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Integral Representations of a Function in the S. L. Sobolev Space and Their Application to Boundary Problems

Ilgar Gurbat Mamedov[1, *], Aynura Jabbar Abdullayeva[2]

[1]Laboratory of Mathematical Analysis of Control in Physical, Biological and Economic Processes, Institute of Control Systems of the Ministry of Science and Education of Republic of Azerbaijan, Baku, Azerbaijan

[2]Military-Scientific Research Institute, Azerbaijan National Defense University, Baku, Azerbaijan

Email address:

ilgar-mamedov-1971@mail.ru (Ilgar Gurbat Mamedov), aynure-huseynova-2015@mail.ru (Aynura Jabbar Abdullayeva)

[*]Corresponding author

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Abstract: First, we prove a theorem on the integral representation of functions of three variables at the middle of a domain in S. L. Sobolev space with a dominant mixed derivative on a three-dimensional parallelepiped. Further, an integral representation of periodic functions of three variables is given at the middle of the domain in the space of S. L. Sobolev with a dominant mixed derivative. A theorem is also given on the integral representation of homogeneous functions of three variables at the middle of a domain in S. L. Sobolev with a dominant mixed derivative. In addition, a theorem is given on the integral representation of odd functions of three variables at the middle of a domain in S. L. Sobolev with a dominant mixed derivative. Next, we present a theorem on the integral representation of even functions of three variables at the middle of a domain in S. L. Sobolev with a dominant mixed derivative. The above theorems are directly applicable to the qualitative theory of differential equations. In this article, in the most general form, an integral representation of functions of several variables at the middle of a domain in S. L. Sobolev with a dominant mixed derivative on a multidimensional parallelepiped. In this article, such an integral representation of functions in Sobolev space is used to study a boundary value problem in the middle of a domain for the Bianchi integro-differential equation, which is a class of dominating mixed differential equations. For the Bianchi integro-differential equation, the boundary value problem in the middle of the domain in the classical form is reduced to a nonclassical boundary value problem. In this setting, no additional conditions such as matching are required. Then the non-classical boundary value problem posed in the middle of the region is reduced to an operator equation. With the method of integral representations of functions for the boundary value problem, an equivalent integral equation is constructed. Using this integral equation, we prove the homeomorphism theorem. By definition, this theorem is demonstrated by the correct solvability of the considered boundary value problem in the middle of the domain.

Keywords: Integral Representation of Functions, S. L. Sobolev Space, Periodic Function, Homogeneous Function, Odd Function, Even Function, Function with Many Variables, Boundary Value Problem

1. Introduction

Currently, the teaching part of the differential and integral calculus of mathematics is called mathematical analysis. The systematic doctrine of differential calculus was developed by the German mathematician and philosopher G. Leibniz (1646-1716) and the English mathematician and founder of modern mathematical science I. Newton (1643-1727).

The term "function" was first used by Leibniz. But for a long time, this name was understood only as functions given by any analytic expression. In Euler's time, a function given by different equations in different parts of an interval was not considered a "real" function. Already in 1822, the French mathematician Fourier actually used the most general concept of a function in his research, but did not clearly define this concept. The modern definition of a numerical function was given independently by the Russian mathematician N. I. Lobachevsky in 1834 and the German mathematician L. Dirichlet in 1837. The main idea of both sides is that: in the concept of a function, it does not matter by what rule a certain value $f(x)$ is assigned to each number x, only the definition of that correspondence is important. The definition of a function with an arbitrary domain of definition and an arbitrary set of values (which may not be numeric), as well as modern terminology and notation, were actually given as recently as the first half of the 20th century.

In the mathematical literature, integral representations of functions from type spaces (from Sobolev spaces with dominant mixed derivatives of a general form) are studied in the works of T. I. Amanov [1], S. S. Akhiev [2], O. V. Besov, V. P. Ilyin and S. M. Nikol'skii [3], A. J. Jabrailov [4], P. I. Lizorkin and S. M. Nikol'skii [5], I. G. Mamedov [8], A. M. Najafov [9], S. M. Nikol'skii [10] and others. In this paper, one integral representation of a function in the space of S. L. Sobolev is found. This formula in a sense generalizes the Newton-Leibniz formula. The result obtained can be applied in problems of the qualitative theory of partial differential equations. Such an integral representation for functions of several variables in S. L. Sobolev spaces was applied in a boundary value problem for one class of 3D Bianchi integro-differential equations.

2. Formulation of the Problem

Let $G = G_1 \times G_2 \times G_3$, where $G_1 = (x_0, x_1)$, $G_2 = (y_0, y_1)$, $G_3 = (z_0, z_1)$; Consider the spaces of S. L. Sobolev $W_p^{(1,1,1)}(G) = \{ u \in L_p(G) / D_x^i\, D_y^j\, D_z^k u \in L_p(G);\ i,\ j, k = 0,1\}$, where $1 \le p \le \infty$. Norm in space $W_p^{(1,1,1)}(G)$ we will define the equality $\|u\|_{W_p^{(1,1,1)}(G)} = \sum_{i,j,k=0}^{1} \|D_x^i D_y^j D_z^k u\|_{L_p(G)}$.

2.1. Integral Representation of Three-Dimensional Functions in Sobolev Space with a Dominant Mixed Derivative

Theorem 1. If $u(x, y, z) \in W_p^{(1,1,1)}(G)$, then the function $u(x, y, z)$ can be represented as

$$u(x,y,z) = u\left(\frac{x_0+x_1}{2}, \frac{y_0+y_1}{2}, \frac{z_0+z_1}{2}\right) + \int_{\frac{x_0+x_1}{2}}^{x} u_x\left(\tau, \frac{y_0+y_1}{2}, \frac{z_0+z_1}{2}\right) d\tau + \int_{\frac{y_0+y_1}{2}}^{y} u_y\left(\frac{x_0+x_1}{2}, \xi, \frac{z_0+z_1}{2}\right) d\xi +$$

$$+ \int_{\frac{z_0+z_1}{2}}^{z} u_z\left(\frac{x_0+x_1}{2}, \frac{y_0+y_1}{2}, \eta\right) d\eta + \int_{\frac{x_0+x_1}{2}}^{x} \int_{\frac{y_0+y_1}{2}}^{y} u_{xy}\left(\tau, \xi, \frac{z_0+z_1}{2}\right) d\tau d\xi + \int_{\frac{y_0+y_1}{2}}^{y} \int_{\frac{z_0+z_1}{2}}^{z} u_{yz}\left(\frac{x_0+x_1}{2}, \xi, \eta\right) d\xi d\eta +$$

$$+ \int_{\frac{x_0+x_1}{2}}^{x} \int_{\frac{z_0+z_1}{2}}^{z} u_{xz}\left(\tau, \frac{y_0+y_1}{2}, \eta\right) d\tau d\eta + \int_{\frac{x_0+x_1}{2}}^{x} \int_{\frac{y_0+y_1}{2}}^{y} \int_{\frac{z_0+z_1}{2}}^{z} u_{xyz}(\tau, \xi, \eta) d\tau d\xi d\eta$$

Proof. From the Newton-Leibniz formula it is clear that

$$u\left(x, \frac{y_0+y_1}{2}, \frac{z_0+z_1}{2}\right) = \int_{\frac{x_0+x_1}{2}}^{x} u_x\left(\tau, \frac{y_0+y_1}{2}, \frac{z_0+z_1}{2}\right) d\tau + u\left(\frac{x_0+x_1}{2}, \frac{y_0+y_1}{2}, \frac{z_0+z_1}{2}\right) \tag{1}$$

$$u\left(\frac{x_0+x_1}{2}, y, \frac{z_0+z_1}{2}\right) = \int_{\frac{y_0+y_1}{2}}^{y} u_y\left(\frac{x_0+x_1}{2}, \xi, \frac{z_0+z_1}{2}\right) d\xi + u\left(\frac{x_0+x_1}{2}, \frac{y_0+y_1}{2}, \frac{z_0+z_1}{2}\right), \tag{2}$$

$$u\left(\frac{x_0+x_1}{2}, \frac{y_0+y_1}{2}, z\right) = \int_{\frac{z_0+z_1}{2}}^{z} u_z\left(\frac{x_0+x_1}{2}, \frac{y_0+y_1}{2}, \eta\right) d\eta + u\left(\frac{x_0+x_1}{2}, \frac{y_0+y_1}{2}, \frac{z_0+z_1}{2}\right), \tag{3}$$

