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Algebraic Points of Any Given Degree on the Affine Equation Curve $y^{11} = x^4(x-1)^4$

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Abstract : The quotients of Fermat curves $\mathcal{C}_{r,s}(p)$ are studied by SALL who extends the work of Gross and Rohlich. Among these studies are the cases $\mathcal{C}_{r,s}(11)$ for $r = s = 1$. COLY and Sall have explicitly determined the algebraic points of degree at most 3 on $\mathbb{Q}$ for the cases $\mathcal{C}_{r,s}(11)$ for $r = s = 2$. Our work focuses we determine explicitly the algebraic points of a given degree over on the curve $\mathcal{C}_{4,4}(11)$ of affine equation $y^{11} = x^4(x-1)^4$ which is a special case of Fermat quotient curves. Our study concerns the cases $\mathcal{C}_{r,s}(11)$ for $r = s = 4$. This note completes pevious work of Gassama and Sall who explicitly determined the algebraic points of degree at most three on the even curve. It seems that the finiteness of the Mordell-Weil group of rational points of the Jacobien $J_{4,4}(11)(\mathbb{Q})$ is an essential condition. So to determine the algebraic points on the curve $\mathcal{C}_{4,4}(11)$ we need a finiteness of the Mordeill-Weill group of rational points of the Jacobien $J_{4,4}(11)(\mathbb{Q})$. The Mordell-Weil group $J_{4,4}(11)(\mathbb{Q})$ of rational points of the Jacobien is finite according to Faddev. Our note is in this framework. Our essential tools in this note are the Mordell-Weil group $J_{4,4}(11)(\mathbb{Q})$ of the Jacobien of $\mathcal{C}_{4,4}(11)$ the Abel-Jacobi theorem and the study of linear systems on the curve $\mathcal{C}_{4,4}(11)$. The result obtained concerns some quotients of Fermat curves. Indeed, the curve of affine equation $y^{11} = x^4(x-1)^4$, we made an extension of the work of work of Gassama and Sall by explicitly determining the algebraic points of given degree on the curve $\mathcal{C}_{4,4}(11)$ and this is what makes this note very interesting.

Keywords : Mordell-Weil Group, Jacobian, Galois Conjugates

1. Introduction

Let $\mathcal{C}$ be a smooth algebraic curve defined on $\mathbb{Q}$. Let K be a field of numbers we note $\mathcal{C}(K)$ the set of points on $\mathcal{C}$ with coordinates in K, and $\bigcup_{[K\,:\,\mathbb{Q}]\leqslant d}\mathcal{C}(K)$ the set of points on $\mathcal{C}$ with coordinates in K of degree at most d on $\mathbb{Q}$. The degree of a point R of $\mathcal{C}$ algebraic on $\mathbb{Q}$ is defined as the degree of its defining field on $\mathbb{Q}$; in other words $deg(R) = [\mathbb{Q}(R) : \mathbb{Q}]$.

When $\mathcal{C}$ is of genus $g \geq 2$, we know since Faltings [5] that the set of rational points $\mathcal{C}(Q)$ is finite. A generalization of this theorem to subvarieties of abelian varieties obtained by Vojta and Faltings [4, 16] can also be used to qualitatively describe $\bigcup_{[K\,:\,\mathbb{Q}]\leqslant d}\mathcal{C}(K)$.

In this note, our work will consist in the study of some particular cases, where particular cases, where we can determine explicitly the algebraic points of any degree on the curve of affine equation $\mathcal{C}_{4,4}(11) : y^{11} = x^4(x-1)^4$. $\mathcal{C}$ corresponds to the curve $\mathcal{C}_{4,4}(11)$. Our curve $\mathcal{C}_{4,4}(11)$ with affine equation $y^{11} = x^4(x-1)^4$ is a special case of the quotients of Fermat curves $\mathcal{C}_{r,s}(p) : y^p = x^r(x-1)^s$, $1 \leq r, s ; r + s \leq p-1$ studied in [10 − 12].

The cases $\mathcal{C}_{r,s}(11)$ for $r = s = 2$ are studied in [1]. See [2, 3, 5, 8, 13] for other explicit examples. Indeed, C corresponds to the curve $\mathcal{C}_{4,4}(11)$. The curves $\mathcal{C}_{r,s}(p)$ are quotients of F_p [7, 16].

