \frac{(-2t^\alpha)^n}{\Gamma(n\alpha+1)} + \cdots\right) \quad (38)$$

For the special case when$\alpha = 1$, we can get the solution in a closed form

$$u(x,y,t) = e^{-2t} \sin x \sin y \quad (39)$$

Example 5.3:

Consider the following three dimensional fractional heat-like equation:

$$D_t^\alpha u(x,y,z,t) = x^4 y^4 z^4 + \frac{1}{36}\left(x^2 u_{xx} + y^2 u_{yy} + z^2 u_{zz}\right), 0 < x, y, z < 1, 0 < \alpha \le 1 \quad (40)$$

With the initial condition;

$$u(x,y,z,t) = 0 \quad (41)$$

Applying the Elzaki transform of both sides of Eq. (40),

$$E[D_t^\alpha u(x,y,t)] = E[x^4 y^4 z^4] + E\left[\frac{1}{36}\left(x^2 u_{xx} + y^2 u_{yy} + z^2 u_{zz}\right)\right] \quad (42)$$

Using the differential property of Elzaki transform Eq.(42), and using initial condition (41), Eq. (42) can be written as:

$$E[u(x,y,z,t)] = v^2 x^4 y^4 z^4 + v^\alpha E\left[\frac{1}{36}\left(u(x,y,z,t)_{xx} + u(x,y,z,t)_{yy} + u(x,y,z,t)_{xx}\right)\right] \quad (43)$$

The inverse Elzaki transform implies that:

$$u(x,y,z,t) = x^4 y^4 z^4 + E^{-1}\left[v^\alpha E\left[\frac{1}{36}\left(u(x,y,z,t)_{xx} + u(x,y,z,t)_{yy} + u(x,y,z,t)_{xx}\right)\right]\right] \quad (44)$$

Now, we apply the homotopy perturbation method, we get:

$$\sum_{n=0}^{\infty} p^n u_n(x,y,z,t) = x^4 y^4 z^4 + pE^{-1}\left[v^\alpha E\left[\left(\sum_{n=0}^{\infty} p^n u_n(x,y,z,t)\right)_{xx} + \left(\sum_{n=0}^{\infty} p^n u_n(x,y,z,t)\right)_{yy} + \left(\sum_{n=0}^{\infty} p^n u_n(x,y,z,t)\right)_{yy}\right]\right] \quad (45)$$

Comparing the coefficient of like powers of p, the following approximations are obtained;

$$p^0 : u_0(x,y,z,t) = x^4 y^4 z^4$$

$$p^1 : u_1(x,y,zt) = E^{-1}\left[v^\alpha E\left[\frac{1}{36}\left(x^2 u_{0_{xx}} + y^2 u_{0_{yy}} + z^2 u_{0_{zz}}\right)\right]\right] = \frac{x^4 y^4 z^4 t^\alpha}{\Gamma(\alpha+1)},$$

$$p^2 : u_2(x,y,zt) = E^{-1}\left[v^\alpha E\left[\frac{1}{36}\left(x^2 u_{1_{xx}} + y^2 u_{1_{yy}} + z^2 u_{1_{zz}}\right)\right]\right] = \frac{x^4 y^4 z^4 t^{2\alpha}}{\Gamma(2\alpha+1)}$$

Proceeding in a similar manner, we have:

$$p^3 : u_3(x,y,z,t) = \frac{x^4 y^4 z^4 t^{3\alpha}}{\Gamma(3\alpha+1)},$$

$$p^n : u_n(x,y,z,t) = \frac{x^4y^4z^4t^{n\alpha}}{\Gamma(n\alpha+1)},$$

Therefore the series solution $u(x,t)$ is given by:

$$u(x,t) = x^4y^4z^4\left(1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots + \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} + \cdots\right) \quad (46)$$

Therefore the approximate solution of equation for the first $N$is given below as:

$$u_n(x,y,z,t) = \sum_{n=1}^{N} \frac{(x^4y^4z^4)t^{n\alpha}}{\Gamma(n\alpha+1)} \quad (47)$$

Now when $N \to \infty$ we obtained the follow solution

$$u_n(x,y,z,t) = \sum_{n=0}^{\infty} \frac{(x^4y^4z^4)t^{n\alpha}}{\Gamma(n\alpha+1)} - (x^4y^4z^4)$$
$$= (x^4y^4z^4)[E_\alpha(t^\alpha) - 1] \quad (48)$$

where $E_\alpha(t^\alpha)$ is the generalized Mittag-Leffler function. Note that in the case $\alpha = 1$

$$u(x,y,z,t) = (xyz)^4[e^t - 1] \quad (49)$$

This is the exact solution for this case

6. Conclusion

The main goal of this paper is to show the applicability of the mixture of new integral transform “ELzaki transform” with the homotopy perturbation method (ETHPM) to construct an analytical solution for Nonlinear Fractional Heat -Like Equations. This combination of two methods successfully worked to give very reliable solutions to the equation. Finally the results tell us that the proposed method is more efficient and easier to handle when is compared with existing other methods in such PDEs.

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