Now we calculate the following double integral:

$$\int_{\frac{x_0+x_1}{2}}^{x} \int_{\frac{y_0+y_1}{2}}^{y} u_{xy}\left(\tau, \xi, \frac{z_0+z_1}{2}\right) d\tau d\xi = \int_{\frac{x_0+x_1}{2}}^{x} \left[u_x\left(\tau, y, \frac{z_0+z_1}{2}\right) - u_x\left(\tau, \frac{y_0+y_1}{2}, \frac{z_0+z_1}{2}\right) \right] d\tau =$$

$$=u(x,y,\frac{z_0+z_1}{2})-u(\frac{x_0+x_1}{2},y,\frac{z_0+z_1}{2})-u(x,\frac{y_0+y_1}{2},\frac{z_0+z_1}{2})+u(\frac{x_0+x_1}{2},\frac{y_0+y_1}{2},\frac{z_0+z_1}{2}).$$

Hence we get that

$$u(x,y,\frac{z_0+z_1}{2})=\int\limits_{\frac{x_0+x_1}{2}}^{x}\int\limits_{\frac{y_0+y_1}{2}}^{y}u_{xy}(\tau,\xi,\frac{z_0+z_1}{2})d\tau d\xi+u(x,\frac{y_0+y_1}{2},\frac{z_0+z_1}{2})+u(\frac{x_0+x_1}{2},y,\frac{z_0+z_1}{2})-$$
$$-u(\frac{x_0+x_1}{2},\frac{y_0+y_1}{2},\frac{z_0+z_1}{2}) \quad (4)$$

Similarly, we find:

$$u(\frac{x_0+x_1}{2},y,z)=\int\limits_{\frac{y_0+y_1}{2}}^{y}\int\limits_{\frac{z_0+z_1}{2}}^{z}u_{yz}(\frac{x_0+x_1}{2},\xi,\eta)d\xi d\eta+u(\frac{x_0+x_1}{2},y,\frac{z_0+z_1}{2})+u(\frac{x_0+x_1}{2},\frac{y_0+y_1}{2},z)-$$
$$-u(\frac{x_0+x_1}{2},\frac{y_0+y_1}{2},\frac{z_0+z_1}{2}), \quad (5)$$

$$u(x,\frac{y_0+y_1}{2},z)=\int\limits_{\frac{x_0+x_1}{2}}^{x}\int\limits_{\frac{z_0+z_1}{2}}^{z}u_{xz}(\tau,\frac{y_0+y_1}{2},\eta)d\tau d\eta+u(x,\frac{y_0+y_1}{2},\frac{z_0+z_1}{2})+u(\frac{x_0+x_1}{2},\frac{y_0+y_1}{2},z)-$$
$$-u(\frac{x_0+x_1}{2},\frac{y_0+y_1}{2},\frac{z_0+z_1}{2}). \quad (6)$$

Calculating the integral

$\int\limits_{\frac{x_0+x_1}{2}}^{x}\int\limits_{\frac{y_0+y_1}{2}}^{y}\int\limits_{\frac{z_0+z_1}{2}}^{z}u_{xyz}(\tau,\xi,\eta)d\tau d\xi d\eta$ and given the expressions (1)-(6), we get:

$$\int\limits_{\frac{x_0+x_1}{2}}^{x}\int\limits_{\frac{y_0+y_1}{2}}^{y}\int\limits_{\frac{z_0+z_1}{2}}^{z}u_{xyz}(\tau,\xi,\eta)d\tau d\xi d\eta=\int\limits_{\frac{x_0+x_1}{2}}^{x}\int\limits_{\frac{y_0+y_1}{2}}^{y}\left[u_{xy}(\tau,\xi,z)-u_{xy}(\tau,\xi,\frac{z_0+z_1}{2})\right]d\tau d\xi=$$
$$=\int\limits_{\frac{x_0+x_1}{2}}^{x}\left[u_x(\tau,y,z)-u_x(\tau,\frac{y_0+y_1}{2},z)-u_x(\tau,y,\frac{z_0+z_1}{2})+u_x(\tau,\frac{y_0+y_1}{2},\frac{z_0+z_1}{2})\right]d\tau=$$
$$=u(x,y,z)-u(\frac{x_0+x_1}{2},y,z)-u(x,\frac{y_0+y_1}{2},z)+u(\frac{x_0+x_1}{2},\frac{y_0+y_1}{2},z)-u(x,y,\frac{z_0+z_1}{2})+$$
$$+u(\frac{x_0+x_1}{2},y,\frac{z_0+z_1}{2})+u(x,\frac{y_0+y_1}{2},\frac{z_0+z_1}{2})-u(\frac{x_0+x_1}{2},\frac{y_0+y_1}{2},\frac{z_0+z_1}{2})=$$
$$=u(x,y,z)-u(\frac{x_0+x_1}{2},\frac{y_0+y_1}{2},\frac{z_0+z_1}{2})+u(x,\frac{y_0+y_1}{2},\frac{z_0+z_1}{2})+u(\frac{x_0+x_1}{2},y,\frac{z_0+z_1}{2})+$$

$$+u(\frac{x_0+x_1}{2},\frac{y_0+y_1}{2},z)-\int\limits_{\frac{x_0+x_1}{2}}^{x}\int\limits_{\frac{y_0+y_1}{2}}^{y}u_{xy}(\tau,\xi,\frac{z_0+z_1}{2})d\tau d\xi-u(\frac{x_0+x_1}{2},y,\frac{z_0+z_1}{2})-$$
$$-u(x,\frac{y_0+y_1}{2},\frac{z_0+z_1}{2})+u(\frac{x_0+x_1}{2},\frac{y_0+y_1}{2},\frac{z_0+z_1}{2})-\int\limits_{\frac{y_0+y_1}{2}}^{y}\int\limits_{\frac{z_0+z_1}{2}}^{z}u_{yz}(\frac{x_0+x_1}{2},\xi,\eta)d\xi d\eta-$$

$$-u(\frac{x_0+x_1}{2},y,\frac{z_0+z_1}{2})-u(\frac{x_0+x_1}{2},\frac{y_0+y_1}{2},z)+u(\frac{x_0+x_1}{2},\frac{y_0+y_1}{2},\frac{z_0+z_1}{2})-$$
$$-\int\limits_{\frac{x_0+x_1}{2}}^{x}\int\limits_{\frac{z_0+z_1}{2}}^{z}u_{xz}(\tau,\frac{y_0+y_1}{2},\eta)d\tau d\eta-u(x,\frac{y_0+y_1}{2},\frac{z_0+z_1}{2})-u(\frac{x_0+x_1}{2},\frac{y_0+y_1}{2},z)+$$
$$+u(\frac{x_0+x_1}{2},\frac{y_0+y_1}{2},\frac{z_0+z_1}{2})=u(x,y,z)+2u(\frac{x_0+x_1}{2},\frac{y_0+y_1}{2},\frac{z_0+z_1}{2})-$$
$$-u(x,\frac{y_0+y_1}{2},\frac{z_0+z_1}{2})-u(\frac{x_0+x_1}{2},y,\frac{z_0+z_1}{2})-u(\frac{x_0+x_1}{2},\frac{y_0+y_1}{2},z)-$$
$$-\int\limits_{\frac{x_0+x_1}{2}}^{x}\int\limits_{\frac{y_0+y_1}{2}}^{y}u_{xy}(\tau,\xi,\frac{z_0+z_1}{2})d\tau d\xi-$$
$$-\int\limits_{\frac{y_0+y_1}{2}}^{y}\int\limits_{\frac{z_0+z_1}{2}}^{z}u_{yz}(\frac{x_0+x_1}{2},\xi,\eta)d\xi d\eta-\int\limits_{\frac{x_0+x_1}{2}}^{x}\int\limits_{\frac{z_0+z_1}{2}}^{z}u_{xz}(\tau,\frac{y_0+y_1}{2},\eta)d\tau d\eta=$$