We denote by $J_{4,4}(11)$ the Jacobien of $\mathcal{C}_{4,4}(11)$ and by $j(P)$ the class denoted $[P - P_\infty]$ of $P - P_\infty$, that is to say j is the Jacobien fold $\mathcal{C}_{4,4}(11) \longrightarrow J_{4,4}(11)$. The Mordell-Weil group $J_{4,4}(11)(\mathbb{Q})$ of the rational points of the Jacobien is finite [6, 7, 14]. The curve $\mathcal{C}_{4,4}(11)$ in projective is $\mathcal{C}_{4,4}(11) : Y^{11} = X^4Z^7(X-Z)^4$. Let us note P_0, P_1 and P_∞ the points defined by : $P_0 = (0,0,1)$; $P_1 = (1,0,1)$ and $P_\infty = (1,0,0)$.

Gassama and Sall [9], determined the set of algebraic points of degree at most 3 on $\mathbb{Q}$ of the curve with affine equation $y^{11} = x^4(x-1)^4$. We extend this result, giving an explicit description of the points of any degree on $\mathbb{Q}$ on the curve $\mathcal{C}_{4,4}(11)$.

In this note we determine the set :

$$\bigcup_{[K:\mathbb{Q}]\leqslant l} \mathcal{C}_{4,4}(11)(K)$$

Our main result is the following :

Théorème 1. The set of algebraic points of at most any degree l on $\mathbb{Q}$ on the curve $\mathcal{C}_{4,4}(11)$ is given by :

$$\bigcup_{[\mathbb{K}:\mathbb{Q}]\leqslant l} \mathcal{C}_{4,4}(11)(\mathbb{K}) = \mathcal{G}_0 \bigcup \left(\bigcup_{1\leq n\leqslant 10} \mathcal{G}_n \right)$$

where :

$$\mathcal{G}_0 = \left\{ \left(-\frac{\sum_{i=0}^{\frac{l}{2}} \alpha_i y^{\frac{i}{2}}}{\sum_{j=0}^{\frac{l-11}{2}} \beta_j y^{\frac{j}{2}}}, y \right) \middle| \begin{array}{c} (\alpha_0 \wedge \beta_0) \neq 0,\ \alpha_{\frac{l}{2}} \neq 0 \text{ if } l \text{ is even, } \beta_{\frac{l-11}{2}} \neq 0 \text{ if } l \text{ is odd and} \\ y \text{ solution of the equation} \\ y^{11}\left(\sum_{j=0}^{\frac{l-11}{2}} \beta_j y^{\frac{j}{2}}\right)^4 = \left(\sum_{i=0}^{\frac{l}{2}} \alpha_i y^{\frac{i}{2}}\right)^2 \left(\sum_{i=0}^{\frac{l}{2}} \alpha_i y^{\frac{i}{2}} + \sum_{j=0}^{\frac{l-11}{2}} \beta_j y^{\frac{j}{2}}\right)^2 \end{array} \right\}$$

$$\mathcal{G}_n = \left\{ \left(-\frac{\sum_{i=0}^{\frac{l}{2}} \alpha_i y^{\frac{i}{2}}}{\sum_{j=0}^{\frac{l-11}{2}} \beta_j y^{\frac{j}{2}}}, y \right) \middle| \begin{array}{c} \beta_0 \neq 0,\ \alpha_{\frac{l+11-n}{2}} \neq 0 \text{ if } l \text{ is even, } \beta_{\frac{l-n}{2}} \neq 0 \text{ if } l \text{ is odd and} \\ y \text{ solution of the equation} \\ y^{n}\left(\sum_{j=0}^{\frac{l-n}{2}} \beta'_j y^{\frac{j}{2}}\right)^4 = \left(\sum_{i=11-n}^{\frac{l+11-n}{2}} \alpha_i y^{\frac{i+n-11}{2}}\right)^2 \left(\sum_{i=11-n}^{\frac{l+11-n}{2}} \alpha_i y^{\frac{i}{2}} + \sum_{j=0}^{\frac{l-n}{2}} \beta_j y^{\frac{j}{2}}\right)^2 \end{array} \right\}$$

2. Auxiliary Results

For a divisor D on C, we denote $\mathcal{L}(D)$ the $\bar{\mathbb{Q}}$-vector space of rational functions f defined on $\mathbb{Q}$ such that $f = 0$ or $div(f) \geq -D$; $l(D)$ denotes the $\bar{\mathbb{Q}}$-dimension of $\mathcal{L}(D)$.