$$=u(x,y,z)-\int\limits_{\frac{x_0+x_1}{2}}^{x}\int\limits_{\frac{y_0+y_1}{2}}^{y}u_{xy}(\tau,\xi,\frac{z_0+z_1}{2})d\tau d\xi-\int\limits_{\frac{x_0+x_1}{2}}^{x}\int\limits_{\frac{z_0+z_1}{2}}^{z}u_{xz}(\tau,\frac{y_0+y_1}{2},\eta)d\tau d\eta-$$

$$-\int\limits_{\frac{y_0+y_1}{2}}^{y}\int\limits_{\frac{z_0+z_1}{2}}^{z}u_{yz}(\frac{x_0+x_1}{2},\xi,\eta)d\xi d\eta+2u(\frac{x_0+x_1}{2},\frac{y_0+y_1}{2},\frac{z_0+z_1}{2})-$$

$$-\int\limits_{\frac{x_0+x_1}{2}}^{x}u_{x}(\tau,\frac{y_0+y_1}{2},\frac{z_0+z_1}{2})d\tau-u(\frac{x_0+x_1}{2},\frac{y_0+y_1}{2},\frac{z_0+z_1}{2})-$$

$$-\int\limits_{\frac{y_0+y_1}{2}}^{y}u_{y}(\frac{x_0+x_1}{2},\xi,\frac{z_0+z_1}{2})d\xi-u(\frac{x_0+x_1}{2},\frac{y_0+y_1}{2},\frac{z_0+z_1}{2})-$$

$$-\int\limits_{\frac{z_0+z_1}{2}}^{z}u_{z}(\frac{x_0+x_1}{2},\frac{y_0+y_1}{2},\eta)d\eta-u(\frac{x_0+x_1}{2},\frac{y_0+y_1}{2},\frac{z_0+z_1}{2})=$$

$$=u(x,y,z)-\int\limits_{\frac{x_0+x_1}{2}}^{x}u_{x}(\tau,\frac{y_0+y_1}{2},\frac{z_0+z_1}{2})d\tau-\int\limits_{\frac{y_0+y_1}{2}}^{y}u_{y}(\frac{x_0+x_1}{2},\xi,\frac{z_0+z_1}{2})d\xi-$$

$$-\int\limits_{\frac{z_0+z_1}{2}}^{z}u_{z}(\frac{x_0+x_1}{2},\frac{y_0+y_1}{2},\eta)d\eta-\int\limits_{\frac{x_0+x_1}{2}}^{x}\int\limits_{\frac{y_0+y_1}{2}}^{y}u_{xy}(\tau,\xi,\frac{z_0+z_1}{2})d\tau d\xi-$$
$$-\int\limits_{\frac{y_0+y_1}{2}}^{y}\int\limits_{\frac{z_0+z_1}{2}}^{z}u_{yz}(\frac{x_0+x_1}{2},\xi,\eta)d\xi d\eta-\int\limits_{\frac{x_0+x_1}{2}}^{x}\int\limits_{\frac{z_0+z_1}{2}}^{z}u_{xz}(\tau,\frac{y_0+y_1}{2},\eta)d\tau d\eta-u(\frac{x_0+x_1}{2},\frac{y_0+y_1}{2},\frac{z_0+z_1}{2}).$$

Hence we have:

$$u(x,y,z)=u(\frac{x_0+x_1}{2},\frac{y_0+y_1}{2},\frac{z_0+z_1}{2})+\int\limits_{\frac{x_0+x_1}{2}}^{x}u_x(\tau,\frac{y_0+y_1}{2},\frac{z_0+z_1}{2})d\tau+\int\limits_{\frac{y_0+y_1}{2}}^{y}u_y(\frac{x_0+x_1}{2},\xi,\frac{z_0+z_1}{2})d\xi+$$
$$+\int\limits_{\frac{z_0+z_1}{2}}^{z}u_z(\frac{x_0+x_1}{2},\frac{y_0+y_1}{2},\eta)d\eta+\int\limits_{\frac{x_0+x_1}{2}}^{x}\int\limits_{\frac{y_0+y_1}{2}}^{y}u_{xy}(\tau,\xi,\frac{z_0+z_1}{2})d\tau d\xi+\int\limits_{\frac{y_0+y_1}{2}}^{y}\int\limits_{\frac{z_0+z_1}{2}}^{z}u_{yz}(\frac{x_0+x_1}{2},\xi,\eta)d\xi d\eta+ \tag{7}$$
$$+\int\limits_{\frac{x_0+x_1}{2}}^{x}\int\limits_{\frac{z_0+z_1}{2}}^{z}u_{xz}(\tau,\frac{y_0+y_1}{2},\eta)d\tau d\eta+\int\limits_{\frac{x_0+x_1}{2}}^{x}\int\limits_{\frac{y_0+y_1}{2}}^{y}\int\limits_{\frac{z_0+z_1}{2}}^{z}u_{xyz}(\tau,\xi,\eta)d\tau d\xi d\eta.$$

The theorem has been proven.

It is obvious that if $G_\xi=(x_\xi^0,h_\xi),\ \xi=\overline{1,3}, G=\prod_{\xi=1}^{3}G_\xi$ and $u(x_1,x_2,x_3)\in W_p^{(1,1,1)}(G)$, then formula (7) can be written in the following compact form

$$u(x_1,x_2,x_3)=u(\frac{x_1^0+h_1}{2},\frac{x_2^0+h_2}{2},\frac{x_3^0+h_3}{2})+\sum_{i=1}^{3}\int\limits_{\frac{x_i^0+h_i}{2}}^{x_i}\left[\frac{\partial u}{\partial x_i}/_{x_j=\frac{x_j^0+h_j}{2},j\neq i,x_i=\alpha_i}\right]d\alpha_i+$$
$$+\sum_{i<j}\int\limits_{\frac{x_i^0+h_i}{2}}^{x_i}\int\limits_{\frac{x_j^0+h_j}{2}}^{x_j}\left[\frac{\partial^2 u}{\partial x_i\partial x_j}/_{x_k=\frac{x_k^0+h_k}{2},k\neq i,j,x_i=\alpha_i,x_j=\alpha_j}\right]d\alpha_i d\alpha_j+\int\limits_{\frac{x_1^0+h_1}{2}}^{x_1}\int\limits_{\frac{x_2^0+h_2}{2}}^{x_2}\int\limits_{\frac{x_3^0+h_3}{2}}^{x_3}\left[\frac{\partial^3 u}{\prod_{\xi=1}^{3}\partial x_\xi}/_{x_{\xi=\alpha_\xi,\xi=\overline{1,3}}}\right]\prod_{\xi=1}^{3}d\alpha_\xi.$$

2.2. Integral Representation of Periodic Functions in Sobolev Space with Dominant Mixed Derivative

Theorem 2. If $u(x_1,x_2,x_3)\in W_p^{(1,1,1)}(G)$ and, besides, $u(x_1+T_1,x_2+T_2,x_3+T_3)=u(x_1,x_2,x_3)$, those. function u is periodic with period T_i relative to variable x_i, $i=\overline{1,3}$, then function $u(x_1,x_2,x_3)$ can be presented in the form

$$u(x_1,x_2,x_3)=u(\frac{x_1^0+h_1}{2},\frac{x_2^0+h_2}{2},\frac{x_3^0+h_3}{2})+\sum_{i=1}^{3}\int\limits_{\frac{x_i^0+h_i}{2}}^{x_i+T_i}\left[\frac{\partial u}{\partial x_i}/_{x_j=\frac{x_j^0+h_j}{2},j\neq i,x_i=\alpha_i}\right]d\alpha_i+$$
$$+\sum_{i<j}\int\limits_{\frac{x_i^0+h_i}{2}}^{x_i+T_i}\int\limits_{\frac{x_j^0+h_j}{2}}^{x_{j+T_j}}\left[\frac{\partial^2 u}{\partial x_i\partial x_j}/_{x_k=\frac{x_k^0+h_k}{2},k\neq i,j,x_i=\alpha_i,x_j=\alpha_j}\right]d\alpha_i d\alpha_j+\int\limits_{\frac{x_1^0+h_1}{2}}^{x_1+T_1}\int\limits_{\frac{x_2^0+h_2}{2}}^{x_2+T_2}\int\limits_{\frac{x_3^0+h_3}{2}}^{x_3+T_3}\left[\frac{\partial^3 u}{\prod_{\xi=1}^{3}\partial x_\xi}/_{x_{\xi=\alpha_\xi,\xi=\overline{1,3}}}\right]\prod_{\xi=1}^{3}d\alpha_\xi.$$

2.3. Integral Representation of Homogeneous Functions in Sobolev Space with Dominant Mixed Derivative

Theorem 3. If $u(x_1,x_2,x_3)\in W_p^{(1,1,1)}(G)$ and, besides, $u(tx_1,tx_2,tx_3)=t^\alpha u(x_1,x_2,x_3)$ for $\forall t\in R$, , those. function u is homogeneous with order α relative to variable x_i, $i=\overline{1,3}$, then function $u(x_1,x_2,x_3)$ can be presented in the form

$$u(x_1,x_2,x_3)=\frac{1}{t^\alpha}$$

$$\left[\begin{array}{l} u(\frac{x_1^0+h_1}{2},\frac{x_2^0+h_2}{2},\frac{x_3^0+h_3}{2})+\sum_{i=1}^{3}\int\limits_{\frac{x_i^0+h_i}{2}}^{tx_i}\left[\frac{\partial u}{\partial x_i}/_{x_j=\frac{x_j^0+h_j}{2},j\neq i,x_{i=\alpha_i}}\right]d\alpha_i+ \\ +\sum_{i<j}\int\limits_{\frac{x_i^0+h_i}{2}}^{tx_i}\int\limits_{\frac{x_j^0+h_j}{2}}^{tx_j}\left[\frac{\partial^2 u}{\partial x_i\partial x_j}/_{x_k=\frac{x_k^0+h_k}{2},k\neq i,j,x_i=\alpha_i,x_j=\alpha_j}\right]d\alpha_i d\alpha_j+ \\ +\int\limits_{\frac{x_1^0+h_1}{2}}^{tx_1}\int\limits_{\frac{x_2^0+h_2}{2}}^{tx_2}\int\limits_{\frac{x_3^0+h_3}{2}}^{tx_3}\left[\frac{\partial^3 u}{\prod_{\xi=1}^{3}\partial x_\xi}/_{x_\xi=\alpha_\xi,\xi=\overline{1,3}}\right]\prod_{\xi=1}^{3}d\alpha_\xi . \end{array}\right]$$

2.4. Integral Representation of Odd Functions in Sobolev Space with Dominating Mixed Derivative

Now suppose that the area G symmetrical. Then it is obvious that the origin of coordinates is included in G.

Theorem 4. If $u(x_1,x_2,x_3)\in W_p^{(1,1,1)}(G)$ and, besides, $u(-x_1,-x_2,-x_3)=-u(x_1,x_2,x_3)$, those. function odd in each variable, then function $u(x_1,x_2,x_3)$ can be presented in the form

$$u(x_1,x_2,x_3)=u(\frac{x_1^0+h_1}{2},\frac{x_2^0+h_2}{2},\frac{x_3^0+h_3}{2})-\sum_{i=1}^{3}\int\limits_{\frac{x_i^0+h_i}{2}}^{-x_i}\left[\frac{\partial u}{\partial x_i}/_{x_j=\frac{x_j^0+h_j}{2},j\neq i,x_{i=\alpha_i}}\right]d\alpha_i-$$

$$-\sum_{i<j}\int\limits_{\frac{x_i^0+h_i}{2}}^{-x_i}\int\limits_{\frac{x_j^0+h_j}{2}}^{-x_j}\left[\frac{\partial^2 u}{\partial x_i\partial x_j}/_{x_k=\frac{x_k^0+h_k}{2},k\neq i,j,x_i=\alpha_i,x_j=\alpha_j}\right]d\alpha_i d\alpha_j-\int\limits_{\frac{x_1^0+h_1}{2}}^{-x_1}\int\limits_{\frac{x_2^0+h_2}{2}}^{-x_2}\int\limits_{\frac{x_3^0+h_3}{2}}^{-x_3}\left[\frac{\partial^3 u}{\prod_{\xi=1}^{3}\partial x_\xi}/_{x_\xi=\alpha_\xi,\xi=\overline{1,3}}\right]\prod_{\xi=1}^{3}d\alpha_\xi .$$

2.5. Integral Representation of Even Functions in Sobolev Space with Dominating Mixed Derivative

Theorem 5. If $u(x_1,x_2,x_3)\in W_p^{(1,1,1)}(G)$ and, $u(-x_1,-x_2,-x_3)=u(x_1,x_2,x_3)$, those. function even in each variable, then the function $u(x_1,x_2,x_3)$ can be presented in the form

$$u(x_1,x_2,x_3)=u(\frac{x_1^0+h_1}{2},\frac{x_2^0+h_2}{2},\frac{x_3^0+h_3}{2})+\sum_{i=1}^{3}\int\limits_{\frac{x_i^0+h_i}{2}}^{-x_i}\left[\frac{\partial u}{\partial x_i}/_{x_j=\frac{x_j^0+h_j}{2},j\neq i,x_{i=\alpha_i}}\right]d\alpha_i+$$

$$+\sum_{i<j}\int\limits_{\frac{x_i^0+h_i}{2}}^{-x_i}\int\limits_{\frac{x_j^0+h_j}{2}}^{-x_j}\left[\frac{\partial^2 u}{\partial x_i\partial x_j}/_{x_k=\frac{x_k^0+h_k}{2},k\neq i,j,x_i=\alpha_i,x_j=\alpha_j}\right]d\alpha_i d\alpha_j+\int\limits_{\frac{x_1^0+h_1}{2}}^{-x_1}\int\limits_{\frac{x_2^0+h_2}{2}}^{-x_2}\int\limits_{\frac{x_3^0+h_3}{2}}^{-x_3}\left[\frac{\partial^3 u}{\prod_{\xi=1}^{3}\partial x_\xi}/_{x_\xi=\alpha_\xi,\xi=\overline{1,3}}\right]\prod_{\xi=1}^{3}d\alpha_\xi .$$

2.6. Integral Representation of Functions of Several Variables in Sobolev Space with Dominating Mixed Derivative

Now suppose that $G_i=(x_i^0,h_i),i=\overline{1,n},\ G=\prod_{i=1}^{n}G_i$.