Lemma 1. we have : $\mathcal{J}_{4,4}(11)(\mathbb{Q}) \cong \mathbb{Z}/11\mathbb{Z}$

Demonstration : According Gross and Rohrlich ([7], page. 219), we have : $\mathcal{J}_{4,4}(11)(\mathbb{Q})_{torsion} \cong \mathbb{Z}/11\mathbb{Z}$ and According to Faddeev [6], we have : $\mathcal{J}_{4,4}(11)(\mathbb{Q})_{torsion} \cong \mathcal{J}_{4,4}(11)(\mathbb{Q})$.

Lemma 2. For the curve $\mathcal{C}_{4,4}(11) : y^{11} = x^4(x-1)^4$, we have :

1. $div(x) = 11P_0 - 11P_\infty$,
2. $div(x-1) = 11P_1 - 11P_\infty$,
3. $div(y) = 4P_0 + 4P_1 - 8P_\infty$.

Demonstration : (see [11], Lemme 1)

Corollary 1. The following results are the consequences of Lemma 2.

1. $4j(P_0) = -4j(P_1)$,
2. $11j(P_0) = 11j(P_1) = 0$

So $j(P_0)$ and $j(P_1)$ generate the same subgroup $\mathcal{J}_{4,4}(11)(\mathbb{Q})$ isomorphic to $\mathbb{Z}/11\mathbb{Z}$. Thus we have :

$$\mathcal{J}_{4,4}(11)(\mathbb{Q}) \cong \mathbb{Z}/11\mathbb{Z} = \{nj(P_0),\ 0 \leq n \leq 10\}.$$

Lemma 3. A $\mathbb{Q}$-base of $\mathcal{L}(mP_\infty)$ is given by :

$$\mathfrak{B}_m = \left\{ \left(\frac{y^3}{x(x-1)}\right)^i,\ 0 \leq i \leq \frac{m}{2} \right\} \bigcup \left\{ x\left(\left(\frac{y^3}{x(x-1)}\right)^j\right),\ 0 \leq j \leq \frac{m-11}{2} \right\}$$

Demonstration :

It is easy to show that $\mathcal{B}_m$ is a free family, it then remains to show that card $\mathcal{B}_m = dim\mathcal{L}(mP_\infty)$. We know that the genus of $\mathcal{C}$ is $g = \dfrac{11-1}{2} = 5$. Since the curve has genus 5, by the Riemann-Roch theorem, we have $\dim\mathcal{L}(mP_\infty) = m-g+1 = m-4$ as soon as $m \geq 2g-1 = 9$. For $m < 9$, $\dim\mathcal{L}(mP_\infty)$ is given by Clifford's theorem, which says that $\dim\mathcal{L}(mP_\infty) \geq \dfrac{1}{2}\deg(mP_\infty) + 1 = \dfrac{1}{2}m + 1$. Two cases are possible :

1. 1^{st} case : suppose that m is even, then $m = 2h$, we obtain :
   $i \leq \frac{m}{2} \Leftrightarrow i \leq \frac{2h}{2} = h$ in the same way $j \leq \frac{m-11}{2} \Leftrightarrow j \leq \frac{2h-11}{2} \Leftrightarrow j \leq h - \frac{11}{2} \Longrightarrow j < h - \frac{10}{2} = h - 5 \Longrightarrow j \leqslant h - 6$. So we have :
   $$\mathcal{B}_m = \left\{1, \left(\frac{y^3}{x(x-1)}\right), \ldots, \left(\frac{y^3}{x(x-1)}\right)^h\right\} \bigcup \left\{x, x\left(\frac{y^3}{x(x-1)}\right), \ldots, x\left(\frac{y^3}{x(x-1)}\right)^{h-6}\right\}.$$
   We deduce that : card $\mathcal{B}_m = h+1+h-6+1 = 2h-4 = m-4 = \dim\mathcal{L}(mP_\infty)$.
2. 2^{nd} case : suppose that m is odd, then $m = 2h+1$, we obtain :
   $i \leq \frac{m}{2} \Leftrightarrow i \leq \frac{2h+1}{2} \Leftrightarrow i \leq h + \frac{1}{2} \Longrightarrow i < h+1 \Longrightarrow i \leqslant h$ in the same way $j \leq \frac{m-11}{2} \Leftrightarrow j \leq \frac{2h-10}{2} = h - 5$. So we have :
   $$\mathcal{B}_m = \left\{1, \left(\frac{y^3}{x(x-1)}\right), \ldots, \left(\frac{y^3}{x(x-1)}\right)^h\right\} \bigcup \left\{x, x\left(\frac{y^3}{x(x-1)}\right), \ldots, x\left(\frac{y^3}{x(x-1)}\right)^{h-6}\right\}.$$
   We deduce that : card $\mathcal{B}_m = h+1+h-5+1 = 2h-3 = m-4 = \dim\mathcal{L}(mP_\infty)$.