Theorem 6. If $u(x_1,x_2,...,x_n)\in W_p^{(1,1,...,1)}(G)$, then function $u(x_1,x_2,...,x_n)$ can be presented in the form

$$
\begin{aligned}
u(x_1,x_2,...,x_n) &= u\left(\frac{x_1^0+h_1}{2},\frac{x_2^0+h_2}{2},...,\frac{x_n^0+h_n}{2}\right)+\sum_{i=1}^{n}\int_{\frac{x_i^0+h_i}{2}}^{x_i}\left[\frac{\partial u}{\partial x_i}/_{x_j=\frac{x_j^0+h_j}{2},j\neq i,x_{i=\alpha_i}}\right]d\alpha+\\
&+\sum_{i<j}\int_{\frac{x_i^0+h_i}{2}}^{x_i}\int_{\frac{x_j^0+h_j}{2}}^{x_j}\left[\frac{\partial^2 u}{\partial x_i\partial x_j}/_{x_k=\frac{x_k^0+h_k}{2},k\neq i,j,x_i=\alpha_i,x_j=\alpha_j}\right]d\alpha_i d\alpha_j+\\
&+\sum_{i_1<i_2<i_3}\int_{\frac{x_{i_1}^0+h_{i_1}}{2}}^{x_{i_1}}\int_{\frac{x_{i_2}^0+h_{i_2}}{2}}^{x_{i_2}}\int_{\frac{x_{i_3}^0+h_{i_3}}{2}}^{x_{i_3}}\left[\frac{\partial^3 u}{\prod_{\xi=1}^{3}\partial x_{i_\xi}}/_{x_j=\frac{x_j^0+h_j}{2},j\neq i_\xi,\xi=\overline{1,3},x_{i_\xi}=\alpha_{i_\xi}}\right]\prod_{\xi=1}^{3}d\alpha_{i_\xi}+...+\\
&+\sum_{i_1<i_2<...<i_{n-1}}\int_{\frac{x_{i_1}^0+h_{i_1}}{2}}^{x_{i_1}}\int_{\frac{x_{i_2}^0+h_{i_2}}{2}}^{x_{i_2}}...\int_{\frac{x_{i_{n-1}}^0+h_{i_{n-1}}}{2}}^{x_{i_{n-1}}}\left[\frac{\partial^{n-1} u}{\prod_{\xi=1}^{n-1}\partial x_{i_\xi}}/_{x_j=\frac{x_j^0+h_j}{2},j\neq i_\xi,\xi=\overline{1,n-1},x_{i_\xi}=\alpha_{i_\xi}}\right]\prod_{\xi=1}^{n-1}d\alpha_{i_\xi}+\\
&+\int_{\frac{x_1^0+h_1}{2}}^{x_1}\int_{\frac{x_2^0+h_2}{2}}^{x_2}...\int_{\frac{x_n^0+h_n}{2}}^{x_n}\left[\frac{\partial^n u}{\prod_{\xi=1}^{n}\partial x_\xi}/_{,x_\xi=\alpha_\xi}\right]\prod_{\xi=1}^{n}d\alpha_\xi.
\end{aligned}
\tag{8}
$$

3. Integral Representations of a Function in the S. L. Sobolev Space and Their Application

Formula (8) has a general form. Note that with the help of this formula, the classical boundary value problem in the middle of the domain is reduced to a new type of boundary value problem [6, 7].

3.1. Boundary-Value Problem in the Non-Classical Interpretation of a Domain Given in the Middle

Consider the 3D Bianchi integro-differential equation

$$
\begin{aligned}
&(V_{1,1,1}u)(x,y,z)\equiv u_{xyz}(x,y,z)+A_{0,0,0}u(x,y,z)+A_{1,0,0}u_x(x,y,z)+\\
&+A_{0,1,0}u_y(x,y,z)+A_{0,0,1}u_z(x,y,z)+A_{1,1,0}u_{xy}(x,y,z)+A_{0,1,1}u_{yz}(x,y,z)+\\
&+A_{1,0,1}u_{xz}(x,y,z)+\int_{\frac{x_0+x_1}{2}}^{x}\int_{\frac{y_0+y_1}{2}}^{y}\int_{\frac{z_0+z_1}{2}}^{z}\Big[K_{0,0,0}(\tau,\xi,\eta;x,y,z)u(\tau,\xi,\eta)+K_{1,0,0}(\tau,\xi,\eta;x,y,z)\times\\
&\times u_x(\tau,\xi,\eta)+K_{0,1,0}(\tau,\xi,\eta;x,y,z)u_y(\tau,\xi,\eta)+K_{0,0,1}(\tau,\xi,\eta;x,y,z)\times\\
&\times u_z(\tau,\xi,\eta)+K_{1,1,0}(\tau,\xi,\eta;x,y,z)u_{xy}(\tau,\xi,\eta+K_{0,1,1}(\tau,\xi,\eta;x,y,z)\times\\
&\times u_{yz}(\tau,\xi,\eta)+K_{1,0,1}(\tau,\xi,\eta;x,y,z)u_{xz}(\tau,\xi,\eta)\Big]d\tau d\xi d\eta=\phi_{1,1,1}(x,y,z),\quad (x,y,z)\in G
\end{aligned}
\tag{9}
$$

Here $u=u(x,y,z)$ desired function defined on G; $A_{i,j,k}=A_{i,j,k}(x,y,z)$ given measurable functions on $G=G_1\times G_2\times G_3$, где $G_1=(x_0,x_1)$, $G_2=(y_0,y_1)$, $G_3=(z_0,z_1)$; $\phi_{1,1,1}(x,y,z)$ given measurable function on G.

Equation (9) is a hyperbolic equation that has three real simple characteristics. This equation arises, for example, when studying the issues of fluid filtration in media with porosity, moisture transfer in soils, propagation of pulsed ray waves, modeling various biological processes of phenomena, as well as in the theory of inverse problems.

In addition, the Riemann function of Eq. (9) has been constructed in the literature see for example [11-19], so far only for the case when $K_{i,j,k}(\tau,\xi,\eta;x,y,z)$ and functions $A_{i,j,k}(x,y,z)$ are sufficiently smooth /i. e. when functions $A_{i,j,k}(x,y,z)$ continuous with derivatives $D_x^i\,D_y^j\,D_z^k\,A_{i,j,k}(x,y,z)$ in area $\overline{G}$/.

The Dirichlet and Neumann problems for partial differential equations with nonsmooth coefficients are studied in the works [20-22].

In this paper, equation (9) is studied for the first time in the general case when the coefficients $A_{i,j,k}(x,y,z)$ and $K_{i,j,k}(\tau,\xi,\eta;x,y,z)$ are nonsmooth functions satisfying only the following conditions:

$$A_{0,0,0}(x,y,z)\in L_p(G),$$
$$A_{1,0,0}(x,y,z)\in L^{x,y,z}_{\infty,p,p}(G),$$
$$A_{0,1,0}(x,y,z)\in L^{x,y,z}_{p,\infty,p}(G),$$
$$A_{0,0,1}(x,y,z)\in L^{x,y,z}_{p,p,\infty}(G),$$
$$A_{1,1,0}(x,y,z)\in L^{x,y,z}_{\infty,\infty,p}(G),$$
$$A_{0,1,1}(x,y,z)\in L^{x,y,z}_{p,\infty,\infty}(G),$$
$$A_{1,0,1}(x,y,z)\in L^{x,y,z}_{\infty,p,\infty}(G),$$
$$K_{i,j,k}(\tau,\xi,\eta;x,y,z).$$

Under these conditions, the solution $u(x,y,z)$ we will look for equations (9) in the space of S. L. Sobolev

$$W_p^{(1,1,1)}(G)=\left\{\,u\in L_p(G)\,/\,D_x^i\,D_y^j\,D_z^k u\in L_p(G);\ i,\ j,k=0,1\right\},$$

where $1\le p\le\infty$. Norm in space $W_p^{(1,1,1)}(G)$ we will define the equality

$$\|u\|_{W_p^{(1,1,1)}(G)}=\sum_{i,j,k=0}^{1}\left\|D_x^i D_y^j D_z^k u\right\|_{L_p(G)}$$

For equation (9), the conditions at the middle of the domain of the classical form can be specified in the form

$$\begin{cases} u\,/_{x=\frac{x_0+x_1}{2}}=\Phi(y,z),\\ u\,/_{y=\frac{y_0+y_1}{2}}=\Psi(x,z),\\ u\,/_{z=\frac{z_0+z_1}{2}}=g(x,y),\end{cases}\tag{10}$$

where $\Phi(y,z)$, $\Psi(x,z)$ and $g(x,y)$ given measurable functions on G. Obviously, in the case of conditions (10) the functions Φ, Ψ, g, in addition to the conditions

$$\Phi\in W_p^{(1,1)}(G_2\times G_3),\quad \Psi\in W_p^{(1,1)}(G_1\times G_3),\quad g\in W_p^{(1,1)}(G_1\times G_2),$$

must also satisfy the following conditions:

$$\begin{cases}\Phi(\frac{y_0+y_1}{2},z)=\Psi(\frac{x_0+x_1}{2},z),\\ \Phi(y\ ,\frac{z_0+z_1}{2})=g(\frac{x_0+x_1}{2},y),\\ \Psi(x\ ,\frac{z_0+z_1}{2})=g(x\ ,\frac{y_0+y_1}{2}),\end{cases}\tag{11}$$

which are terms of agreement.