3. Demonstration of the Theorem

Let $R \in \mathcal{C}_{4,4}(11)(\bar{\mathbb{Q}})$ with $[\mathbb{Q}(R):\mathbb{Q}] = l$. Notons $R_1, \ldots\ldots, R_l$ Let R and l be the conjugates of R, and work with $t = [R_1 + \ldots\ldots + R_l - lP_\infty]$ which is a point of $\mathcal{J}_{4,4}(11)(\mathbb{Q}) = \{nj(P_0),\ 0 \leq n \leq 10\}$; donc $t = nj(P_0)$ with $0 \leq n \leq 10$, ainsi

$$[R_1 + \ldots\ldots + R_l - lP_\infty] = nj(P_0) \text{ with } 0 \leq n \leq 10 \qquad (i)$$

We discuss according to the values of $n \in \{0, \ldots\ldots, 10\}$:

1. *case : $n = 0$.* Th formula (i) becomes : $[R_1 + \ldots\ldots + R_l - lP_\infty] = 0$.
   According to the Abel-Jacobi theorem there exists then a rational function f defined on $\mathbb{Q}$ such that $div(f) = R_1 + \ldots\ldots + R_l - lP_\infty$, donc $f \in \mathcal{L}(lP_\infty)$.
   According to Lemma 3, we have, :
   $$f = \sum_{i=0}^{\frac{l}{2}} a_i \left(\frac{y^3}{x(x-1)}\right)^i + \sum_{j=0}^{\frac{l-11}{2}} b_j x \left(\frac{y^3}{x(x-1)}\right)^j$$
   with $a_0 \neq 0$ (otherwise one of the R_i should be equal to P_0, which would be absurd), $a_{\frac{l}{2}} \neq 0$ if l is even (otherwise one of R_i would be equal to P_∞, which would be absurd) and $b_{\frac{l-11}{2}} \neq 0$ if l is odd (otherwise one of R_i would be equal to P_∞, which would be absurd).
   At the points R_i, we have :
   $$\sum_{i=0}^{\frac{l}{2}} a_i \left(\frac{y^3}{x(x-1)}\right)^i + \sum_{j=0}^{\frac{l-11}{2}} b_j x \left(\frac{y^3}{x(x-1)}\right)^j = 0$$
   Hence $x = -\dfrac{\sum_{i=0}^{\frac{l}{2}} a_i \left(\frac{y^3}{x(x-1)}\right)^i}{\sum_{j=0}^{\frac{l-11}{2}} b_j \left(\frac{y^3}{x(x-1)}\right)^j}$.

The relation $y^{11} = x^4(x-1)^4$, involves $\frac{y^3}{x(x-1)} = \pm y^{\frac{1}{4}}$. Thus we have :

$$x = -\frac{\sum_{i=0}^{\frac{l}{2}} \alpha_i y^{\frac{i}{4}}}{\sum_{j=0}^{\frac{l-11}{2}} \beta_j y^{\frac{j}{4}}} \text{ where } \begin{cases} \alpha_i = \pm a_i \\ \beta_j = \pm b_j \end{cases}.$$

And as a result :

$$y^{11} = x^4(x-1)^4 \iff y^{11} = \left(-\frac{\sum_{i=0}^{\frac{l}{2}} \alpha_i y^{\frac{i}{4}}}{\sum_{j=0}^{\frac{l-11}{2}} \beta_j y^{\frac{j}{4}}} \right)^4 \left(-\frac{\sum_{i=0}^{\frac{l}{2}} \alpha_i y^{\frac{i}{4}}}{\sum_{j=0}^{\frac{l-11}{2}} \beta_j y^{\frac{j}{4}}} - 1 \right)^4$$

$$\iff y^{11} \left(\sum_{j=0}^{\frac{l-11}{2}} \beta_j y^{\frac{j}{4}} \right)^8 = \left(\sum_{i=0}^{\frac{l}{2}} \alpha_i y^{\frac{i}{4}} \right)^4 \left(\sum_{i=0}^{\frac{l}{2}} \alpha_i y^{\frac{i}{4}} + \sum_{j=0}^{\frac{l-11}{2}} \beta_j y^{\frac{j}{4}} \right)^4$$

The expression :

$$y^{11} \left(\sum_{j=0}^{\frac{l-11}{2}} \beta_j y^{\frac{j}{4}} \right)^8 = \left(\sum_{i=0}^{\frac{l}{2}} \alpha_i y^{\frac{i}{4}} \right)^4 \left(\sum_{i=0}^{\frac{l}{2}} \alpha_i y^{\frac{i}{4}} + \sum_{j=0}^{\frac{l-11}{2}} \beta_j y^{\frac{j}{4}} \right)^4$$

is an equation of degree l in y ; indeed

The first member of the equation is of degree at most equal to $11 + 8\left(\frac{\frac{l-11}{2}}{4}\right) = l$ and the second member of the equation is of degree equal to $2 \times 4\left(\frac{\frac{l}{2}}{4}\right) = l$.

We thus obtain a family of points of degree l :

$$\mathcal{G}_0 = \left\{ \left(-\frac{\sum_{i=0}^{\frac{l}{2}} \alpha_i y^{\frac{i}{4}}}{\sum_{j=0}^{\frac{l-11}{2}} \beta_j y^{\frac{j}{4}}}, y \right) \middle| \begin{array}{c} \alpha_0 \neq 0,\ \alpha_{\frac{l}{2}} \neq 0 \text{ if } l \text{ is even, } \beta_{\frac{l-11}{2}} \neq 0 \text{ if } l \text{ is odd and} \\ y \text{ solution of the equation} \\ y^{11} \left(\sum_{j=0}^{\frac{l-11}{2}} \beta_j y^{\frac{j}{4}} \right)^8 = \left(\sum_{i=0}^{\frac{l}{2}} \alpha_i y^{\frac{i}{4}} \right)^4 \left(\sum_{i=0}^{\frac{l}{2}} \alpha_i y^{\frac{i}{4}} + \sum_{j=0}^{\frac{l-11}{2}} \beta_j y^{\frac{j}{4}} \right)^4 \end{array} \right\}$$

2. *case :* $n \in \{1, \ldots, 10\}$. Formula (i) becomes : $[R_1 + \ldots\ldots + R_l - lP_\infty] = nj(P_0)$, we deduce that $[R_1 + \ldots\ldots + R_l + (11-n)P_0 - (l+11-n)P_\infty] = 0$.
According to the Abel-Jacobi theorem, there exists then a rational function f defined on $\mathbb{Q}$ such that :

$$div(f) = R_1 + \ldots\ldots + R_l + (11-n)P_0 - (l+11-n)P_\infty,$$

so $f \in \mathcal{L}((l+11-n)P_\infty)$.
According to Lemma 3, we have :

$$f = \sum_{i=0}^{\frac{l+11-n}{2}} a_i \left(\frac{y^6}{x(x-1)}\right)^i + \sum_{j=0}^{\frac{l-n}{2}} b_j x \left(\frac{y^6}{x(x-1)}\right)^j$$

with $b_0 \neq 0$ (otherwise one of R_i should be equal to P_0, which would be absurd), $a_{\frac{l}{2}} \neq 0$ if l is even (otherwise one of R_i would be equal to P_∞, which would be absurd) and $b_{\frac{l-n}{2}} \neq 0$ if l is odd (otherwise one of R_i would be equal to P_∞, which would be absurd).
At points R_i, we have :

$$\sum_{i\,=\,11-n}^{\frac{l+11-n}{2}} a_i\left(\frac{y^6}{x(x\,-\,1)}\right)^i + \sum_{j\,=\,0}^{\frac{l-n}{2}} b_j x\left(\frac{y^6}{x(x\,-\,1)}\right)^j \,=\, 0$$

Hence $x \,=\, -\dfrac{\displaystyle\sum_{i\,=\,11-n}^{\frac{l+11-n}{2}} a_i\left(\frac{y^6}{x(x\,-\,1)}\right)^i}{\displaystyle\sum_{j\,=\,0}^{\frac{l-n}{2}} b_j\left(\frac{y^6}{x(x\,-\,1)}\right)^j}$.