The presence of agreement conditions in the formulation of problem (9), (10) means that conditions (11) also specify some redundant information about the solution of this problem. Therefore, the question arises of finding boundary conditions that do not contain redundant information about the solution and do not require the fulfillment of some additional conditions such as agreement.

In this regard, consider the following boundary conditions

$$\begin{cases} V_{0,0,0}u\equiv u(\frac{x_0+x_1}{2},\frac{y_0+y_1}{2},\frac{z_0+z_1}{2})=\phi_{0,0,0}\\ (V_{1,0,0}u)(x)\equiv u_x(x,\frac{y_0+y_1}{2},\frac{z_0+z_1}{2})=\phi_{1,0,0}(x),\\ (V_{0,1,0}u)(y)\equiv u_y(\frac{x_0+x_1}{2},y,\frac{z_0+z_1}{2})=\phi_{0,1,0}(y),\\ (V_{0,0,1}u)(z)\equiv u_z(\frac{x_0+x_1}{2},\frac{y_0+y_1}{2},z)=\phi_{0,0,1}(z),\\ (V_{1,1,0}u)(x,y)\equiv u_{xy}(x,y,\frac{z_0+z_1}{2})=\phi_{1,1,0}(x,y),\\ (V_{0,1,1}u)(y,z)\equiv u_{yz}(\frac{x_0+x_1}{2},y,z)=\phi_{0,1,1}(y,z),\\ (V_{1,0,1}u)(x,z)\equiv u_{xz}(x,\frac{y_0+y_1}{2},z)=\phi_{1,0,1}(x,z),\end{cases}\tag{12}$$

where $\phi_{0,0,0}\in R$ is a given number, and the rest $\phi_{i,j,k}$ are given functions that satisfy the conditions:

$$\phi_{1,0,0}(x)\in L_p(G_1),\ \phi_{0,0,1}(z)\in L_p(G_3),\ \phi_{0,1,0}(y)\in L_p(G_2),$$
$$\phi_{1,1,0}(x,y)\in L_p(G_1\times G_2),\phi_{0,1,1}(y,z)\in L_p(G_2\times G_3),\ \phi_{1,0,1}(x,z)\in L_p(G_1\times G_3).$$

If the function $u \in W_p^{(1,1,1)}(G)$ is a solution to a problem with conditions in the middle of a domain of the classical form (9), (10), then it is also a solution to problem (9), (12) for $\phi_{i,j,k}$ defined by the following equalities

$$\begin{cases} \phi_{0,0,0} = \Phi(\frac{y_0+y_1}{2}, \frac{z_0+z_1}{2}) = \Psi\left(\frac{x_0+x_1}{2}, \frac{z_0+z_1}{2}\right) = g(\frac{x_0+x_1}{2}, \frac{y_0+y_1}{2}), \\ \phi_{1,0,0}(x) = \Psi_x\left(x\ , \frac{z_0+z_1}{2}\right) = g_x(x\ , \frac{y_0+y_1}{2}), \\ \phi_{0,1,0}(y) = g_y(\frac{x_0+x_1}{2}, y\ ) = \Phi_y(y, \frac{z_0+z_1}{2}), \\ \phi_{0,0,1}(z) = \Phi_z(\frac{y_0+y_1}{2}, z\ ) = \Psi_z\left(\frac{x_0+x_1}{2}, z\ \right), \\ \phi_{1,1,0}(x,y) = g_{xy}(x, y\ ), \\ \phi_{0,1,1}(y,z) = \Phi_{yz}(y\ , z\ ), \\ \phi_{1,0,1}(x,z) = \Psi_{xz}\left(x\ , z\ \right), \end{cases} \tag{13}$$

It is easy to prove that the converse is also true. In other words, if the function $u \in W_p^{(1,1,1)}(G)$ is a solution to problem (9), (12), then it is also a solution to problem (9), (10), for the following functions Φ, Ψ, g:

$$\begin{cases} \Phi(y,z) = \phi_{0,0,0} + \int\limits_{\frac{y_0+y_1}{2}}^{y} \phi_{0,1,0}(\beta)d\beta + \int\limits_{\frac{z_0+z_1}{2}}^{z} \phi_{0,0,1}(\gamma)d\gamma + \int\limits_{\frac{y_0+y_1}{2}}^{y}\int\limits_{\frac{z_0+z_1}{2}}^{z} \phi_{0,1,1}(\beta,\gamma)d\beta d\gamma, \\ \Psi(x,z) = \phi_{0,0,0} + \int\limits_{\frac{x_0+x_1}{2}}^{x} \phi_{1,0,0}(\alpha)d\alpha + \int\limits_{\frac{z_0+z_1}{2}}^{z} \phi_{0,0,1}(\gamma)d\gamma + \int\limits_{\frac{x_0+x_1}{2}}^{x}\int\limits_{\frac{z_0+z_1}{2}}^{z} \phi_{1,0,1}(\alpha,\gamma)d\alpha d\gamma, \\ g(x,y) = \phi_{0,0,0} + \int\limits_{\frac{x_0+x_1}{2}}^{x} \phi_{1,0,0}(\alpha)d\alpha + \int\limits_{\frac{y_0+y_1}{2}}^{y} \phi_{0,1,0}(\beta)d\beta + \int\limits_{\frac{x_0+x_1}{2}}^{x}\int\limits_{\frac{y_0+y_1}{2}}^{y} \phi_{1,1,0}(\alpha,\beta)d\alpha d\beta \end{cases} \tag{14}$$

Note that, in this case, functions (14) have one important property, related to the fact that for them the matching conditions (11) are satisfied automatically for all $\phi_{i,j,k}$, having the above properties having the above properties.

$$\Phi(y,z) \in W_p^{(1,1)}(G_2 \times G_3),$$
$$\Psi(x,z) \in W_p^{(1,1)}(G_1 \times G_3),$$
$$g(x,y) \in W_p^{(1,1)}(G_1 \times G_2),$$

satisfying the matching conditions (11).

Thus, problems with conditions in the middle of the region of the classical form (9), (10) and of the form (9), (12) are equivalent in the general case. However, problem (9),

(9) (12) is more natural in formulation than problem (9), (10). This is due to the fact that in the formulation of problem (9),

(12) on the right-hand sides of the boundary conditions, no additional conditions of the matching type are required.

Therefore, problem (9), (12) can be considered as a problem with conditions in the middle of a new type of domain.

3.2. Operator View of a Boundary Value Problem Given in the Middle of a Domain

Problem (9), (12) we will study by the method of operator equations. We first write problem (9), (12) as an operator equation

$$Vu = \phi, \tag{15}$$

where V is a vector operator defined by the equality
$V = (V_{0,0,0}, V_{1,0,0}, V_{0,1,0}, V_{0,0,1}, V_{1,1,0}, V_{0,1,1}, V_{1,0,1}, V_{1,1,1}) : W_p^{(1,1,1)}(G) \to E_p^{(1,1,1)}$
and ϕ there is a given vector element of the form

$\phi=(\phi_{0,0,0},\phi_{1,0,0},\phi_{0,1,0},\phi_{0,0,1},\phi_{1,1,0},\phi_{0,1,1},\phi_{1,0,1},\phi_{1,1,1})$ out of space

$$E_p^{(1,1,1)} \equiv R\times L_p(x_0,x_1)\times L_p(y_0,y_1)\times L_p(z_0,z_1)\times L_p(G_1\times G_2)\times L_p(G_2\times G_3)\times L_p(G_1\times G_3)\times L_p(G).$$

Note that in space $E_p^{(1,1,1)}$ we will define the norm in a natural way, using the equality

$$\|\phi\|_{E_p^{(1,1,1)}}=\|\phi_{0,0,0}\|_R+\|\phi_{1,0,0}\|_{L_p(x_0,x_1)}+\|\phi_{0,1,0}\|_{L_p(y_0,y_1)}+\|\phi_{0,0,1}\|_{L_p(z_0,z_1)}++\|\phi_{1,1,0}\|_{L_p(G_1\times G_2)}+\|\phi_{0,1,1}\|_{L_p(G_2\times G_3)}+\|\phi_{1,0,1}\|_{L_p(G_1\times G_3)}+\|\phi_{1,1,1}\|_{L_p(G)}.$$

To study the boundary value problem (9),(12), we will use the integral representation from (7), according to which any function $u(x,y,z)\in W_p^{(1,1,1)}(G)$ uniquely representable in the form

$$u(x,y,z)=(Qb)(x,y,z)\equiv b_{0,0,0}+\int_{\frac{x_0+x_1}{2}}^{x} b_{1,0,0}(\alpha)d\alpha+\int_{\frac{y_0+y_1}{2}}^{y} b_{0,1,0}(\beta)d\beta+\int_{\frac{z_0+z_1}{2}}^{z} b_{0,0,1}(\gamma)d\gamma+$$
$$+\int_{\frac{x_0+x_1}{2}}^{x}\int_{\frac{y_0+y_1}{2}}^{y} b_{1,1,0}(\alpha,\beta)d\alpha d\beta+\int_{\frac{y_0+y_1}{2}}^{y}\int_{\frac{z_0+z_1}{2}}^{z} b_{0,1,1}(\beta,\gamma)d\beta d\gamma+\int_{\frac{x_0+x_1}{2}}^{x}\int_{\frac{z_0+z_1}{2}}^{z} b_{1,0,1}(\alpha,\gamma)d\alpha d\gamma+$$
$$+\int_{\frac{x_0+x_1}{2}}^{x}\int_{\frac{y_0+y_1}{2}}^{y}\int_{\frac{z_0+z_1}{2}}^{z} b_{1,1,1}(\alpha,\beta,\gamma)d\alpha d\beta d\gamma \tag{16}$$

through a single element

$$b=(b_{0,0,0},b_{1,0,0},b_{0,1,0},b_{0,0,1},b_{1,1,0},b_{0,1,1},b_{1,0,1},b_{1,1,1})\in E_p^{(1,1,1)}.$$

There are positive constants M_1^0 and M_2^0 such that

$$M_1^0\|b\|_{E_p^{(1,1,1)}}\le\|Qb)(x,y,z)\|_{W_p^{(1,1,1)}(G)}\le M_2^0\|b\|_{E_p^{(1,1,1)}},\ \text{ for any }\ b\in E_p^{(1,1,1)} \tag{17}$$

It is obvious that the operator $Q:E_p^{(1,1,1)}\to W_p^{(1,1,1)}(G)$ is a linear bounded operator. Inequality (17) shows that the operator Q also has a bounded inverse operator defined on the space $W_p^{(1,1,1)}(G)$. Hence the operator Q is a homeomorphism between Banach spaces $E_p^{(1,1,1)}$ and $W_p^{(1,1,1)}(G)$. Therefore, the solution of equation (15) is equivalent to the solution of the equation

$$VQb=\phi, \tag{18}$$

Equation (18) will be called the canonical form of equation (15).

In addition, formula (16) shows that any function $u\in W_p^{(1,1,1)}(G)$ has traces:

$$u(\frac{x_0+x_1}{2},\frac{y_0+y_1}{2},\frac{z_0+z_1}{2}), u_x(x,\frac{y_0+y_1}{2},\frac{z_0+z_1}{2}), u_y(\frac{x_0+x_1}{2},y,\frac{z_0+z_1}{2}),$$
$$u_z(\frac{x_0+x_1}{2},\frac{y_0+y_1}{2},z), u_{xy}(x,y,\frac{z_0+z_1}{2}), u_{yz}(\frac{x_0+x_1}{2},y,z), u_{xz}(x,\frac{y_0+y_1}{2},z)$$

and the operations of taking these traces are continuous from $W_p^{(1,1,1)}(G)$ в R,

$$L_p(x_0,x_1),\ L_p(y_0,y_1), L_p(z_0,z_1), L_p(G_1\times G_2), L_p(G_2\times G_3), L_p(G_1\times G_3)$$

respectively.

Further, for these traces, the equalities are also valid:

$$u(\frac{x_0+x_1}{2},\frac{y_0+y_1}{2},\frac{z_0+z_1}{2})=b_{0,0,0},$$
$$u_x(x,\frac{y_0+y_1}{2},\frac{z_0+z_1}{2})=b_{1,0,0}(x),$$
$$u_y(\frac{x_0+x_1}{2},y,\frac{z_0+z_1}{2})=b_{0,1,0}(y),$$
$$u_z(\frac{x_0+x_1}{2},\frac{y_0+y_1}{2},z)=b_{0,0,1}(z),$$
$$u_{xy}(x,y,\frac{z_0+z_1}{2})=b_{1,1,0}(x,y),$$
$$u_{yz}(\frac{x_0+x_1}{2},y,z)=b_{0,1,1}(y,z),$$
$$u_{xz}(x,\frac{y_0+y_1}{2},z)=b_{1,0,1}(x,z).$$

3.3. Equivalent Integral Equation for a Boundary Value Problem

We will study problem (9), (12) using the integral representation (7) of the functions $u \in W_p^{(1,1,1)}(G)$. Formula (7) shows that the function that satisfies conditions (12) has the form:

$$u(x,y,z) = g_0(x,y,z) + \int\limits_{\frac{x_0+x_1}{2}}^{x_1} \int\limits_{\frac{y_0+y_1}{2}}^{y_1} \int\limits_{\frac{z_0+z_1}{2}}^{z_1} u_{xyz}(\alpha,\beta,\gamma) R_0(\alpha,\beta,\gamma;x,y,z)\, d\alpha\, d\beta\, d\gamma,$$

where

$$g_0(x,y,z) = \phi_{0,0,0} + \int\limits_{\frac{x_0+x_1}{2}}^{x} \phi_{1,0,0}(\alpha)d\alpha + \int\limits_{\frac{y_0+y_1}{2}}^{y} \phi_{0,1,0}(\beta)d\beta + \int\limits_{\frac{z_0+z_1}{2}}^{z} \phi_{0,0,1}(\gamma)d\gamma +$$

$$+ \int\limits_{\frac{x_0+x_1}{2}}^{x} \int\limits_{\frac{y_0+y_1}{2}}^{y} \phi_{1,1,0}(\alpha,\beta)d\alpha d\beta + \int\limits_{\frac{y_0+y_1}{2}}^{y} \int\limits_{\frac{z_0+z_1}{2}}^{z} \phi_{0,1,1}(\beta,\gamma)d\beta d\gamma + \int\limits_{\frac{x_0+x_1}{2}}^{x} \int\limits_{\frac{z_0+z_1}{2}}^{z} \phi_{1,0,1}(\alpha,\gamma)d\alpha d\gamma$$

and

$$R_0(\alpha,\beta,\gamma;x,y,z) = \theta(x-\alpha)\theta(y-\beta)\theta(z-\gamma),$$

$\theta(z)$ where is the Heaviside function on R, itc $\theta(z) = \begin{cases} 1, z>0 \\ 0, z \le 0 \end{cases}$. Then after replacing

$u = g_0 + \hat{u},$ where

$$\hat{u}(x,y,z) = \int\limits_{\frac{x_0+x_1}{2}}^{x_1} \int\limits_{\frac{y_0+y_1}{2}}^{y_1} \int\limits_{\frac{z_0+z_1}{2}}^{z_1} R_0(\alpha,\beta,\gamma;x,y,z) U_{xyz}(\alpha,\beta,\gamma)\, d\alpha\, d\beta\, d\gamma,$$

equation (9) can be written as

$$(V_{1,1,1}\hat{u})(x,y,z) = \hat{Z}(x,y,z), \tag{19}$$

where $\hat{Z} = \phi_{1,1,1} - V_{1,1,1} g_0$.