In addition $\dfrac{y^6}{x(x\,-\,1)} \,=\, \pm y^{\frac{1}{2}}$, we deduce that :

$$x \,=\, -\frac{\displaystyle\sum_{i\,=\,0}^{\frac{l+11-n}{2}} \alpha'_i y^{\frac{i}{2}}}{\displaystyle\sum_{j\,=\,0}^{\frac{l-n}{2}} \beta'_j y^{\frac{j}{2}}} \text{ where } \begin{cases} \alpha'_i \,=\, \pm a_i \\ \beta'_j \,=\, \pm b_i \end{cases}$$

From the relation $y^{11} \,=\, x^2(x\,-\,1)^2$, we deduce, using the same procedure as the previous case, the following equation in y :

$$y^{11}\left(\sum_{j\,=\,0}^{\frac{l-n}{2}} \beta'_j y^{\frac{j}{2}}\right)^4 \,=\, \left(\sum_{i\,=\,11-n}^{\frac{l+11-n}{2}} \alpha_i y^{\frac{i}{2}}\right)^2\left(\sum_{i\,=\,11-n}^{\frac{l+11-n}{2}} \alpha_i y^{\frac{i}{2}} + \sum_{j\,=\,0}^{\frac{l-n}{2}} \beta_j y^{\frac{j}{2}}\right)^2$$

This equation becomes :

$$y^{n}\left(\sum_{j\,=\,0}^{\frac{l-n}{2}} \beta'_j y^{\frac{j}{2}}\right)^4 \,=\, \left(\sum_{i\,=\,11-n}^{\frac{l+11-n}{2}} \alpha_i y^{\frac{i+n-11}{2}}\right)^2\left(\sum_{i\,=\,11-n}^{\frac{l+11-n}{2}} \alpha_i y^{\frac{i}{2}} + \sum_{j\,=\,0}^{\frac{l-n}{2}} \beta_j y^{\frac{j}{2}}\right)^2$$

which is an equation of degree l. Indeed :

For l is even or odd, the first member of the equation is of degree equal to $n+4\left(\dfrac{\frac{l\,-\,n}{2}}{2}\right) \,=\, l$ and the second member of the equation is of degree equal to $2\times\left(\dfrac{\frac{l\,+11\,-\,n}{2}\,+\,n\,-11}{2}\right) + 2\times\left(\dfrac{\frac{l\,+\,11\,-n}{2}}{2}\right) \,=\, l.$

We thus obtain a family of points of degree l :

$$\mathcal{G}_n \,=\, \left\{\left(-\frac{\displaystyle\sum_{i\,=\,0}^{\frac{l}{2}} \alpha_i y^{\frac{i}{2}}}{\displaystyle\sum_{j\,=\,0}^{\frac{l-11}{2}} \beta_j y^{\frac{j}{2}}},\, y\right) \,\middle|\, \begin{array}{c} \beta_0 \,\neq\, 0,\ \alpha_{\frac{l+11-n}{2}} \,\neq\, 0 \text{ si } l \text{ est pair}, \beta_{\frac{l-n}{2}} \,\neq\, 0 \text{ if } l \text{ is odd and} \\ y \text{ solution of the equation} \\ y^n\left(\displaystyle\sum_{j\,=\,0}^{\frac{l-n}{2}} \beta'_j y^{\frac{j}{2}}\right)^4 = \left(\displaystyle\sum_{i\,=\,11-n}^{\frac{l+11-n}{2}} \alpha_i y^{\frac{i+n-11}{2}}\right)^2\left(\displaystyle\sum_{i\,=\,11-n}^{\frac{l+11-n}{2}} \alpha_i y^{\frac{i}{2}} + \displaystyle\sum_{j\,=\,0}^{\frac{l-n}{2}} \beta_j y^{\frac{j}{2}}\right)^2 \end{array}\right\}$$

4. Conclusion

Our note focuses on the determination of algebraic points on the curve $\mathcal{C}_{4,4}(11)$ of affine equation $y^{11} \,=\, x^4(x\,-\,1)^4$. The curve $\mathcal{C}_{4,4}(11)$ is a special case of the quotients of Fermat curves.

We have determined the algebraic points of any given degree on the given curve. To be able to do it one of the bases was determined and then the determination of given degree l of algebraic points. The determination of the algebraic points of degree exactly l on the given curve remains to be studied.

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