It is obvious that the derivative functions $\hat{u}$ can be calculated using the equalities

$$\hat{u}_x(x,y,z) = \int\limits_{\frac{y_0+y_1}{2}}^{y} \int\limits_{\frac{z_0+z_1}{2}}^{z} u_{xyz}(x,\beta,\gamma)\, d\beta\, d\gamma,$$

$$\hat{u}_y(x,y,z) = \int\limits_{\frac{x_0+x_1}{2}}^{x} \int\limits_{\frac{z_0+z_1}{2}}^{z} u_{xyz}(\alpha,y,\gamma)\, d\alpha\, d\gamma,$$

$$\hat{u}_z(x,y,z) = \int\limits_{\frac{x_0+x_1}{2}}^{x} \int\limits_{\frac{y_0+y_1}{2}}^{y} u_{xyz}(\alpha,\beta,z)\, d\alpha\, d\beta,$$

$$\hat{u}_{xy}(x,y,z) = \int\limits_{\frac{z_0+z_1}{2}}^{z} u_{xyz}(x,y,\gamma)\, d\gamma,$$

$$\hat{u}_{yz}(x,y,z)=\int_{\frac{x_0+x_1}{2}}^{x} u_{xyz}(\alpha,y,z)d\alpha,$$

$$\hat{u}_{xz}(x,y,z)=\int_{\frac{y_0+y_1}{2}}^{y} u_{xyz}(x,\beta,z)d\beta,$$

$$\hat{u}_{xyz}(x,y,z)=u_{xyz}(x,y,z).$$

Now we consider the dominant derivative as an unknown function, in other words, we make the change $u_{xyz}(x,y,z)=b(x,y,z)$. Then equation (9) can be written as:

$$(Nb)(x,y,z)\equiv b(x,y,z)+\int_{\frac{x_0+x_1}{2}}^{x}\int_{\frac{y_0+y_1}{2}}^{y}\int_{\frac{z_0+z_1}{2}}^{z} A_{0,0,0}(x,y,z)R_0(\alpha,\beta,\gamma;x,y,z)b(\alpha,\beta,\gamma)d\alpha d\beta d\gamma+$$

$$+\int_{\frac{y_0+y_1}{2}}^{y}\int_{\frac{z_0+z_1}{2}}^{z} A_{1,0,0}(x,y,z)b(x,\beta,\gamma)d\beta d\gamma+\int_{\frac{x_0+x_1}{2}}^{x}\int_{\frac{z_0+z_1}{2}}^{z} A_{0,1,0}(x,y,z)b(\alpha,y,\gamma)d\alpha d\gamma+$$

$$+\int_{\frac{x_0+x_1}{2}}^{x}\int_{\frac{y_0+y_1}{2}}^{y} A_{0,0,1}(x,y,z)b(\alpha,\beta,z)d\alpha d\beta+\int_{\frac{z_0+z_1}{2}}^{z} A_{1,1,0}(x,y,z)b(x,y,\gamma)d\gamma+\int_{\frac{x_0+x_1}{2}}^{x} A_{0,1,1}(x,y,z)b(\alpha,y,z)d\alpha+$$

$$+\int_{\frac{y_0+y_1}{2}}^{y} A_{1,0,1}(x,y,z)b(x,\beta,z)d\beta+\int_{\frac{x_0+x_1}{2}}^{x}\int_{\frac{y_0+y_1}{2}}^{y}\int_{\frac{z_0+z_1}{2}}^{z}\left[\int_{\frac{x_0+x_1}{2}}^{\tau}\int_{\frac{y_0+y_1}{2}}^{\xi}\int_{\frac{z_0+z_1}{2}}^{\eta} K_{0,0,0}(\tau,\xi,\eta;x,y,z)R_0(\alpha,\beta,\gamma;x,y,z)\times\right.$$

$$\times b(\alpha,\beta,\gamma)d\alpha d\beta d\gamma+\int_{\frac{y_0+y_1}{2}}^{\xi}\int_{\frac{z_0+z_1}{2}}^{\eta} K_{1,0,0}(\tau,\xi,\eta;x,y,z)b(\tau,\beta,\gamma)d\beta d\gamma+$$

$$+\int_{\frac{x_0+x_1}{2}}^{\tau}\int_{\frac{z_0+z_1}{2}}^{\eta} K_{0,1,0}(\tau,\xi,\eta;x,y,z)b(\alpha,\xi,\gamma)d\alpha d\gamma+\int_{\frac{x_0+x_1}{2}}^{\tau}\int_{\frac{y_0+y_1}{2}}^{\xi} K_{0,0,1}(\tau,\xi,\eta;x,y,z)b(\alpha,\beta,\eta)\times$$

$$\times d\alpha d\beta+\int_{\frac{z_0+z_1}{2}}^{\eta} K_{1,1,0}(\tau,\xi,\eta;x,y,z)b(\tau,\xi,\gamma)d\gamma+\int_{\frac{x_0+x_1}{2}}^{\tau} K_{0,1,1}(\tau,\xi,\eta;x,y,z)b(\alpha,\xi,\eta)d\alpha+$$

$$\left.+\int_{\frac{y_0+y_1}{2}}^{\eta} K_{1,0,1}(\tau,\xi,\eta;x,y,z))b(\tau,\beta,\eta)d\beta\right]d\tau d\xi d\eta=\hat{Z}(x,y,z),\qquad (x,y,z)\in G, \tag{20}$$

Operator N equation (20) is linear. Using the conditions imposed on the coefficients $A_{i,j,k}$, one can prove that this operator is a bounded operator from $L_p(G)$ в $L_p(G)$, $1\le p\le\infty$.

Definition. If problem (9), (12) for any $\phi=(\phi_{0,0,0},\phi_{1,0,0},\phi_{0,1,0},\phi_{0,0,1},\phi_{1,1,0},\phi_{0,1,1},\phi_{1,0,1},\phi_{1,1,1})\in \mathrm{E}_p^{(1,1,1)}$ has the only solution $u\in W_p^{(1,1,1)}(G)$ such that

$$\|u\|_{W_p^{(1,1,1)}(G)}\le M_1\|\phi\|_{\mathrm{E}_p^{(1,1,1)}},$$

then we will say that the operator V problem (9), (12) (or equation (15)) is a homeomorphism from $W_p^{(1,1,1)}(G)$ on or problem (9), (12) is everywhere correctly solvable. Here constant independent of.

Obviously, if the operator problem (9), (12) is a homeomorphism from in, then there is a bounded inverse operator.

The operator is a Volterra operator with respect to a point. This means that if the functions in area satisfy the condition, then the condition for almost everyone, where arbitrary point.

Using the Volterra property of the operator, using, for example, method of successive approximations can be proved, that equation (20) for any right-hand side has a unique solution, where, and this solution satisfies the condition this solution satisfies the condition, where constant independent of. Further, it is obvious that if, то. In addition, if is a solution to equation (20), then the solution to problem (9), (12) can be found using the equality. Therefore

Theorem 7. The operator of problem (9), (12) is a homeomorphism from and.

4. Conclusion

In this work, an integral representation is found for a function of several variables in the space of S. L. Sobolev with dominant mixed derivatives. The result obtained was applied to study one boundary value problem in the middle of the domain for the 3D Bianchi integro-differential equations.

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